



HAESE MATHEMATICS

Mathematics

Analysis and Approaches SL



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for use with

IB Diploma Programme

REVISION GUIDE

MATHEMATICS: ANALYSIS AND APPROACHES SL REVISION GUIDE

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FOREWORD

The aim of this Guide is to help you prepare for tests and the final examination for the Mathematics: Analysis and Approaches SL course.

This Guide covers all five Topics in the Mathematics: Analysis and Approaches SL syllabus. All of the relevant material from the Mathematics: Core Topics SL and the Mathematics: Analysis and Approaches SL textbooks is covered.

For each Topic, there is a theory summary and a set of skill builder questions.

- The theory summaries highlight the important facts and concepts. They are intended to complement your textbook and International Baccalaureate booklet. When a formula can be found in the formula booklet, it may not be repeated in this Guide.
- The set of skill builder questions are designed to help consolidate your understanding of each Topic. They should be used to reinforce key ideas, and to identify any areas of weakness. Within each Topic, the questions are logically ordered according to the chapters of the textbook, so they can be used for test preparation.

Following the coverage of all five Topics, the Guide has ten mixed questions sets, each containing 12 questions. Each set contains questions from every Topic, as well as cross-topic questions. It is recommended that you attempt all of the questions in a mixed questions set in one sitting, as this will give you practice in answering questions from a range of topics in a short time frame.

The Guide concludes with five trial examinations, written by IB teachers from around the world. Each trial examination contains two papers: Paper 1, where calculators are not permitted, and Paper 2, where calculators are required. This format is consistent with the Mathematics: Analysis and Approaches SL final examination. Full solutions are provided, but it is recommended that you work through a complete paper before checking the solutions.

We recommend completing each trial examination under exam conditions. You are encouraged to print the formulae summary (see page 5), and have it alongside you as you complete the trial examinations.

- If you are having trouble with a question, it is often a good strategy to move on to other questions, and return to it later. Time management is very important during the examination, and too much time spent on a difficult question may mean that you do not leave yourself sufficient time to complete other questions.
- Set out your work clearly with full explanations. A correct answer with no working will not necessarily receive full marks.
- If you make a mistake, draw a single line through the work you want to replace. Do not cross out work until you have replaced it with something you consider better.
- Diagrams and graphs should be sufficiently large, well labelled, and clearly drawn.

- Remember to leave answers correct to three significant figures unless an exact answer is more appropriate or a different level of accuracy is requested in the question.
- Check for key words. If the word “hence” appears, then you must use the result you have just obtained. “Hence, or otherwise” means that you can use any method you like, although it is likely that the best method uses the previous result.
- It is important to read the question carefully. Rushing into a question may mean that you miss subtle points. Underlining key words may help.
- Remember that questions in the examination are often set so that, even if you cannot complete one part, the question can still be picked up in a later part.

After completing a trial examination, you should identify areas of weakness.

- Return to your notes or textbook and review any material you found challenging.
- Ask your teacher or a friend for help if further explanation is needed.
- Summarise each Topic. Summaries that you make yourself are the most valuable.
- If you have had difficulty with a question, try it again later. Do not just assume that you know how to do it once you have read the solution.

In addition to the formula booklet, your graphics display calculator is an essential aid.

- Make sure you are familiar with the model you will be using.
- In trigonometry questions, remember to check whether the graphics calculator should be in degrees or radians.
- Important features of graphs may be revealed by zooming in or out.
- When using your graphics calculator, it is always important to reflect on the reasonableness of the results.

We hope this Guide will help you structure your revision program effectively. Remember that good examination techniques will come from good examination preparation.

We welcome your feedback:

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FORMULAE SUMMARY



PRIOR LEARNING

Area of a parallelogram	$A = bh$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(bh)$, where b is the base, h is the height
Area of a trapezoid	$A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a cuboid	$V = lwh$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Volume of a prism	$V = Ah$, where A is the area of the cross-section, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

TOPIC 1: NUMBER AND ALGEBRA

ARITHMETIC SEQUENCES

$$u_n = u_1 + (n - 1)d$$

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d) \quad \text{or} \quad S_n = \frac{n}{2}(u_1 + u_n)$$

GEOMETRIC SEQUENCES

$$u_n = u_1 r^{n-1}$$

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

$$S_\infty = \frac{u_1}{r - 1}, \quad |r| < 1$$

COMPOUND INTEREST

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}, \quad \text{where}$$

FV is the future value
 PV is the present value
 n is the number of years
 k is the number of compounding periods per year
 $r\%$ is the nominal annual rate of interest

EXPONENTS AND LOGARITHMS

$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$

$\log_a xy = \log_a x + \log_a y$

$\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^m = m \log_a x$

$\log_a x = \frac{\log_b x}{\log_b a}$

BINOMIAL THEOREM

$(a + b)^n = a^n + {}^nC_1 a^{n-1}b + + {}^nC_r a^{n-r}b^r + + b^n$

${}^nC_r = \frac{n!}{r!(n-r)!}$

TOPIC 2: FUNCTIONS

STRAIGHT LINES

$y = mx + c$ or $ax + by + d = 0$ or $y - y_1 = m(x - x_1)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

QUADRATIC FUNCTIONS AND EQUATIONS

$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$

$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$

$\Delta = b^2 - 4ac$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$a^x = e^{x \ln a}$ and $\log_a a^x = x = a^{\log_a x}$ where $a, x > 0, a \neq 1$

TOPIC 3: GEOMETRY AND TRIGONOMETRY

MEASUREMENT

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Volume of a right-pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base, h is the height
Volume of a right cone	$V = \frac{1}{3}\pi r^2h$, where r is the radius, h is the height
Area of the curved surface of a cone	$A = \pi rl$, where r is the radius, l is the slant height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
Surface area of a sphere	$A = 4\pi r^2$, where r is the radius
Length of an arc	$l = r\theta$, where r is the radius, θ is the angle measured in radians
Area of a sector	$A = \frac{1}{2}r^2\theta$, where r is the radius, θ is the angle measured in radians

TRIGONOMETRY

Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab \cos C$ and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Area of a triangle	$A = \frac{1}{2}ab \sin C$
Identity for $\tan \theta$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
Pythagorean identity	$\cos^2 \theta + \sin^2 \theta = 1$
Double angle identities	$\sin 2\theta = 2 \sin \theta \cos \theta$ $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ $ = 2 \cos^2 \theta - 1$ $ = 1 - 2 \sin^2 \theta$

TOPIC 4: STATISTICS AND PROBABILITY

Interquartile range = $Q_3 - Q_1$

Mean of a set of data $\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$, where $n = \sum_{i=1}^k f_i$

PROBABILITY

Probability of an event A $P(A) = \frac{n(A)}{n(U)}$

$P(A) + P(A') = 1$
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $P(A \cup B) = P(A) + P(B)$ for mutually exclusive events
 $P(A | B) = \frac{P(A \cap B)}{P(B)}$
 $P(A \cap B) = P(A) P(B)$ for independent events

Expected value of a discrete random variable X , $E(X) = \sum x P(X = x)$

BINOMIAL DISTRIBUTION

For $X \sim B(n, p)$:

- Mean $E(X) = np$
- Variance $\text{Var}(X) = np(1 - p)$

STANDARDISED NORMAL VARIABLE

$z = \frac{x - \mu}{\sigma}$

TOPIC 5: CALCULUS

DIFFERENTIATION

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

Chain rule	$y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

INTEGRATION

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$
$$\int \frac{1}{x} dx = \ln|x| + C$$
$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$
$$\int e^x dx = e^x + C$$

Area between a curve $y = f(x)$, where $f(x) > 0$, and the x -axis $= \int_a^b y dx$

Area of region enclosed by a curve and x -axis $= \int_a^b |y| dx$

KINEMATICS

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Distance travelled from t_1 to $t_2 = \int_{t_1}^{t_2} |v(t)| dt$

Displacement from t_1 to $t_2 = \int_{t_1}^{t_2} v(t) dt$

TOPIC 1: NUMBER AND ALGEBRA

SCIENTIFIC NOTATION (STANDARD FORM)

A number is in **scientific notation** if it is written in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule. Often, the rule is a formula for the **general term** or **n th term** of the sequence.

A sequence which continues forever is called an **infinite sequence**. A sequence which terminates is called a **finite sequence**.

Arithmetic sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all $n \in \mathbb{Z}^+$, where d is a constant called the **common difference**.

For an arithmetic sequence with first term u_1 and common difference d , the n th term is $u_n = u_1 + (n - 1)d$.

Geometric sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant, called the **common ratio** r .

$u_{n+1} = ru_n$, so we can find $r = \frac{u_{n+1}}{u_n}$ for all $n \in \mathbb{Z}^+$.

For a geometric sequence with first term u_1 and common ratio r , the n th term is $u_n = u_1 r^{n-1}$.

Series

A **series** is the sum of the terms of a sequence.

For a finite sequence with n terms, the corresponding series is $S_n = u_1 + u_2 + \dots + u_n$.

For an infinite sequence, the corresponding series $u_1 + u_2 + \dots + u_n + \dots$ can only be calculated in some cases.

Using **sigma notation** or **summation notation** we write $u_1 + u_2 + u_3 + \dots + u_n$ as $\sum_{k=1}^n u_k$.

For a **finite arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n - 1)d)$.

For a **finite geometric series** with $r \neq 1$, $S_n = \frac{u_1(r^n - 1)}{r - 1}$.

The sum of an **infinite geometric series** is $S = \frac{u_1}{1 - r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent**.

Compound interest

The value of a compound interest investment after n time periods is

$$u_n = u_0(1 + i)^n$$

where u_0 is the initial value of the investment

and i is the interest rate per compounding period.

To find the **real value** of the investment, we divide by the inflation multiplier each year.

Depreciation

Depreciation is the loss in value of an item over time.

The value of an item after n years is $u_n = u_0(1 - d)^n$

where u_0 is the initial value of the item

and d is the rate of depreciation per year.

EXPONENTIALS AND LOGARITHMS

Laws of exponents	
$a^m \times a^n = a^{m+n}$	$a^0 = 1, a \neq 0$
$\frac{a^m}{a^n} = a^{m-n}$	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
$(a^m)^n = a^{mn}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(ab)^n = a^n b^n$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	

If $a^x = a^k$ then $x = k$. So, if the base numbers are the same, we can **equate indices**.
If $a^x = b$, $a \neq 1$, $a, b > 0$, we say that x is the **logarithm** of b in base a , and that $a^x = b \Leftrightarrow x = \log_a b$.
 $x = \log_a(a^x)$ and $x = a^{\log_a x}$ provided $x > 0$.
The **natural logarithm** is the logarithm in base e . $\ln x \equiv \log_e x$

Laws of logarithms	
Base a , $a \neq 1, a > 0$	Base e
$\log_a xy = \log_a x + \log_a y$	$\ln xy = \ln x + \ln y$
$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\ln \left(\frac{x}{y}\right) = \ln x - \ln y$
$\log_a (x^m) = m \log_a x$	$\ln (x^m) = m \ln x$
$\log_a 1 = 0$	$\ln 1 = 0$
$\log_a a = 1$	$\ln e = 1$

To change the base of a logarithm, use the rule $\log_a x = \frac{\log_b x}{\log_b a}$.

PROOF

A **mathematical proof** is a correct argument which establishes the truth of a mathematical statement.
In a **deductive proof**, we start with a **hypothesis**, and then perform a series of **implications** to arrive at the **conclusion**.
To prove that two statements A and B are **equivalent**, we must show that $A \Rightarrow B$ and $B \Rightarrow A$. We write $A \Leftrightarrow B$ to indicate that A and B are equivalent.

THE BINOMIAL THEOREM

$a + b$ is called a **binomial** as it contains two terms.
Any expression of the form $(a + b)^n$ is called a **power of a binomial**.
The coefficients of the terms of $(a + b)^n$, $n \in \mathbb{N}$, are row n of Pascal’s triangle.

1	1			row 1
1	2	1		row 2
1	3	3	1	row 3
⋮	⋮	⋮	⋮	⋮

The **binomial coefficient** nC_r or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
where $n! = n(n-1)(n-2) \dots \times 3 \times 2 \times 1$
and $0! = 1$

You should also know how to calculate binomial coefficients using your calculator.

The **general binomial expansion** is $(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$
 $= \sum_{r=0}^n \binom{n}{r}a^{n-r}b^r$

where $\binom{n}{r}$ is the binomial coefficient of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, \dots, n$.

The **general term** in the binomial expansion is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

SKILL BUILDER QUESTIONS

- 1 Write without brackets:
 - a $(-3m^3)^4$
 - b $\left(\frac{xy^2}{2}\right)^5$
 - c $7s^2t \times (4st^3)^3$
- 2 Simplify:
 - a $4^0 + 4^{-1}$
 - b $\left(2\frac{3}{4}\right)^{-2}$
 - c $2^2 + 2^1 + 2^{-1}$
- 3 Expand the brackets and write in simplest form:
 - a $(x^2 + x^{-2})^2$
 - b $(x^4 - x^2)(x^3 + 3)$
- 4 Write without negative exponents:
 - a a^2b^{-3}
 - b $\frac{2m^{-2}n^3}{m^5n^{-5}}$
 - c $\frac{12a^{-3}}{b^{-5}}$
- 5 Use your calculator to evaluate the following, giving your answer in scientific notation:
 - a $(3.57 \times 10^6) \times (2.38 \times 10^3)$
 - b $\frac{4.61 \times 10^{-7}}{3.45 \times 10^8}$
 - c $(0.000\,08)^4$
- 6 The fluoride concentration of lakes in a particular region was found to be 3×10^{-4} g per litre.
 - a One lake has 5.6×10^8 litres of water. Find the amount of fluoride in the lake, giving your answer in scientific notation.
 - b Another lake contains 4.13×10^7 g of fluoride. Find the volume of the lake.
- 7 Find k given the consecutive arithmetic terms:
 - a 3, k , 11
 - b -2 , $k + 4$, $k^2 + 11$
 - c $k - 5$, $2k$, $2k^2$
- 8 An empty hamster cage has mass 800 g. When 5 hamsters are placed in the cage, the total mass is 1400 g.
 - a Find the average mass of the hamsters in the cage.
 - b Hence write an arithmetic sequence for u_n , the approximate total mass when n hamsters are placed in the cage.
- 9 Find the general term u_n of the geometric sequence which has:
 - a $u_5 = 324$ and $u_{10} = 78\,732$
 - b $u_8 = -10$ and $u_{12} = -160$
- 10 Consider the sequence $2, 2\sqrt{3}, 6, 6\sqrt{3}, \dots$
 - a Show that the sequence is geometric.
 - b Find the formula for its general term.
 - c Find the 10th term.
 - d Find the first term which exceeds 1000.
- 11 An endangered species of bird has population 217. However, with a successful breeding program it is expected to increase by 42% each year.
 - a Find the expected population size after:
 - i 5 years
 - ii 10 years.
 - b How long will it take for the population to reach 30 000?
- 12 Paige invests €500 in an account that pays 7.2% p.a. compounded monthly.

The amount of money in Paige's account at the end of each month follows a geometric sequence with common ratio r .

 - a Find the value of r .
 - b Find the value of the account after 3 years.
 - c Given that inflation averages 2% p.a. over the 3 years, find the real value of the investment after 3 years.
- 13 A television was purchased for £2000, and depreciates at 30% p.a. for 3 years.
 - a Find the value of the television at the end of this period.
 - b By how much has the television depreciated?
- 14 Find the sum of:
 - a $11 + 15 + 19 + 23 + \dots$ to 20 terms
 - b $7 + 12.5 + 18 + 23.5 + \dots + 106$
 - c $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms
 - d the integers from 1 to 200 not divisible by 3.

- 15** An arithmetic sequence has terms $u_7 = 1$ and $u_{15} = -23$.
- Find the first term u_1 and common difference d .
 - Find the 27th term u_{27} .
 - Find the sum of the first 27 terms of the series.
- 16** The first term of a finite arithmetic series is 18 and the sum of the series is -210 . The common difference is -3 . Suppose there are n terms in the series.
- Show that $\frac{n}{2}(39 - 3n) = -210$.
 - Hence find n .
- 17** Consider the arithmetic sequence 7, 10, 13, 16, 19,
- Write down an expression for the sum of the first n terms S_n .
 - Find n such that $S_n = 140$.
- 18** The first four terms of a geometric sequence are 0.125, 0.25, 0.5, and 1.
- Write down the common ratio r .
 - Find the 20th term u_{20} .
 - Find the sum of the first 10 terms.
- 19** Kapil invested 2000 rupees in a bank account on January 1st 2012. The account pays 8.25% per annum compounded annually.
- Find the total value of Kapil's investment on January 1st 2019.
 - Would it have been a better option for Kapil to invest his money in an account paying 8% per annum interest compounded monthly? Justify your answer.
- 20** **a** An infinite geometric series is defined by $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k$.
- Find the first term u_1 and common ratio r .
 - Find the sum of the series.
- b** A finite arithmetic series is defined by $\sum_{k=1}^n (k - 4)$.
- Find the first term u_1 and common difference d .
 - Find the sum of the series, in terms of n .
- c** Find n such that the sums of the series in **a** and **b** are equal.
- 21** **a** Find the sum to infinity of the infinite geometric series $1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$
- b** When a ball is dropped from a height of 1 m, on each bounce it returns to 60% of the height it reached previously. Find the total distance travelled by the ball until it stops bouncing.
- 22** Consider the series $\sum_{k=1}^{\infty} 12(x - 2)^{k-1}$.
- For what values of x will the series converge?
 - Evaluate the sum of the series when $x = \sqrt{5}$.
- 23** Find x if $\sum_{k=1}^{\infty} \left(\frac{4x}{3}\right)^{k-1} = \frac{5}{2}$.
- 24** Write using factorial notation: **a** $10 \times 9 \times 8 \times 7$ **b** $\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$
- 25** **a** Use Pascal's triangle to find the binomial expansion of $(a + b)^4$.
- b** Hence find the binomial expansion of:
- $(2 + x)^4$
 - $(3x - y)^4$
 - $\left(x - \frac{2}{x}\right)^4$
- 26** **a** Write down the first 5 rows of Pascal's triangle.
- b** Hence expand and simplify: **i** $\left(x + \frac{1}{x}\right)^5$ **ii** $(1 - \sqrt{2})^5$
- 27** If $(a + bx)^n = 1 - 12x + 54x^2 - \dots$, $a > 0$, $n \in \mathbb{Z}^+$, find the values of a , b , and n .
- 28** Consider the binomial expansion of $(a + b)^6$.
- Write down the general term in the expansion.
 - Given that $\binom{6}{4} = 15$, find the coefficient of a^4b^2 .
- 29** Find the coefficient of x^5 in the expansion of $(x + 2)(1 - x)^{10}$.
- 30** **a** Use technology to find 7C_r for $r = 0, 1, \dots, 7$. **b** Hence find r such that ${}^7C_r = 35$.
- c** In the expansion of $(2x + k)^7$, $k > 0$, the coefficient of x^3 is 10 times the coefficient of x . Find the value of k .

- 31** **a** Expand $(x - 2)^3$. **b** Hence find the coefficient of x^3 in $(3x^2 - 7)(x - 2)^3$.
- 32** **a** Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate $\binom{6}{2}$. **b** Hence state the value of $\binom{6}{4}$.
c Given that $\binom{6}{3} = 20$, write down the expansion of $(x - 2)^6$. Simplify your answer.
- 33** Without using a calculator, write in simplest rational form:
a $4^{\frac{5}{2}}$ **b** $49^{-\frac{3}{2}}$ **c** $27^{\frac{5}{3}}$
- 34** Expand and simplify:
a $x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}})$ **b** $5^x(5^{-x} + 5^{3x})$ **c** $2^{-2x}(2^{2x+3} - 2^{-4x} + 3)$
- 35** Factorise:
a $36 - 16^x$ **b** $9^x + 4(3^x) - 12$ **c** $25^x - 5^{x+1} - 24$
- 36** Solve for x :
a $5 \times 2^x = 160$ **b** $8^{2x-3} = 16^{2-x}$ **c** $(\frac{1}{3})^{2x-5} = 27$ **d** $25^x + 2(5^x) = 35$
- 37** Simplify:
a $\log(10^9 \times 1000^b)$ **b** $\log\left(\frac{10^n}{100}\right)$ **c** $\log(2^t \times 5^t)$
- 38** Find:
a $\log_4 8$ **b** $\log_9\left(\frac{1}{27}\right)$ **c** $\log_9\left(\frac{1}{3\sqrt{3}}\right)$ **d** $\log_{49} 7$
- 39** Solve for x :
a $\log_3 x = 2$ **b** $\log_x 27 = 3$ **c** $\log_5(2x - 1) = -1$
- 40** Find x in terms of a if $a > 1$ and $\log_a(x + 2) = \log_a x + 2$.
- 41** Simplify:
a $\log_m(m^4)$ **b** $\log_n(n^2\sqrt{n})$ **c** $\log_a\left(\frac{1}{a^3}\right)$
- 42** Simplify by writing as a single logarithm or as a rational number:
a $\frac{1}{4}\ln 81 + \ln 12 - \ln 4$ **b** $3\log_9 2 - \log_9 24$ **c** $5 + \log_2 3 - \frac{1}{2}\log_2 49$
- 43** If $x = \log_a 5$, write in terms of x :
a $\log_a(5a)$ **b** $\log_a\left(\frac{125}{a^2}\right)$ **c** $\log_{25a} 5$
- 44** Suppose $A = \log_{10} P$, $B = \log_{10} Q$, and $C = \log_{10} R$. Write in terms of A , B , and C :
a $\log_{10}(PQ)$ **b** $\log_{10}(P^2Q\sqrt{R})$ **c** $\log_{10}\left(\frac{PQ^3}{R^2}\right)$
- 45** Simplify without using a calculator:
a $\frac{\log_2 9}{\log_2 3}$ **b** $\frac{\log_5 8}{\log_5 4}$ **c** $\frac{\log_3(0.25)}{\log_3 64}$
- 46** Write as a single logarithm:
a $\ln 20 - \ln 10$ **b** $-\ln 13 - 3$ **c** $\frac{1}{3}\ln 64 + 2\ln 2$
- 47** Write as a logarithmic equation in base b :
a $M = ab^3$ **b** $D = \frac{a}{b^2}$ **c** $F = \sqrt{\frac{b}{a^3}}$
- 48** Solve for x :
a $3\log_5 x = \log_5 24 + \log_5\left(\frac{1}{3}\right)$ **b** $\log_2 x = \log_2 12 - \log_2(7 - x)$
c $\ln(x^2 - 3) - \ln(2x) = 0$ **d** $\log_3 x + \log_3(x - 2) = 1$
- 49** Solve for x :
a $\log_{\frac{1}{9}} x = \log_9 5$ **b** $\log_2 x - \log_8 x = 3$ **c** $\log_{27}(x^4) = \log_9 x - \log_3(\sqrt[5]{9})$
- 50** Write $\frac{8}{\log_5 9}$ in the form $a \log_3 b$ where $a, b \in \mathbb{Z}$.
- 51** Solve for x exactly:
a $3^x = 15$ **b** $3^{x+1} = 8$ **c** $e^{2x} - 20 = e^x$ **d** $3 \times 4^x - 2^x = 0$

52 Solve for x :

a $9^x - 6(3^x) + 8 = 0$

b $25^x - 5^{x+1} + 6 = 0$

c $2 \times 3^{2x} + 3^{x+1} = 5$

53 The population of kangaroos on an island is given by $K(t) = 3200 \times (0.85)^t$, where t is the time in years.

a What was the initial kangaroo population?

b How many kangaroos were on the island after 5 years?

c Use technology to estimate how many years it will take for the kangaroo population to fall to 1000.

d Check your answer to **c** using logarithms.

54 State, with justification, whether each statement is true or false:

a If $x > 1$ then $\frac{1}{x} < 1$.

b If $\frac{1}{x} < 1$ then $x > 1$.

c $x > 1$ if and only if $\frac{1}{x} < 1$.

55 Prove that the sum of three consecutive odd integers is divisible by 3.

56 The following “proof” by deduction is incorrect. Identify the incorrect step and write the correct solution to the inequality.

$$2x^3 \geq x$$

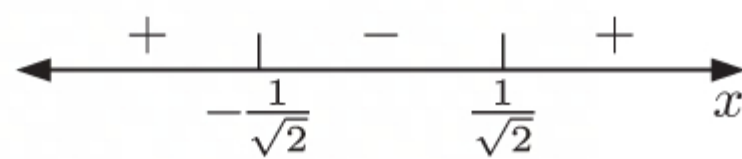
$$\therefore 2x^2 \geq 1$$

$$\therefore x^2 \geq \frac{1}{2}$$

$$\therefore x^2 - \frac{1}{2} \geq 0$$

$$\therefore \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0$$

$$\therefore x \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad x \leq -\frac{1}{\sqrt{2}}$$



57 a Show that $(a + b)^3 - (a - b)^3 = 2b(3a^2 + b^2)$.

b Check this result for $a = 2$, $b = 1$.

58 Suppose $x, y, z > 0$. Show that $xyz = 1 \Leftrightarrow \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = x^2z + y^2x + z^2y$.

TOPIC 2: FUNCTIONS

PROPERTIES OF LINES

The **gradient** of the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is $m = \frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}$.

The gradient of any horizontal line is zero. The gradient of any vertical line is undefined.

The **y-intercept** of a line is the value of y where the line cuts the y -axis.

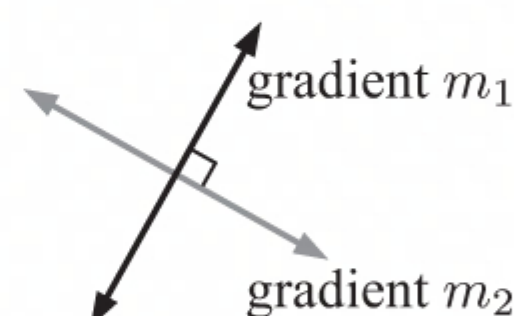
The **x-intercept** of a line is the value of x where the line cuts the x -axis.

PARALLEL AND PERPENDICULAR LINES

The gradients of parallel lines are equal.

The gradients of perpendicular lines are negative reciprocals.

$$m_1 = -\frac{1}{m_2}$$



EQUATION OF A LINE

The equation of a line can be presented in:

- **gradient-intercept form** $y = mx + c$ where m is the gradient and c is the y -intercept.
- **general form** $ax + by = d$
- **point-gradient form** $y - y_1 = m(x - x_1)$

You should be able to find the equation of a line given:

- its gradient and the coordinates of any point on the line
- the coordinates of two distinct points on the line.

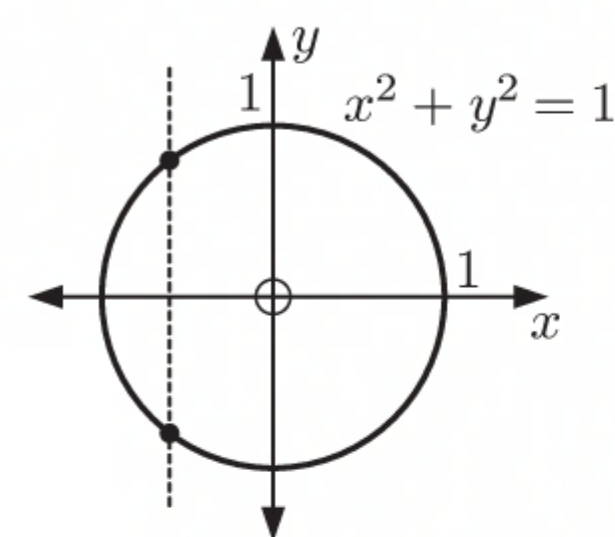
FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** between variables x and y is any set of points in the (x, y) plane.

A **function** is a relation in which no two different ordered pairs have the same x -coordinate or first component. For each value of x there is only one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **image** of x .

We test for functions using the **vertical line test**. A graph is a function if no vertical line intersects the graph more than once.

For example, the graph of the circle $x^2 + y^2 = 1$ shows that this relation is not a function.



The **domain** of a relation is the set of values that x can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a relation is the set of values that y or $f(x)$ can take.

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g is $f \circ g : x \mapsto f(g(x))$.

In general, $f(g(x)) \neq g(f(x))$, so $f \circ g \neq g \circ f$.

The **identity function** is $f(x) = x$.

INVERSE FUNCTIONS

A function is:

- **one-to-one** if there is only one value of x for each value of y
- **many-to-one** if there is more than one value of x with the same value of y .

The function $y = f(x)$ has an **inverse function** $y = f^{-1}(x)$ if and only if it is one-to-one.

Many-to-one functions do not have an inverse function.

If $y = f(x)$ has an inverse function $y = f^{-1}(x)$, then the inverse function:

- must satisfy the vertical line test
- is a reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- has range equal to the domain of $f(x)$
- has domain equal to the range of $f(x)$.

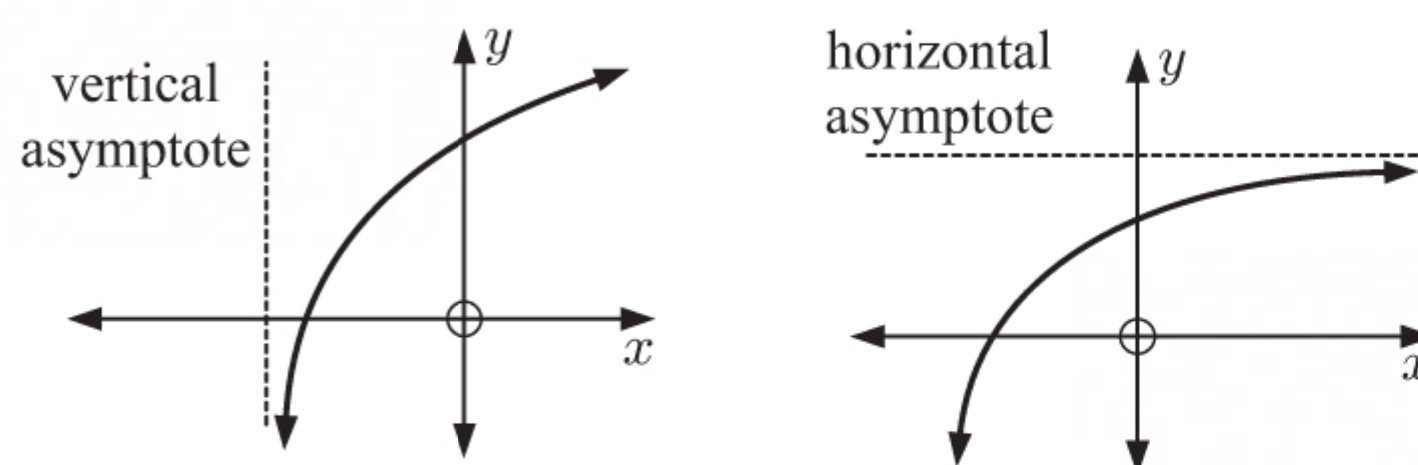
An invertible function f is **self-inverse** if $f^{-1} = f$. The graph of a self-inverse function is symmetrical about the line $y = x$.

GRAPHS OF FUNCTIONS

The **x -intercepts** of a function are the values of x for which $y = 0$. They are the **zeros** of the function.

The **y -intercept** of a function is the value of y when $x = 0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.



To find vertical asymptotes, look for values of x for which the function is undefined:

- If $y = \frac{f(x)}{g(x)}$, find where $g(x) = 0$.
- If $y = \log_a(f(x))$, find where $f(x) = 0$.

To find horizontal asymptotes, consider the behaviour as $x \rightarrow \pm\infty$.



You should be able to use technology to solve equations graphically.

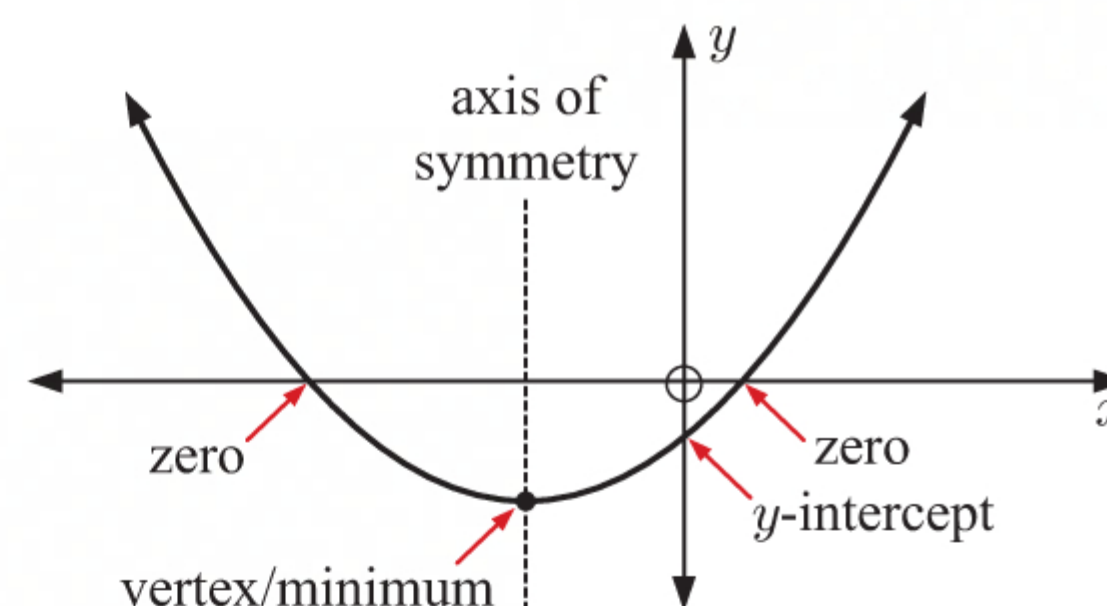
QUADRATICS

Quadratic functions

A **quadratic function** has the form $y = ax^2 + bx + c$, $a \neq 0$.

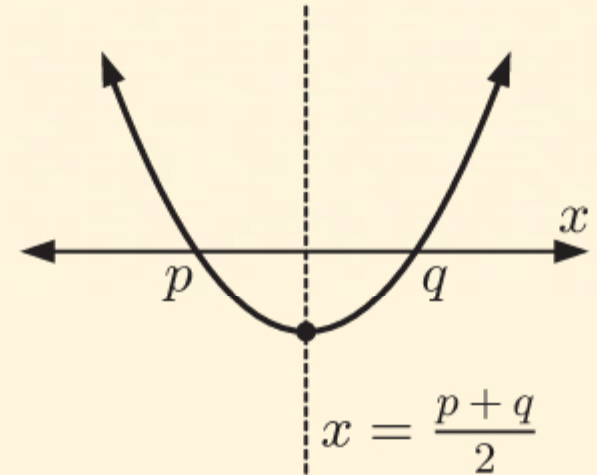
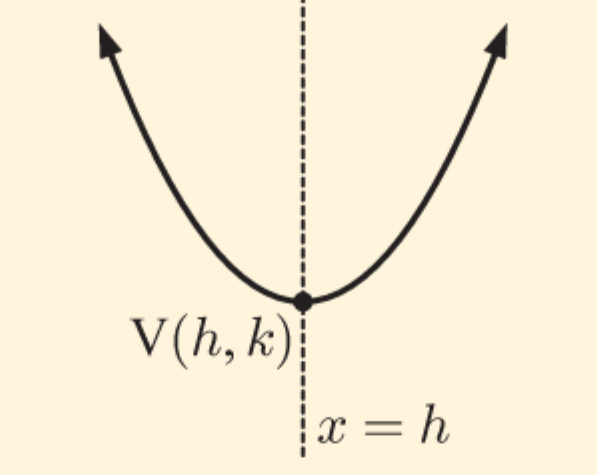
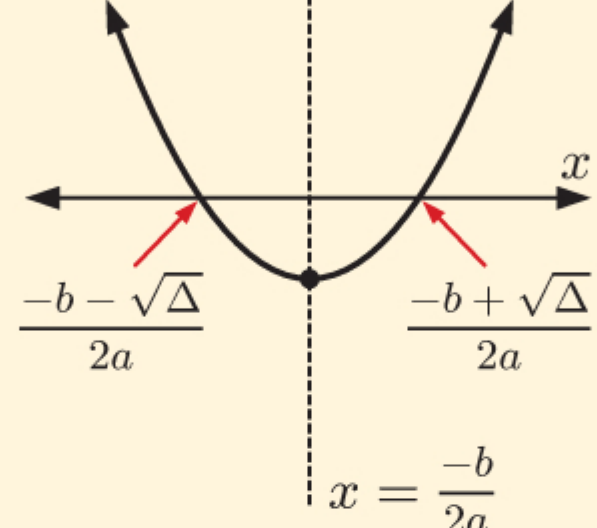
The graph is a parabola with the following properties:

- It is *concave up* if $a > 0$  and *concave down* if $a < 0$. 
- Its axis of symmetry is $x = \frac{-b}{2a}$.
- Its vertex has x -coordinate $\frac{-b}{2a}$. The y -coordinate of its vertex is found by substituting $x = \frac{-b}{2a}$ into the function.
 - ▶ If $a > 0$ the vertex is a minimum turning point.
 - ▶ If $a < 0$ the vertex is a maximum turning point.



The quadratic function has **discriminant** $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the graph cuts the x -axis twice.
- If $\Delta = 0$, the graph *touches* the x -axis.
- If $\Delta < 0$, the graph does not cut the x -axis.

$y = a(x - p)(x - q)$ x -intercepts p, q axis of symmetry $x = \frac{p+q}{2}$	
$y = a(x - h)^2 + k$ vertex (h, k) axis of symmetry $x = h$	
$y = ax^2 + bx + c$ axis of symmetry $x = \frac{-b}{2a}$ x -intercepts $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac \geq 0$	

Quadratic equations

A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, can be solved by:

- factorisation
- completing the square
- the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant of the quadratic equation is $\Delta = b^2 - 4ac$.

The quadratic equation has:

- *two real solutions* if $\Delta > 0$
- *one real (repeated) solution* if $\Delta = 0$
- *no real solutions* if $\Delta < 0$.

Quadratic inequalities

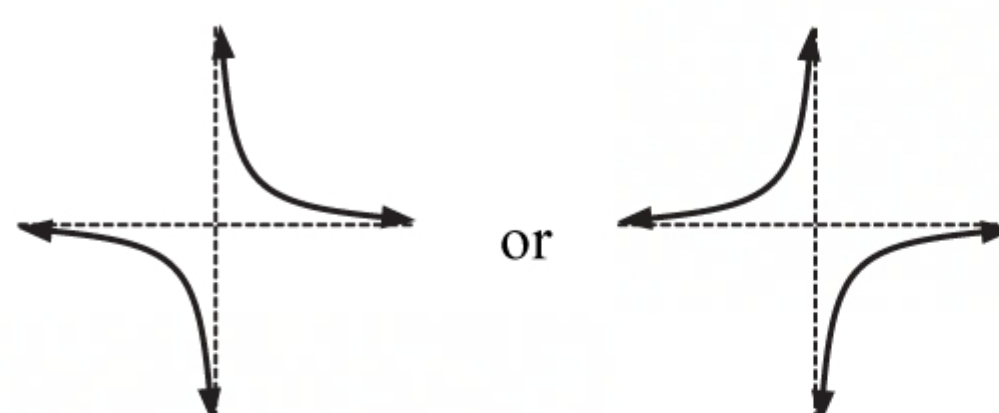
A **quadratic inequality** can be written in either the form $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$ where $a \neq 0$.

You should be able to use sign diagrams to solve quadratic inequalities.

RATIONAL FUNCTIONS

The **rational functions** we consider in this course can be written in the form $y = \frac{ax+b}{cx+d}$, $c \neq 0$.

They are characterised by a vertical asymptote $x = -\frac{d}{c}$ and a horizontal asymptote $y = \frac{a}{c}$.



A **reciprocal function** is a function of the form $f(x) = \frac{k}{x}$, $k \neq 0$.

All reciprocal functions are self-inverse functions.

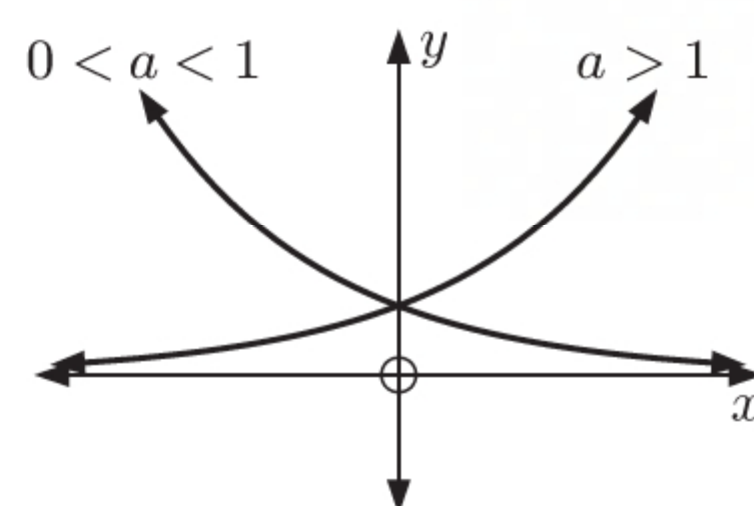
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The simplest **exponential function** is $f(x) = a^x$, $a > 0$, $a \neq 1$.

If $a > 1$ we have *growth*.

If $0 < a < 1$ we have *decay*.

The graph of $y = a^x$ has the horizontal asymptote $y = 0$.



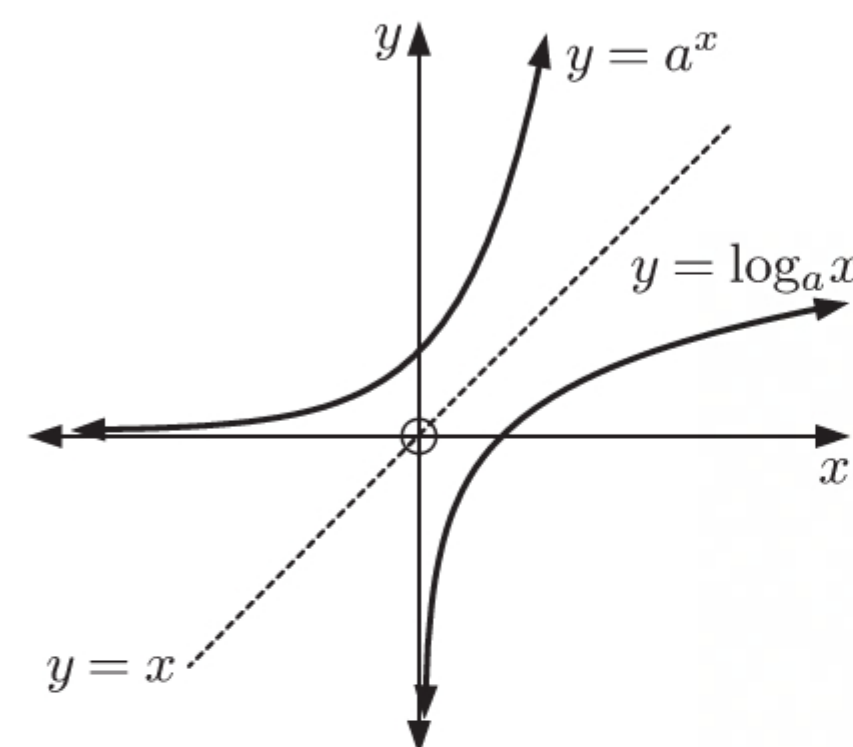
For the general exponential function $y = a^{x-h} + k$ where $a > 0$, $a \neq 1$:

- a controls how steeply the graph increases or decreases.
- h controls horizontal translation.
- k controls vertical translation.
- The equation of the horizontal asymptote is $y = k$.

The **logarithmic function** $y = \log_a x$, $x > 0$ is the inverse function of $y = a^x$.

The graph of $y = \log_a x$ has the vertical asymptote $x = 0$.

The natural logarithmic function $y = \ln x$, $x > 0$ is the inverse function of $y = e^x$.

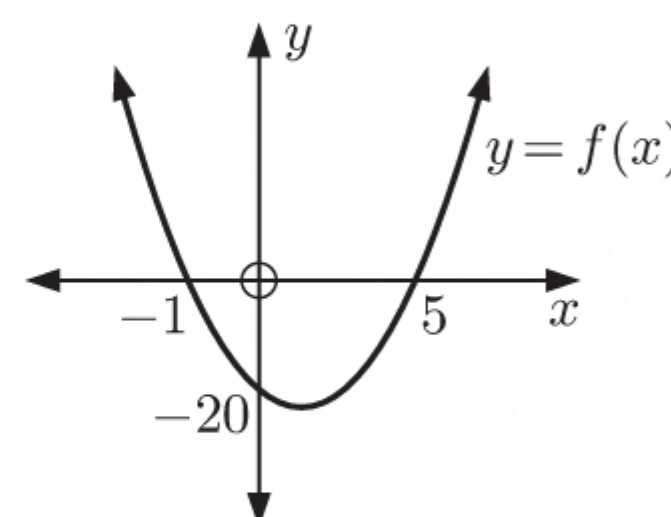


TRANSFORMATIONS OF FUNCTIONS

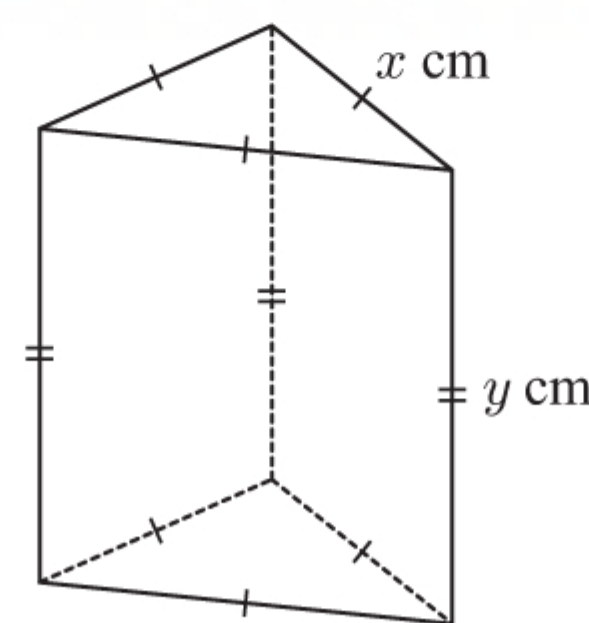
- $y = f(x) + b$ **translates** $y = f(x)$ vertically b units.
- $y = f(x - a)$ **translates** $y = f(x)$ horizontally a units.
- $y = f(x - a) + b$ **translates** $y = f(x)$ by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.
- $y = pf(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with scale factor p .
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with scale factor $\frac{1}{q}$.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the x -axis.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the y -axis.
- If $f^{-1}(x)$ exists, $y = f^{-1}(x)$ is a **reflection** of $y = f(x)$ in the line $y = x$.

SKILL BUILDER QUESTIONS

- 1 A line segment has equation $4x - 3y + 2 = 0$. Its midpoint is $(4, 6)$.
 - a State the gradient of:
 - i the line segment
 - ii its perpendicular bisector.
 - b State the equation of the perpendicular bisector. Write your answer in the form $ax + by + d = 0$.
- 2 Find the equation of the line which is:
 - a parallel to $2x - y = -3$ and passes through $(5, 3)$
 - b perpendicular to $y = -4x + 3$ and passes through $(-1, 5)$.
- 3 Line L has equation $y = 3 - 2x$.
 - a If the point $P(3, k)$ lies on line L , determine the value of k .
 - b Write down the gradient of line L .
 - c Find the equation of the line perpendicular to L which passes through point P .
- 4 Tammy buys tickets to a stage show. Tickets cost \$30 for adults, and \$15 for children. She spends a total of \$120 buying tickets for x adults and y children.
 - a Explain why $30x + 15y = 120$.
 - b If Tammy bought tickets for 4 children, how many adult tickets did she buy?
 - c Find the x -intercept of the line $30x + 15y = 120$, and interpret your answer.
 - d Draw the graph of $30x + 15y = 120$. Mark two points on your graph to indicate your answers to **b** and **c**.
- 5 Solve for x :
 - a $2x^2 - 9x = 0$
 - b $x^2 + 8x - 20 = 0$
 - c $4x^2 + 11x = 3$
 - d $(x + 3)(1 - 2x) = -9$
- 6 $x = -2$ is a solution to $x^2 + bx + (b - 2) = 0$.
 - a Find the value of b .
 - b Find the other solution to the equation.
- 7 Find m given that $mx^2 + (m - 2)x + m = 0$ has a repeated root.
- 8 Use technology to solve:
 - a $\frac{2}{x} = 5x - 3$
 - b $2^x - x^3 = 0$
- 9 For each of the following functions:
 - i Find the x -intercepts.
 - ii Find the equation of the axis of symmetry.
 - iii Find the coordinates of the vertex.
 - iv State the y -intercept.
 - v Sketch the function.
 - a $y = -4x(x + 3)$
 - b $y = \frac{1}{2}(x + 6)(x - 4)$
 - c $y = -3(x - 2)^2$
 - d $y = 2(x + 5)^2 - 4$
- 10 For each of the following quadratics:
 - i Find the y -intercept.
 - ii Write the function in the form $y = a(x - h)^2 + k$.
 - iii Find the coordinates of the vertex.
 - iv Sketch the graph of the quadratic.
 - a $y = x^2 - 4x + 9$
 - b $y = 4x^2 + 16x + 11$
 - c $y = -3x^2 + 12x - 10$
- 11 For each of the following quadratics:
 - i Find the coordinates of the vertex.
 - ii State whether the vertex is a minimum or a maximum.
 - iii State the range of the function.
 - iv Find the axes intercepts.
 - v Sketch the function.
 - a $y = x^2 - 3x - 4$
 - b $y = -2x^2 - 5x + 7$
- 12 Find the value(s) of k for which the graph of $y = (k + 3)x^2 - 2kx + (k - 2)$:
 - a cuts the x -axis twice
 - b touches the x -axis
 - c misses the x -axis.
- 13 The function f can be written in the form $f(x) = a(x - p)(x - q)$ where $p > q$.
 - a Write down the values of p and q .
 - b Find a .
 - c Write down the equation of the axis of symmetry.



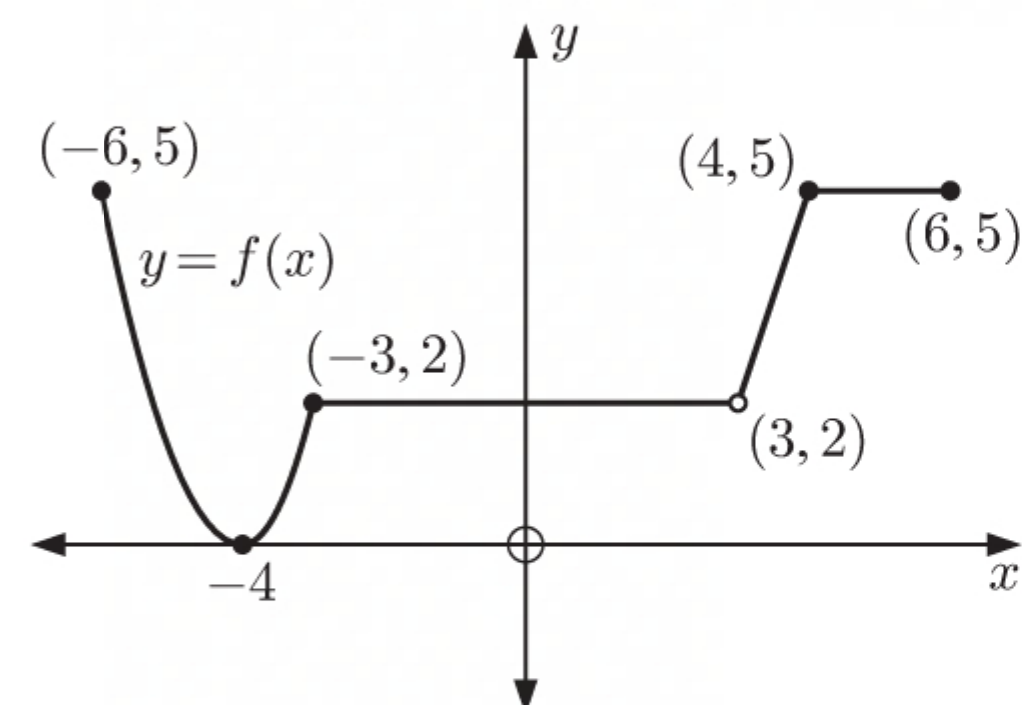
- 14** Consider the graph of $y = -3x^2 - x + 1$.
- Describe the shape of the graph.
 - Use the discriminant to show that the graph cuts the x -axis twice.
 - Find the x -intercepts, rounding your answers to 2 decimal places.
 - State the y -intercept.
 - Hence sketch the function.
- 15** For what values of m does the graph of $y = mx^2 + 4x + 6$ lie entirely above the x -axis?
- 16** The line with equation $y = kx - 2$ is a tangent to the quadratic $y = 3x^2 + x + 1$. Find k .
- 17** Find the coordinates of the point(s) of intersection of:
- $y = x^2 - 4x - 5$ and $y = 3x - 11$
 - $y = -2x^2 + 5x$ and $y = 5 - 2x$
- 18** For what values of c does the line $y = 2x + c$ never meet the parabola with equation $y = 3x^2 + 5x + 7$?
- 19** The graph of the quadratic function $y = f(x)$ has axis of symmetry $x = -3$, y -intercept -3 , and *touches* the x -axis.
- Find the quadratic function.
 - $y = kx - \frac{9}{4}$ is a tangent to the graph of $y = f(x)$. Find the possible values of k , and the points at which these tangents meet the curve.
 - Illustrate your answers to **a** and **b**.
- 20** A quadratic function cuts the y -axis at 4, and touches the lines $y = x - 5$ and $y = -2x$. Find the quadratic function.
- 21** A factory manufactures x radios per day.
- The production cost of each radio is $\left(26 + \frac{10}{x}\right)$ euros, and the income from selling each radio is $\left(42 - \frac{x}{15}\right)$ euros.
- Write down a formula for the *total profit* P from selling all the radios made each day.
 - Find the number of radios that should be made per day to maximise profit.
 - Calculate the maximum profit per day.
- 22** Andreas is making an aquarium in the shape of an equilateral triangular prism. The sum of all side lengths of the prism must be 1.8 m.
- Let the equilateral triangle ends have sides of length x cm, and the aquarium have height y cm.
- Show that the area of the end is $\frac{\sqrt{3}}{4}x^2$ cm².
 - Hence show that the total surface area of the aquarium is $A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x$ cm².
 - What dimensions should Andreas choose for the aquarium to maximise its surface area?
- 23** Solve for x :
- $(x - 1)(5 - x) \leq 0$
 - $x^2 + 8x - 20 < 0$
 - $-9x^2 + 4x + 5 \geq 0$
- 24** Solve for x :
- $x^2 > 9$
 - $x^2 - 15 \leq 2x$
 - $3x^2 < 2(5x + 4)$
- 25** Find the value(s) of k for which the graph of the quadratic function $y = kx^2 - (k - 6)x + (k - 6)$:
- cuts the x -axis twice
 - touches the x -axis
 - misses the x -axis.
- 26** Jacob's rainwater tank started leaking. The amount of water in the tank after t hours is given by $W(t) = 1000 - 0.5t$ litres.
- Find $W(0)$, and interpret your answer.
 - Find t when $W(t) = 700$, and explain what this represents.
 - How long will it take for the tank to empty?
- 27** If $f(x) = \frac{x-2}{x-3}$, find in simplest form:
- $f(-x)$
 - $f(x+2)$
 - $f\left(\frac{1}{x}\right)$



28 Consider the graph of $y = f(x)$ alongside.

Decide whether each statement is true or false:

- a** 0 is in the domain of f . **b** 0 is in the range of f .
c 6 is in the range of f . **d** 3 is in the domain of f .
e 2 is in the range of f .



29 State the domain and range of each function:

- a** $f(x) = \sqrt{3 - 2x}$ **b** $f : x \mapsto \frac{2}{x - 3}$ **c** $f : x \mapsto \frac{1}{\sqrt{x} - 2}$

30 Consider the function $k(t) = 2t - 4$ for $0 \leq t < 4$, $t \in \mathbb{Z}$.

- a** List the elements of the domain of $k(t)$. **b** List the elements of the range of $k(t)$.
c Sketch the function k on a set of axes, showing all elements in the domain and range.

31 Consider the function $f(x) = \frac{x + 2}{x - 1}$.

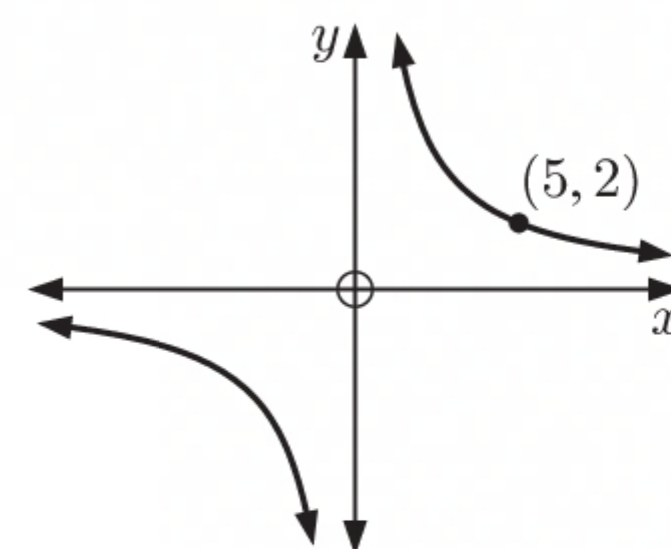
- a** Find the domain and range of f . **b** Write down the equations of the asymptotes of $y = f(x)$.
c Find the axes intercepts. **d** Draw a sign diagram for $f(x)$.
e Describe the behaviour of the function near the asymptotes.
f Sketch the function.

32 Consider the graph of $y = f(x)$ where $f(x) = 2 + \frac{4}{x + 1}$.

- a** Find the axes intercepts. **b** Calculate $f(-2)$.
c Determine the equation of the: **i** horizontal asymptote **ii** vertical asymptote.
d Sketch the graph of $y = 2 + \frac{4}{x + 1}$. Label the axes intercepts and asymptotes clearly.

33 Consider the graph of the reciprocal function $f(x) = \frac{k}{x}$.

- a** Find k .
b State the domain and range of the function.
c Find $f(-\frac{1}{2})$.
d Is the function self-inverse? Explain your answer.



34 Consider the functions $f(x) = \frac{1}{x - 1} + \sqrt{x + 1}$ and $g(x) = x^2$.

- a** State the domain of f . **b** Find $(f \circ g)(x)$.
c Is the domain of $(f \circ g)$ the same as the domain of either f or g ? Explain your answer.

35 Functions f and g are given by $f : x \mapsto e^{x+1}$ and $g : x \mapsto \ln x - 1$.

- a** Find $(f \circ g)(x)$ and state its domain and range.
b Find $(g \circ f)(x)$ and state its domain and range.
c Graph $y = f(x)$ and $y = g(x)$ on the same set of axes.
d State the relationship between f and g .

36 Suppose $f(x) = \sqrt[3]{x}$. Find $g(x)$ given that:

- a** $(f \circ g)(x) = 2x - 1$ **b** $(g \circ f)(x) = 2x - 1$

37 Let $f(x) = \sqrt{x + 4}$ and $g(x) = x^2 - 3$.

- a** Find $(f \circ g)(x)$ and state its domain and range.
b Find $(g \circ f)(x)$ and state its domain and range.

38 Functions f and g are defined by $f : x \mapsto 3x + 1$ and $g : x \mapsto 4 - x$. Find:

- a** $f(g(x))$ **b** $(g \circ f)(-4)$ **c** $f^{-1}(\frac{1}{2})$

39 Suppose $f : x \mapsto \ln x$ and $g : x \mapsto 3 + x$. Find:

a $f^{-1}(2) \times g^{-1}(2)$

b $(f \circ g)^{-1}(2)$

c a given that $(g \circ f)^{-1}(a) = \sqrt{e}$.

40 Suppose $f : x \mapsto x + 5$ and $g : x \mapsto 7 - 3x$.

a Find: **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$ **iii** $(f \circ g)(x)$

b Show that $(g^{-1} \circ f^{-1})(x) = (f \circ g)^{-1}(x)$.

41 Consider $y = -1 + 2^{-x}$.

a Find the axes intercepts.

b Find any asymptotes of the function.

c State the domain and range of the function.

d Hence sketch the function.

42 The exponential function $y = a \times 2^x + b$ passes through the points alongside:

x	0	1	2	3
y	20	p	35	q

a Write down two linear equations which could be used to determine the values of a and b .

b Solve the linear equations simultaneously to find a and b .

c Hence find the values of p and q .

43 Consider the exponential function $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-1)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

44 The graph alongside shows the percentage P of radioactive Carbon-14 remaining in an organism t thousands of years after it dies.

a Use the graph to estimate:

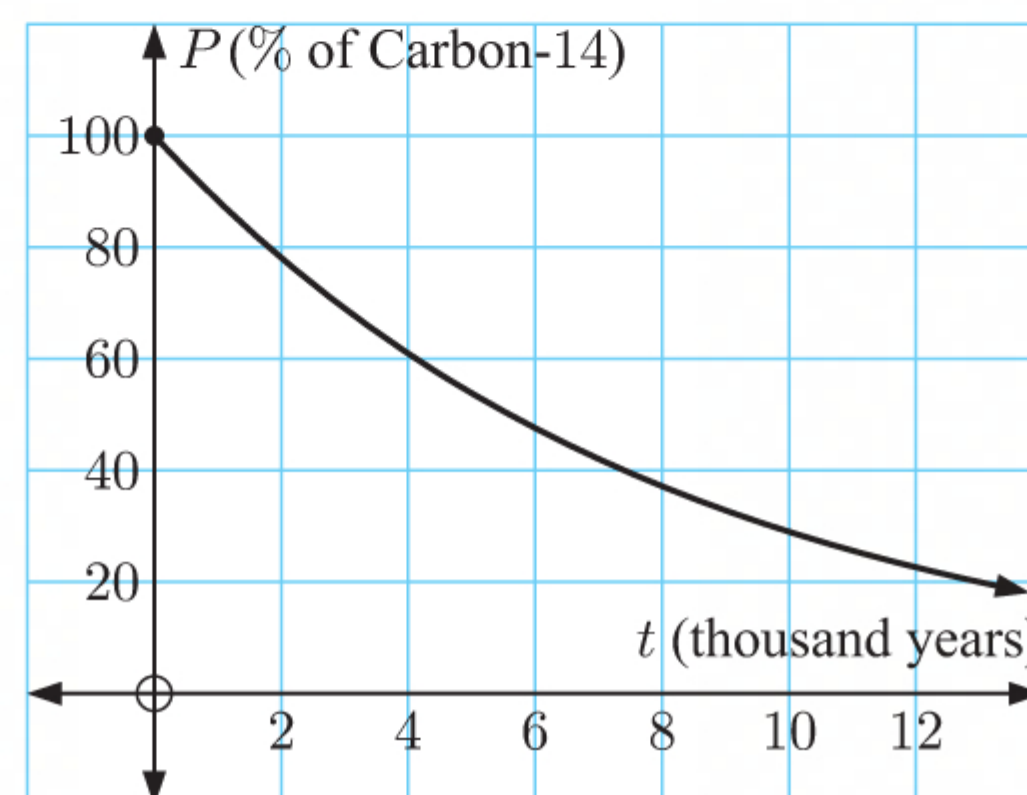
i the percentage of Carbon-14 remaining after 4000 years

ii the number of years for the percentage of Carbon-14 to fall to 50%.

b The equation of the graph is $P = 100 \times (1.1318)^{-t}$, $t \geq 0$.

i Calculate the percentage of Carbon-14 remaining after 8000 years.

ii How long will it take for the percentage to fall to 15%?



45 The number of people N on a small island t years after settlement, increases according to the formula $N = 120 \times (1.04)^t$.

a Find the number of people who started the settlement.

b Find the number of people on the island after 4 years.

c How many years will it take for the number of people to double?

46 Before it is turned on, a refrigerator has an internal temperature of 27°C . Three hours later it has cooled to 6°C .

The internal temperature T (in $^\circ\text{C}$) of the refrigerator t hours after being turned on is given by the function $T(t) = A \times B^{-t} + 3$, where A and B are constants.

a Determine the value of: **i** A **ii** B .

b Find the internal temperature of the refrigerator 5 hours after being turned on.

c Write down the minimum temperature that the refrigerator could be expected to reach.

47 Consider $f : x \mapsto e^{x-1}$.

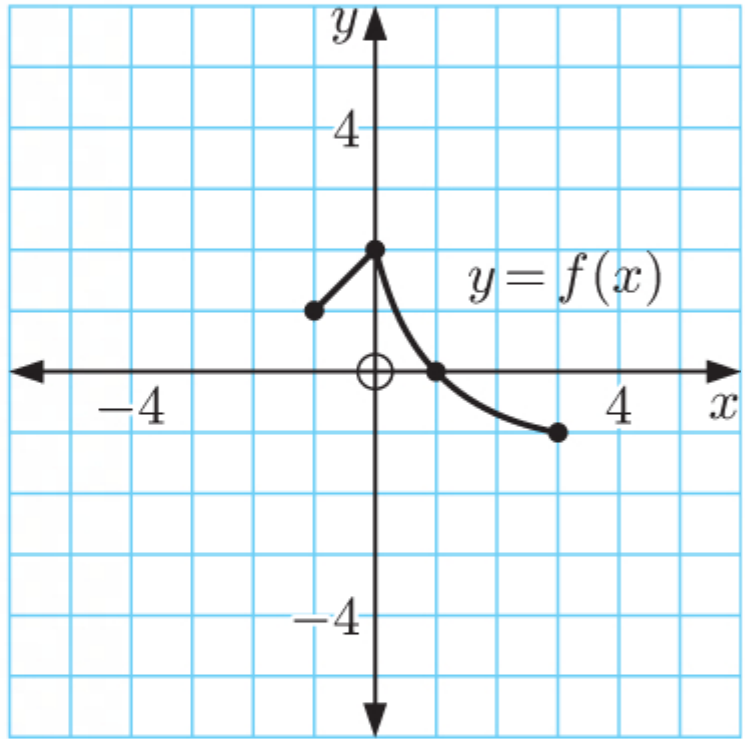
a Graph $y = f(x)$.

b State the domain and range of f .

48 The population P of a species after n months follows the rule $P = 1000 + ae^{kn}$. The initial population was 2000.

After 1 year the population was 4000. Find how long it will take for the population to reach 10 000.

49 Show that $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$.

- 50** Consider $f(x) = 2^{x-1}$.
- Find $f(1)$ and $f(2)$.
 - Graph $y = f(x)$ and its inverse function on the same set of axes.
 - Find $f^{-1}(x)$.
 - State the domain and range of $f(x)$ and $f^{-1}(x)$.
- 51** Consider $f : x \mapsto \ln(x+2) - 5$, $x > -2$.
- Find f^{-1} .
 - On the same set of axes, sketch the graphs of f and f^{-1} .
 - State the domain and range of f^{-1} .
- 52** Find the equation of the resulting graph $g(x)$ when:
- $f(x) = x^2 - 5x + 6$ is translated 8 units upwards
 - $f(x) = -2x^2 + x + 3$ is translated 1 unit to the right.
- 53** The graph of $f(x) = \frac{5}{x+2}$ is translated by $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ to give $g(x)$. Find $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.
- 54**
- State the domain and range of $f(x) = \frac{1}{\sqrt{x-4}} + 3$.
 - What transformation maps $y = \frac{1}{\sqrt{x}}$ onto the function f ?
 - Write down the equations of the asymptotes of $y = f(x)$.
- 55** Copy this graph of $y = f(x)$, and draw the graph of:
- $y = f(x+2)$
 - $y = 2f(x) - 3$
 - $y = 4 - f(x)$
 - $y = f(-x)$
 - $y = f(2x)$
- 
- 56** Consider the function $g : x \mapsto 4 - \ln(x-2)$.
- State the domain and range of g .
 - Write down the equation of the asymptote of $y = g(x)$.
 - Write down the function h which is a horizontal stretch of g with scale factor $\frac{1}{2}$.
 - Write down the equation of the asymptote of $y = h(x)$.
- 57** Suppose f and g are functions such that $g(x) = 3f(\frac{1}{2}x)$.
- What transformations are needed to map $y = f(x)$ onto $y = g(x)$?
 - Given that $(-6, 3)$ lies on $y = f(x)$, find the coordinates of the corresponding point on $y = g(x)$.
 - Given that $(4, -9)$ lies on $y = g(x)$, find the coordinates of the corresponding point on $y = f(x)$.
- 58** Find the equation of the resulting image when $y = \frac{2}{x}$ is:
- reflected in the y -axis
 - translated through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$
 - stretched horizontally with scale factor 3.
- 59** The function $f(x)$ has domain $\{x \mid x < 0, x \geq 2\}$ and range $\{y \mid y \geq 3\}$.
Find the domain and range of:
- $g(x) = f(x-3) + 2$
 - $g(x) = 4 - \frac{1}{2}f(5x)$
- 60** Let T_A be a horizontal translation 4 units to the right, T_B be a reflection in the x -axis, and T_C be a translation through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Find the resulting function when $y = f(x)$ has the following transformations applied:
- T_A then T_B
 - T_B then T_A
 - T_B then T_C
 - T_C then T_B
- 61** Consider the function $f : x \mapsto \ln(x-2)$.
- State the domain and range of f .
 - Write down the equation of the asymptote of f .
 - Find the resulting function when f is stretched vertically with scale factor 3, then reflected in the y -axis.

TOPIC 3: GEOMETRY AND TRIGONOMETRY

GEOMETRY OF 3-DIMENSIONAL FIGURES

The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

The **volume** of a solid is the amount of space it occupies.

The **capacity** of a container is the quantity of fluid it is capable of holding. You should understand how the units of volume and capacity are related.

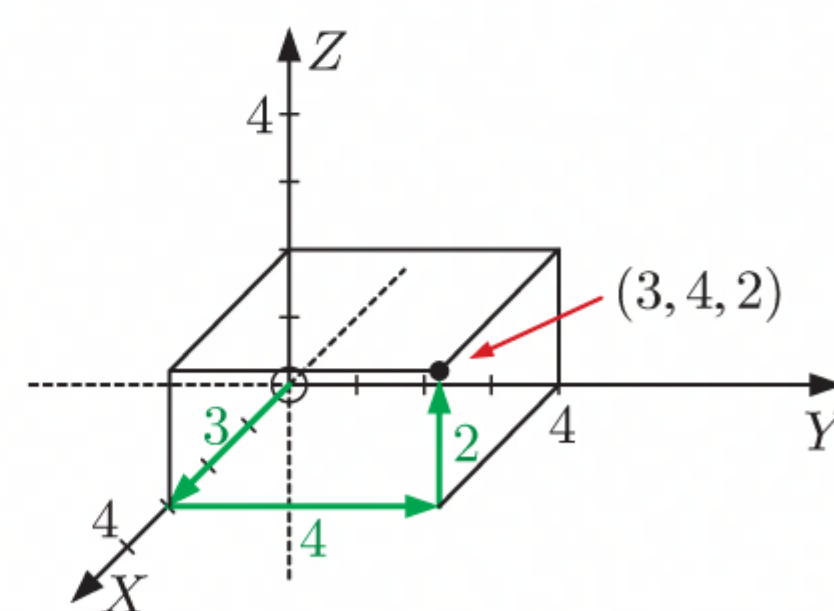
You should be able to calculate the surface area and volume of 3-dimensional figures, including solids of uniform cross-section, pyramids, spheres, and cones.

3-DIMENSIONAL COORDINATE GEOMETRY

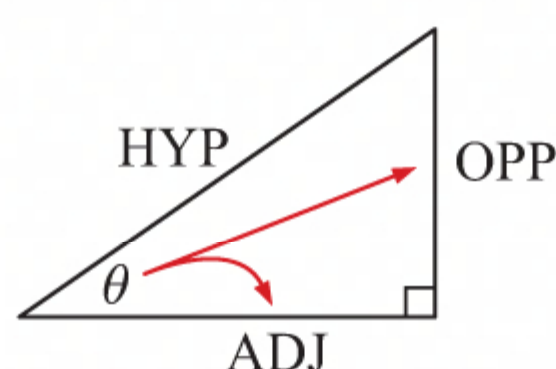
In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the *X*-axis, the *Y*-axis, and the *Z*-axis.

For points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the **distance** $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- the **midpoint** of $[AB]$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.



RIGHT ANGLED TRIANGLE TRIGONOMETRY



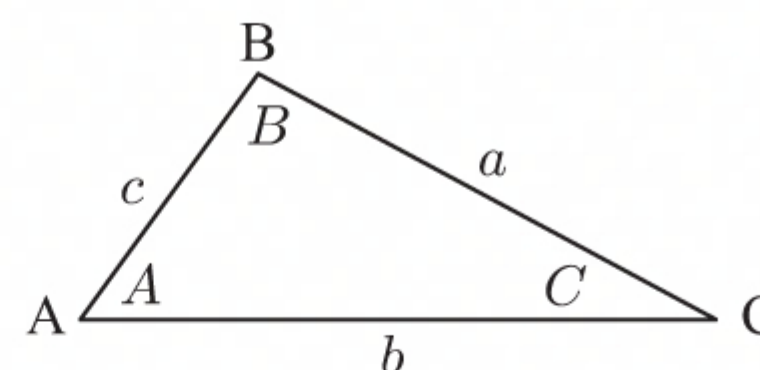
$$\begin{aligned}\cos \theta &= \frac{\text{ADJ}}{\text{HYP}} \\ \sin \theta &= \frac{\text{OPP}}{\text{HYP}} \\ \tan \theta &= \frac{\text{OPP}}{\text{ADJ}}\end{aligned}$$

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

Area formula: $\text{Area} = \frac{1}{2}ab \sin C$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



RADIAN MEASURE

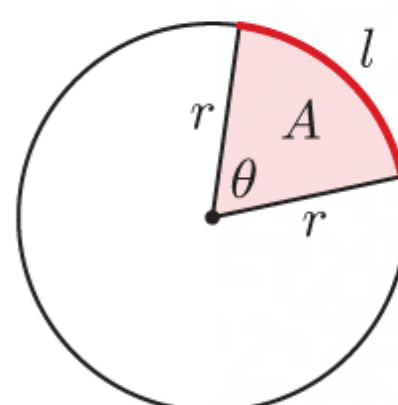
There are $360^\circ \equiv 2\pi$ radians in a circle.

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

For θ in radians:

- the length of an arc of radius r and angle θ is $l = \theta r$
- the area of a sector of radius r and angle θ is $A = \frac{1}{2}\theta r^2$.



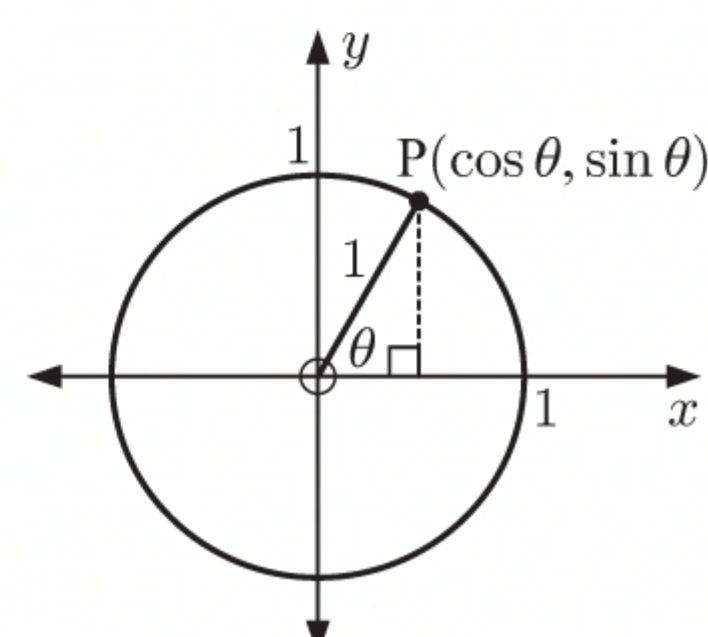
THE UNIT CIRCLE

The **unit circle** is the circle centred at the origin O and with radius 1 unit.

Consider point P on the unit circle where $[OP]$ makes angle θ with the positive *x*-axis. The coordinates of P are $(\cos \theta, \sin \theta)$.

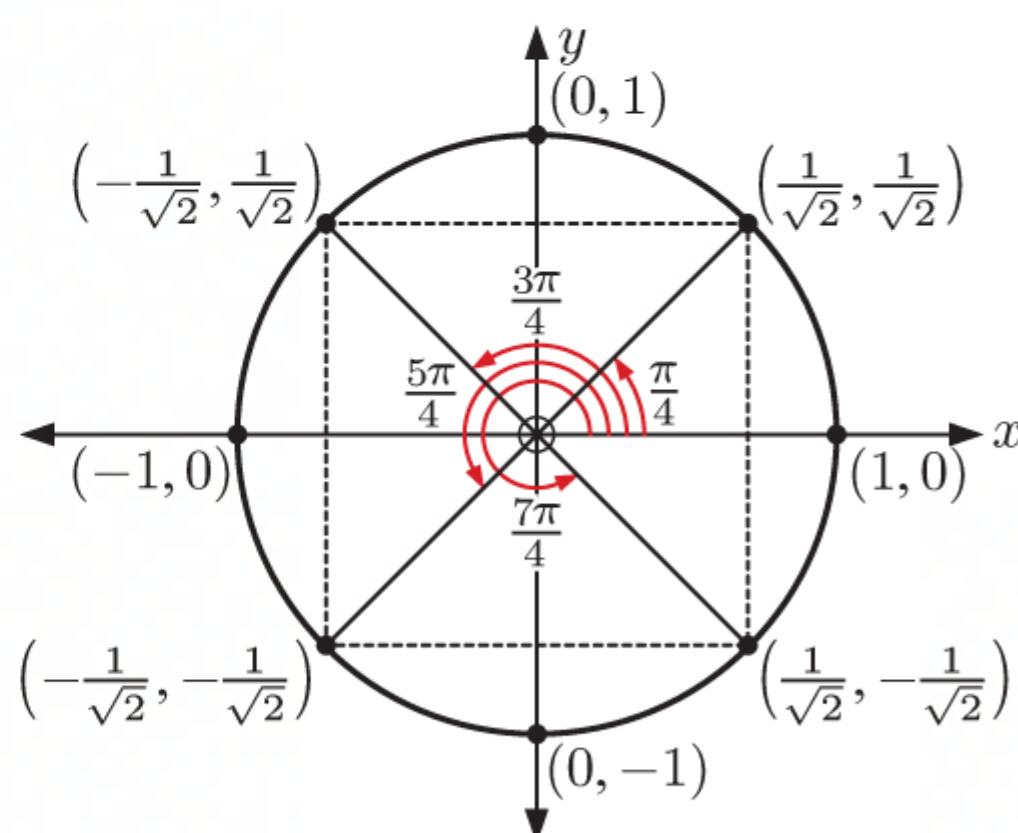
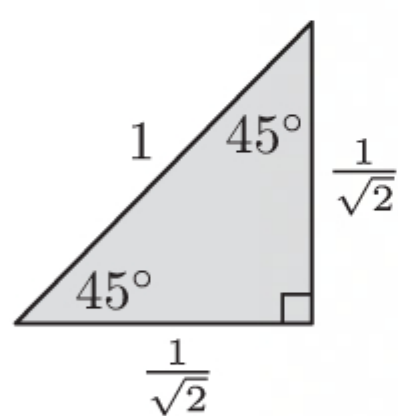
θ is **positive** when measured in an **anticlockwise** direction from the positive *x*-axis.

$\tan \theta$ is defined as $\frac{\sin \theta}{\cos \theta}$. $\tan \theta$ is the **gradient** of $[OP]$.

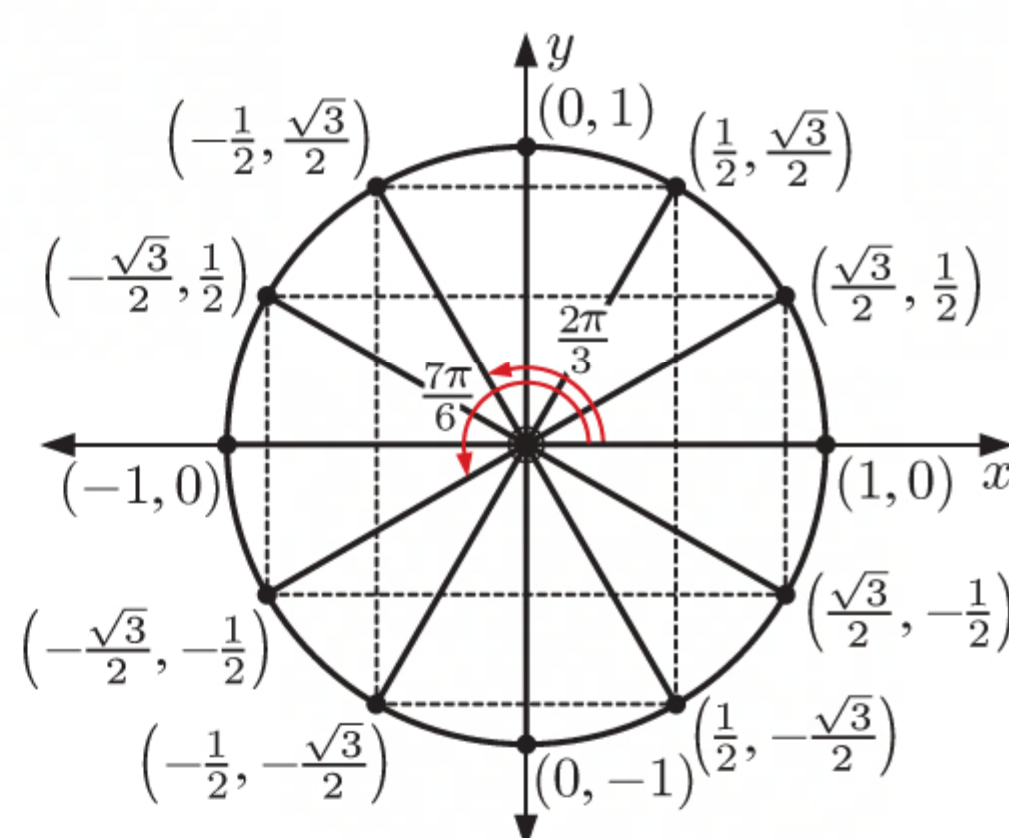
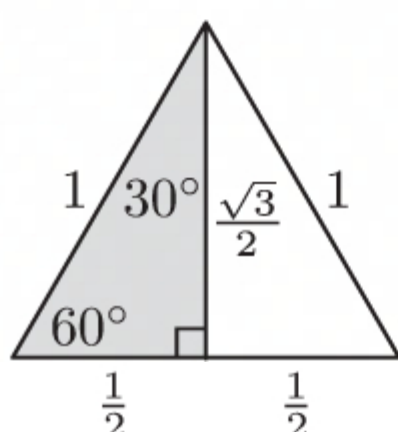


You should memorise or be able to quickly find the values of $\cos \theta$, $\sin \theta$, and $\tan \theta$ that are multiples of $\frac{\pi}{4}$ or $\frac{\pi}{6}$.

Multiples of $\frac{\pi}{4}$ or 45°

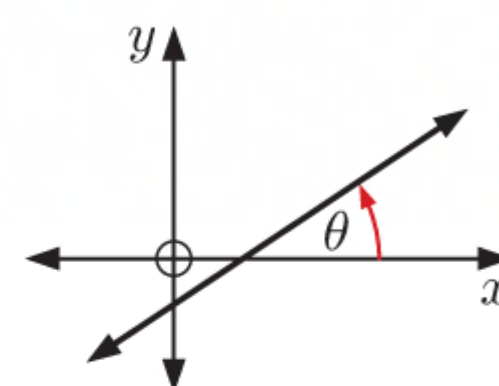


Multiples of $\frac{\pi}{6}$ or 30°



The **Pythagorean identity** $\cos^2 \theta + \sin^2 \theta = 1$ can be used to find one trigonometric ratio from another.

If a straight line makes an angle of θ with the positive x -axis, then its gradient is $m = \tan \theta$.



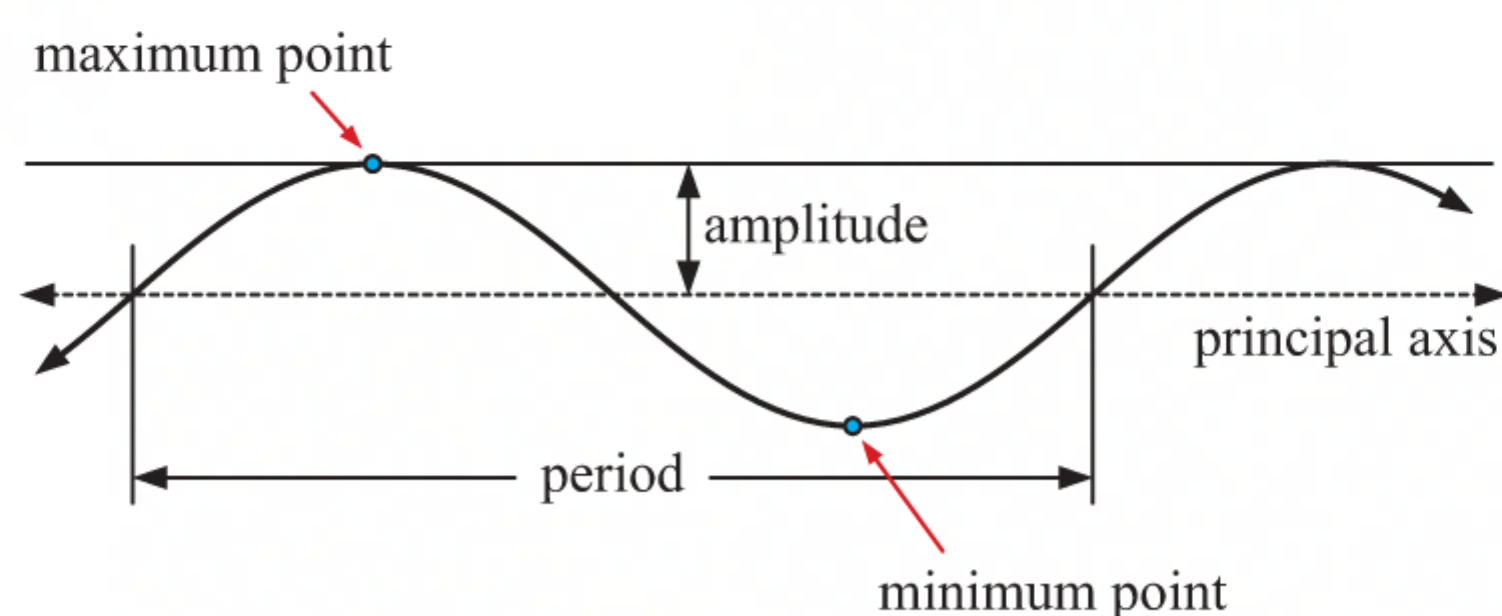
PERIODIC FUNCTIONS

A **periodic function** is one which repeats itself over and over in a horizontal direction.

The **period** of a periodic function is the length of one cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow p$ is the smallest positive value such that $f(x + p) = f(x)$ for all x .

For example, a **wave** is a periodic function which oscillates about a horizontal line called the **principal axis**.



The **amplitude** is the distance between a maximum or minimum point and the principal axis.

THE SINE FUNCTION

If we begin with $y = \sin x$, we can perform transformations to produce the **general sine function** $f(x) = a \sin(b(x - c)) + d$, where $b > 0$.

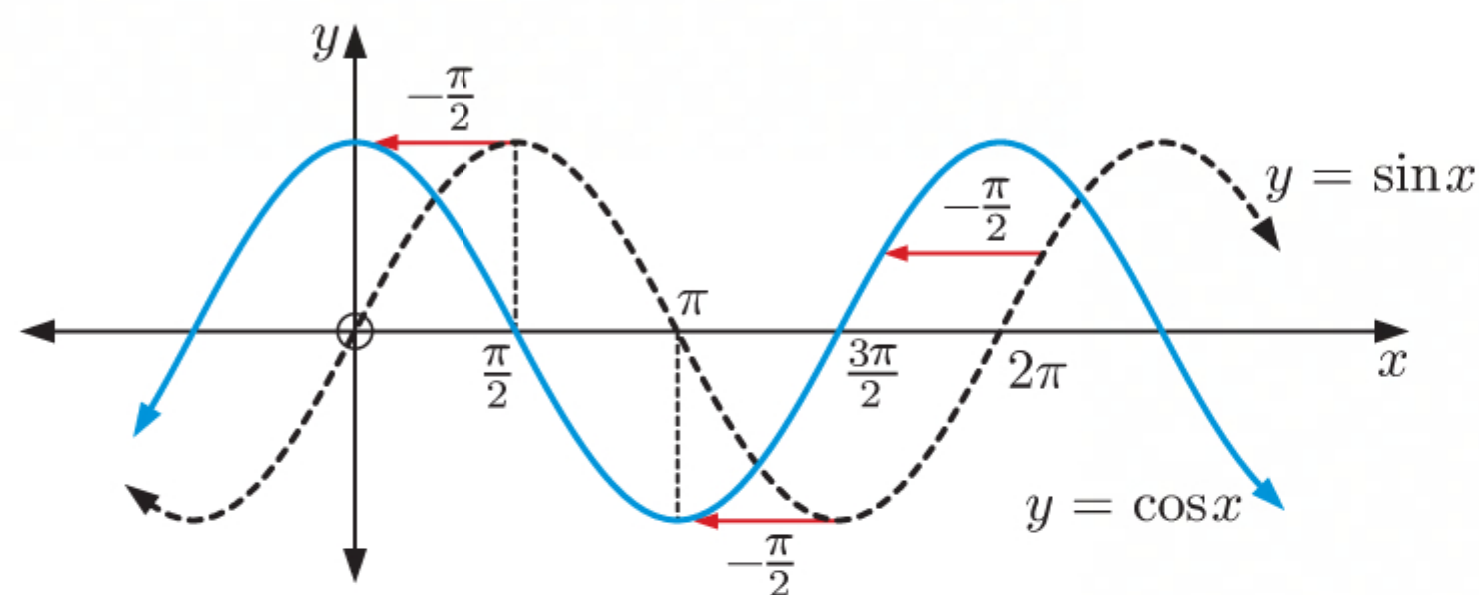
We have a vertical stretch with scale factor $|a|$ and a horizontal stretch with scale factor $\frac{1}{b}$, a reflection in the x -axis if $a < 0$, and a translation through $\begin{pmatrix} c \\ d \end{pmatrix}$.

The general sine function has the following properties:

- the **amplitude** is $|a|$
- the **principal axis** is $y = d$
- the **period** is $\frac{2\pi}{b}$.

THE COSINE FUNCTION

Since $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, the graph of $y = \cos x$ is a horizontal translation of $y = \sin x$, $\frac{\pi}{2}$ units to the left.

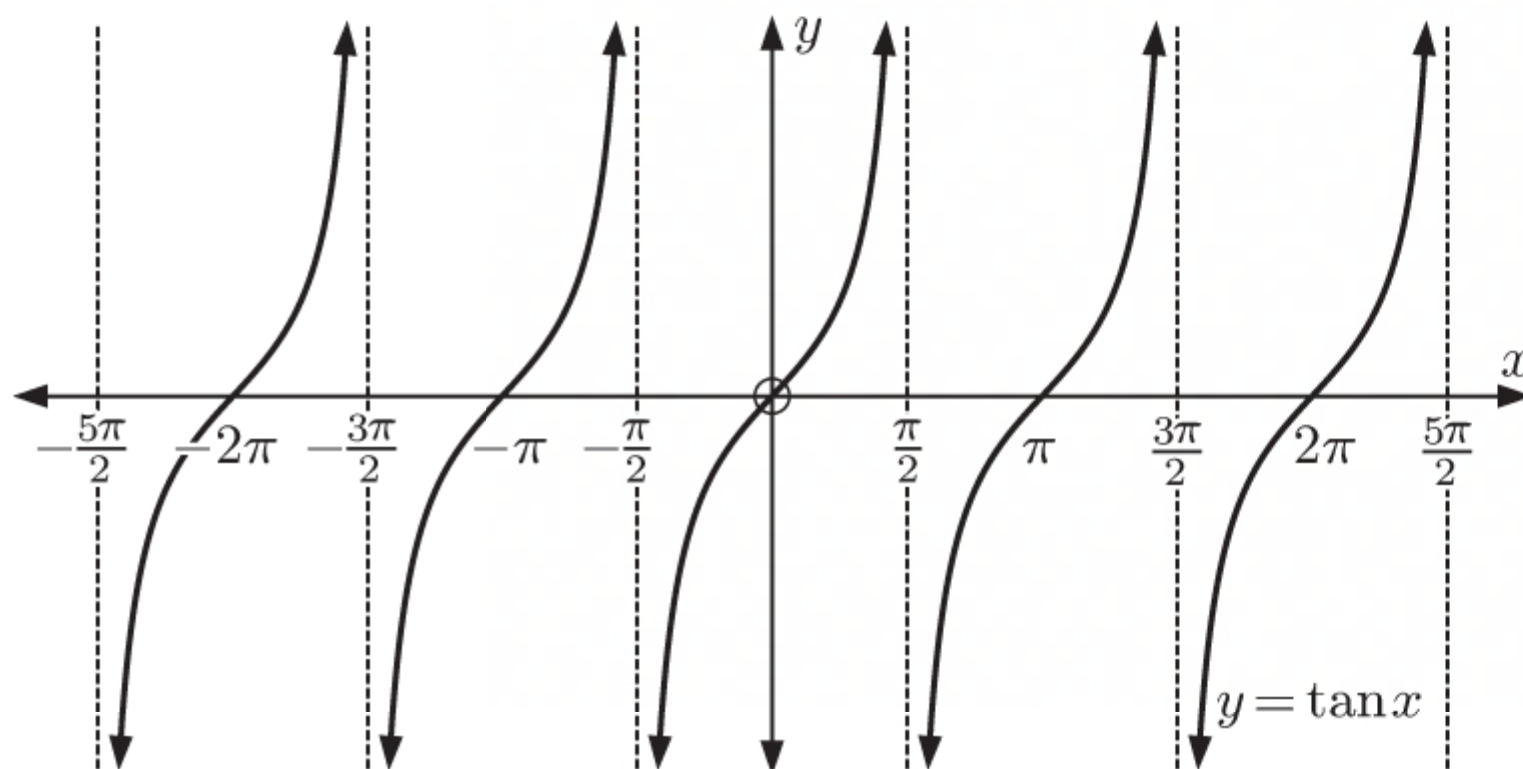


The properties of the **general cosine function** $y = a \cos(b(x - c)) + d$ are the same as those of the general sine function.

THE TANGENT FUNCTION

$y = \tan x = \frac{\sin x}{\cos x}$ is undefined when $\cos x = 0$.

\therefore the graph of $y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.



For the **general tangent function** $y = a \tan(b(x - c)) + d$ where $b > 0$:

- the **principal axis** is $y = d$
- the **period** is $\frac{\pi}{b}$
- the **amplitude** is undefined.

TRIGONOMETRIC EQUATIONS

Trigonometric equations may be solved:

- graphically (using pre-prepared graphs or technology)
- algebraically

When solving trigonometric equations algebraically, we use the unit circle and the *periodicity* of the trigonometric functions to give us all solutions in the required domain.

An equation of the form $a \sin x = b \cos x$ can be solved as $\tan x = \frac{b}{a}$.

TRIGONOMETRIC IDENTITIES

$\cos(\theta + 2k\pi) = \cos \theta$ and $\sin(\theta + 2k\pi) = \sin \theta$ for all $k \in \mathbb{Z}$.

Negative angles

$\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$, $\tan(-\theta) = -\tan \theta$

Complementary angles

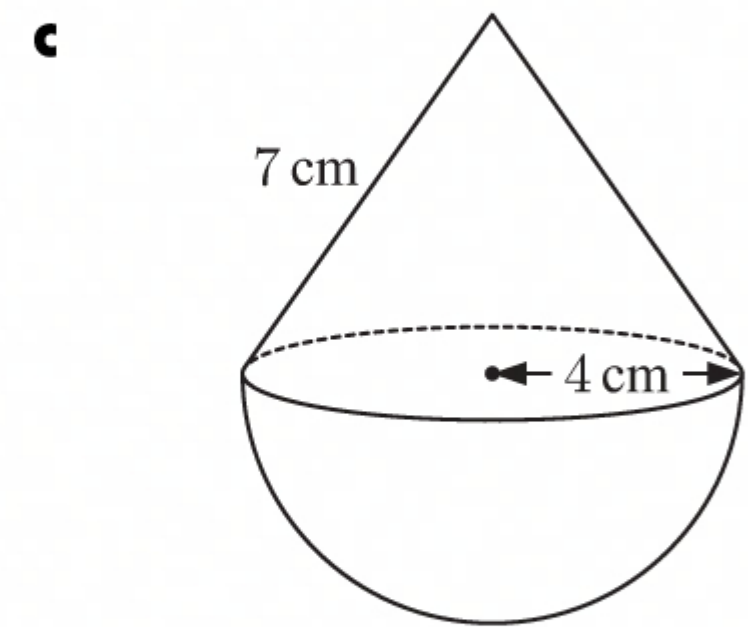
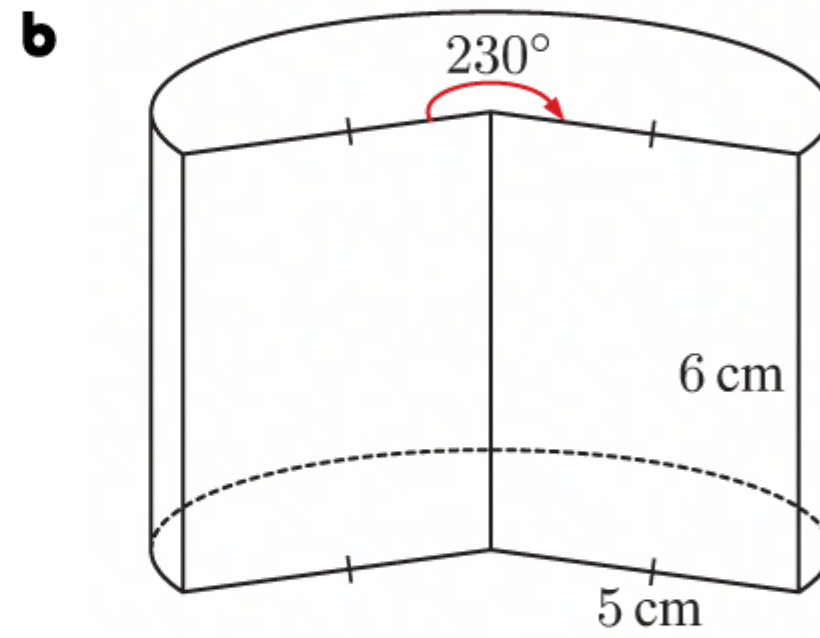
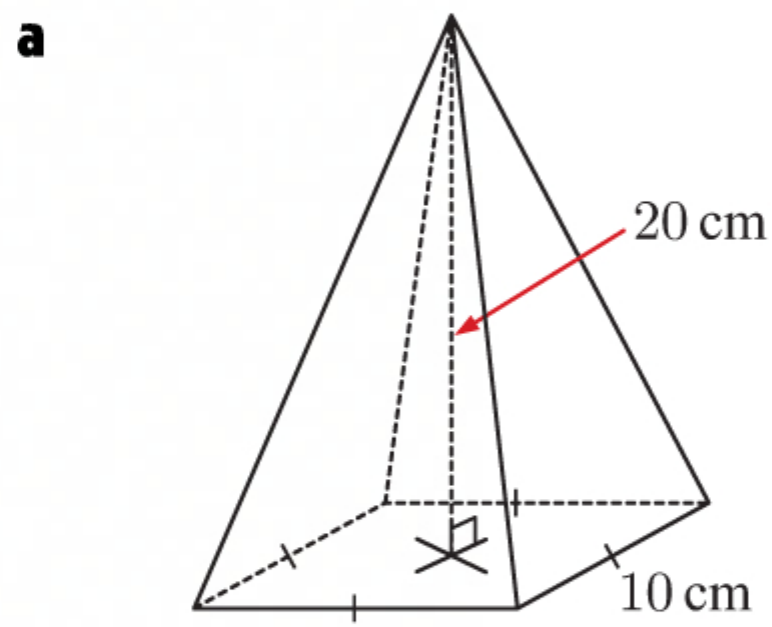
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$, $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$

Double angle formulae

$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = \begin{cases} \cos^2 \theta - \sin^2 \theta \\ 1 - 2 \sin^2 \theta \\ 2 \cos^2 \theta - 1 \end{cases}$$

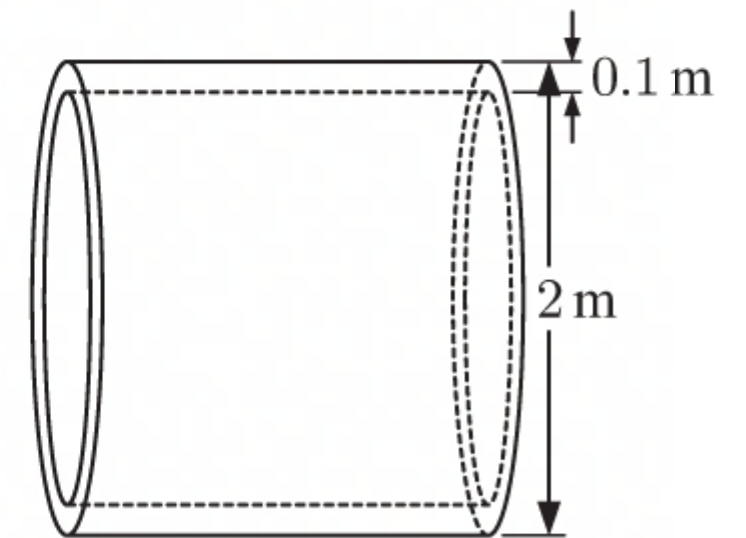
SKILL BUILDER QUESTIONS

1 Find the surface area of each solid:



2 The surface area of a beach ball is 2800 cm^2 . Find the radius of the beach ball.

3 A pipe used to drain stormwater is made from 3 m^3 of concrete. Find the length of the pipe.



4 A sector of a circle of radius 10 cm has perimeter 40 cm. Find:

a the arc length of the sector

b the area of the sector.

5 A large artificial ice cream for a shop front display is to be made with a hemisphere on top of an inverted cone.

The total height of the structure is 7 m, and the cone is 4 m high.

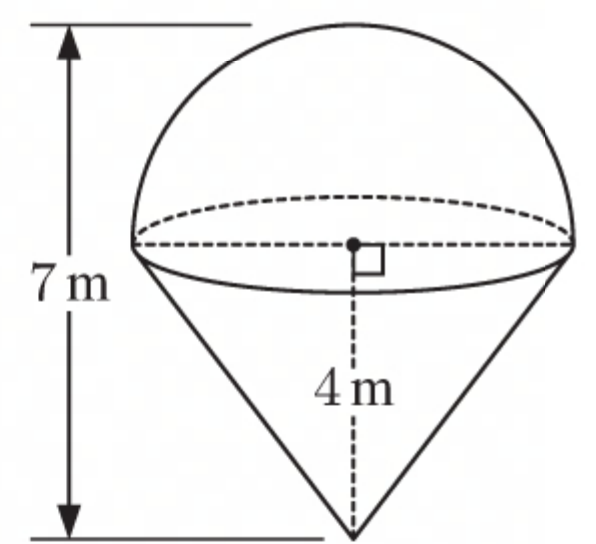
a Show that the radius of the cone is 3 m.

b Calculate the total volume of the ice cream.

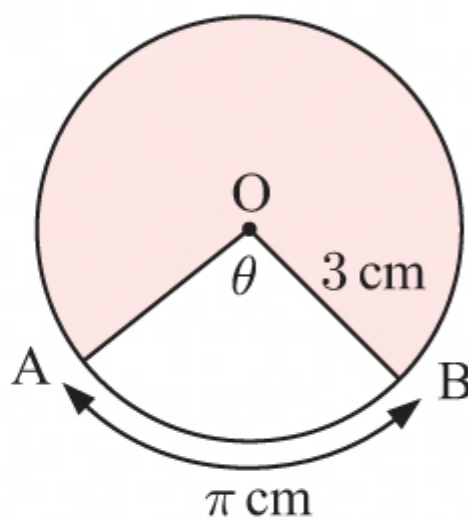
c Find the slant height of the cone.

d Find the total surface area of the ice cream.

e The ice cream is to be made from a lightweight polymer, weighing 1.23 kg per m^2 . Calculate the total weight of the ice cream.



6

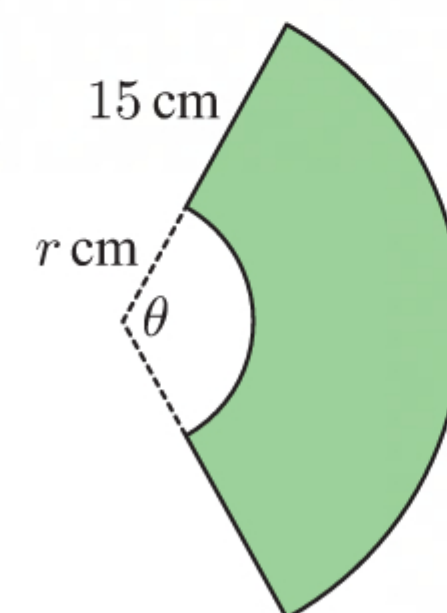
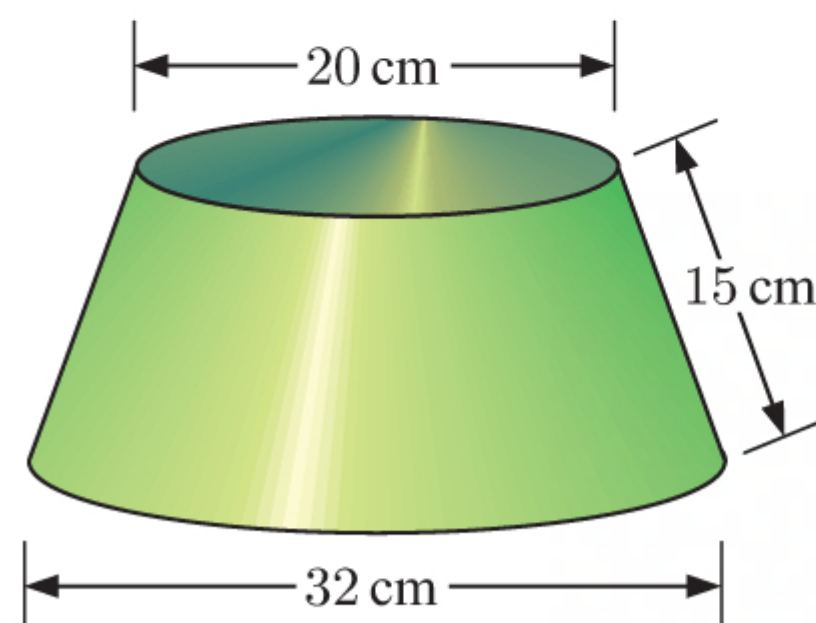


Find:

a θ

b the shaded area.

7 The lampshade on the left is made from the sheet of metal on the right. Find r and θ .



8 For each pair of points, find:

i the distance AB

ii the midpoint of [AB].

a $A(2, 4, 1)$ and $B(4, 0, 7)$

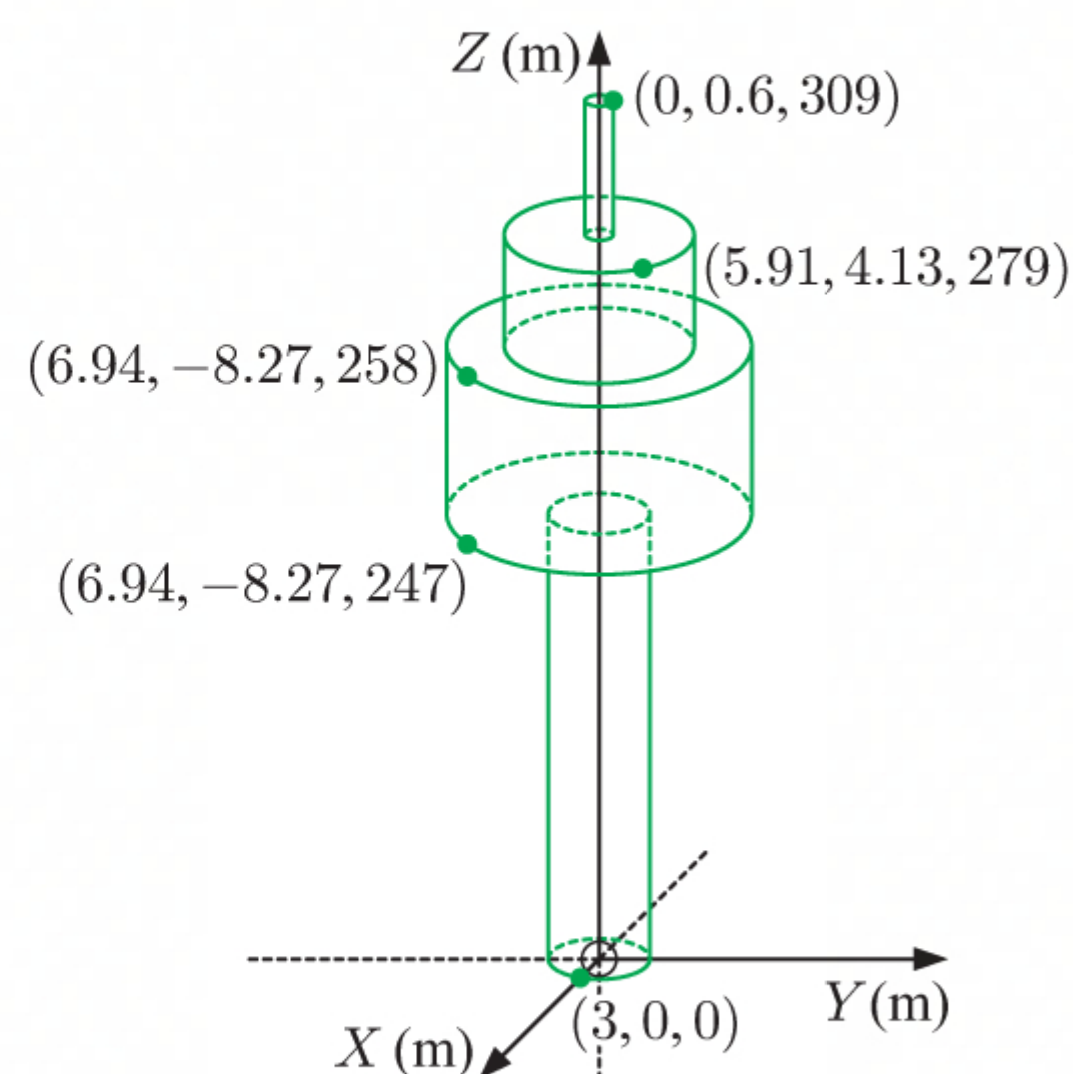
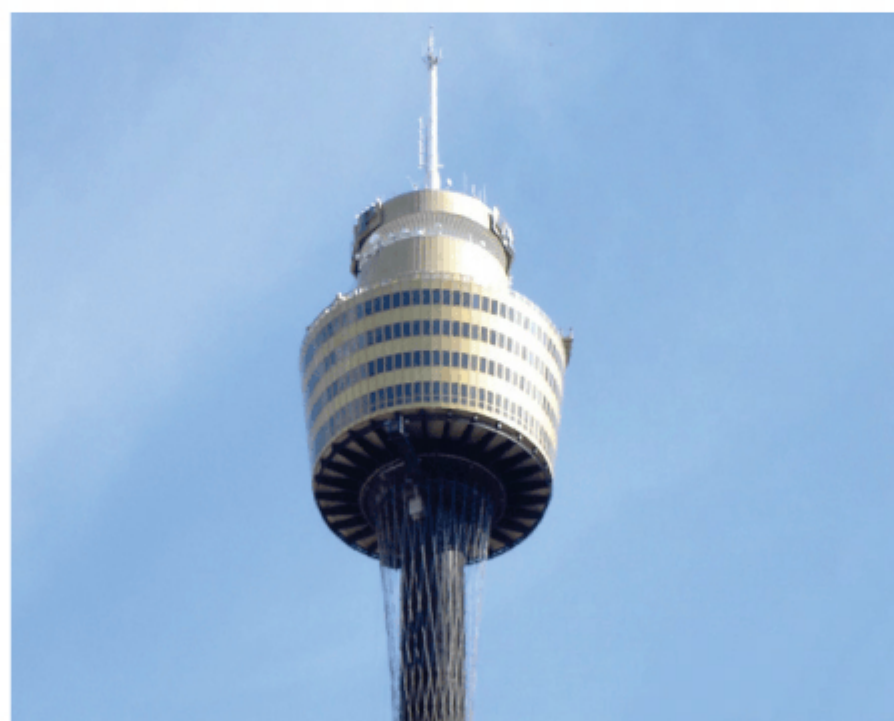
b $A(3, -5, 2)$ and $B(-1, 2, -3)$

c $A(-6, 0, 5)$ and $B(-3, -3, 1)$

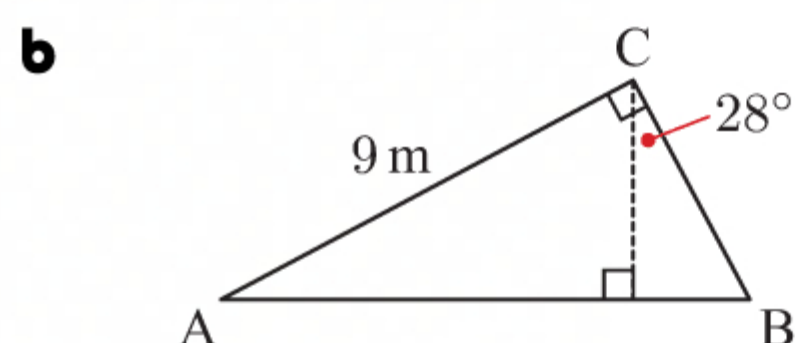
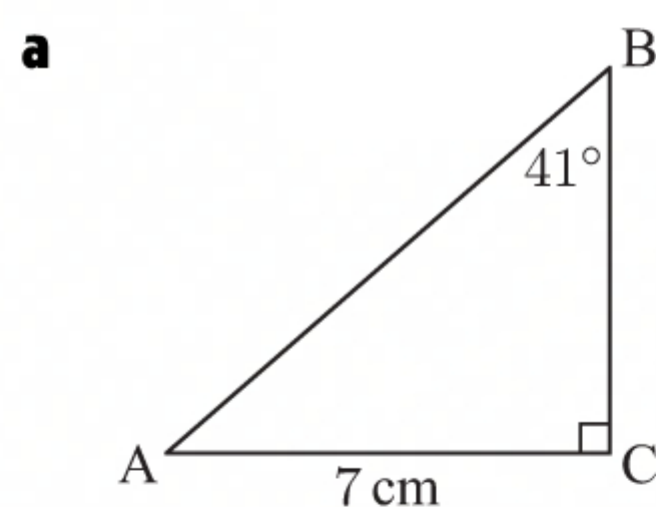
9 The distance from $P(k, 6, -5)$ to $Q(2, -1, -8)$ is 9 units. Find the possible values of k .

- 10** Sydney tower in Australia is the second tallest observation tower in the Southern Hemisphere.

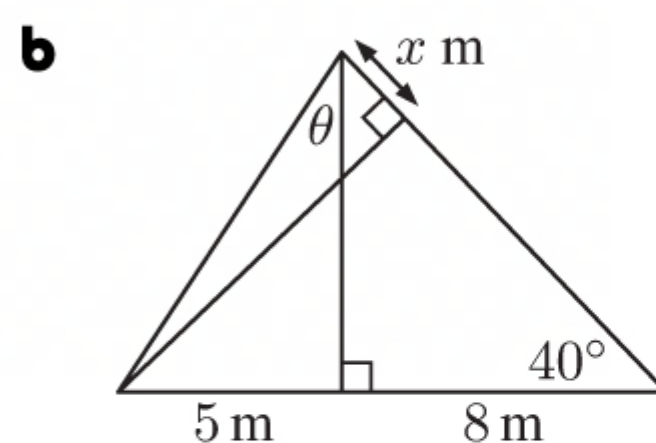
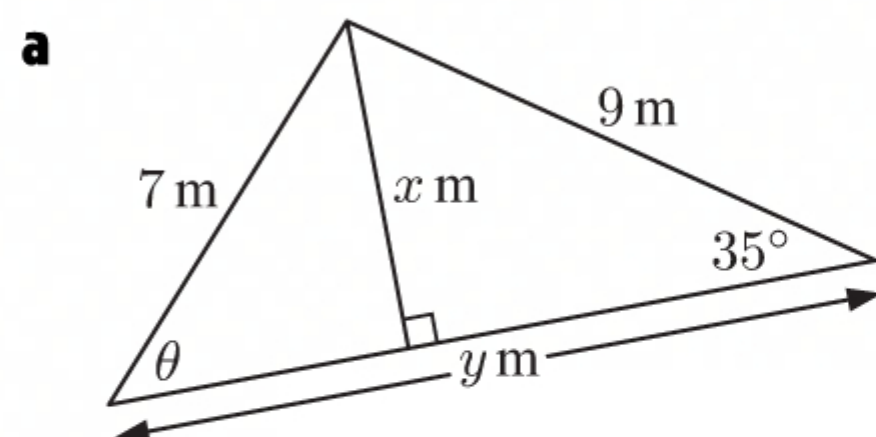
Find the volume of the tower.



- 11** Find the perimeter and area of triangle ABC:



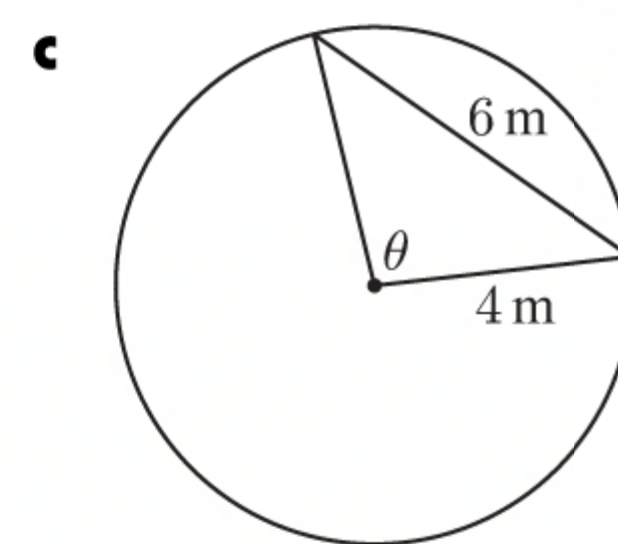
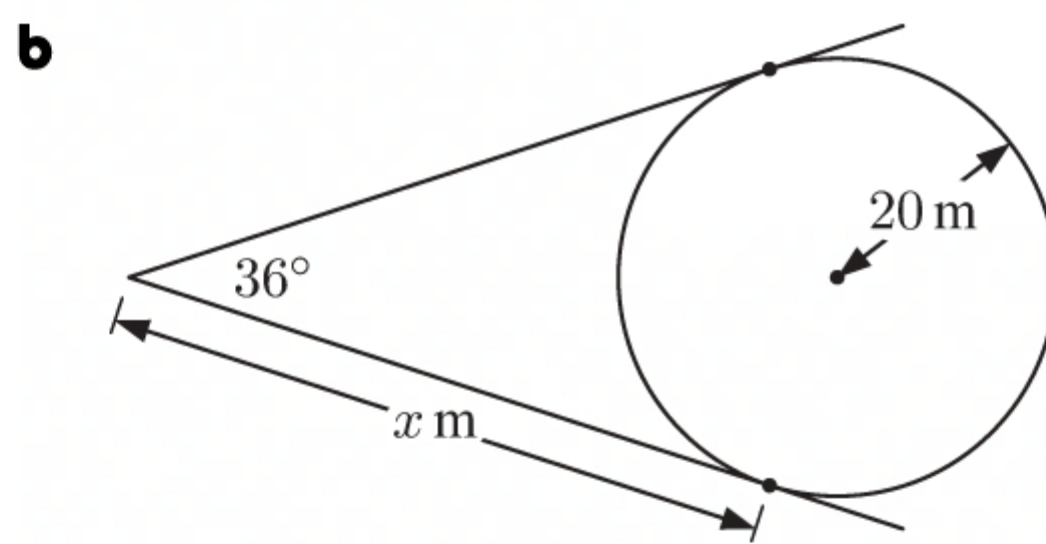
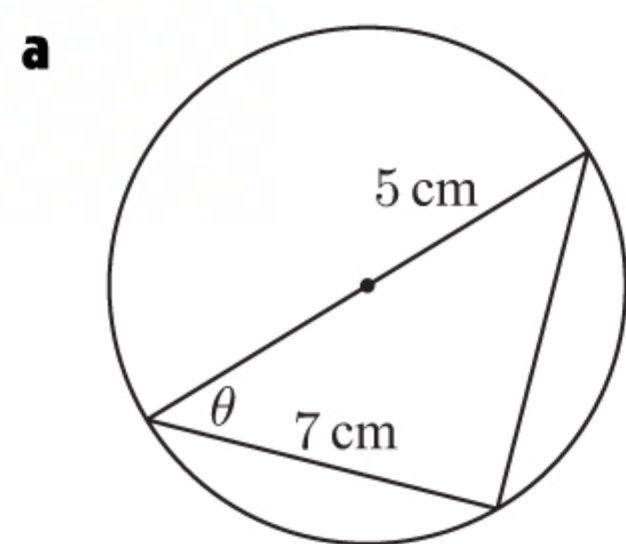
- 12** Find all unknowns in these figures:



- 13** A rhombus has diagonals of length 5 cm and 8 cm.

- Draw a diagram and label it with the given information.
- Find the length of the sides of the rhombus.
- Find the measure of the larger angle in the rhombus.

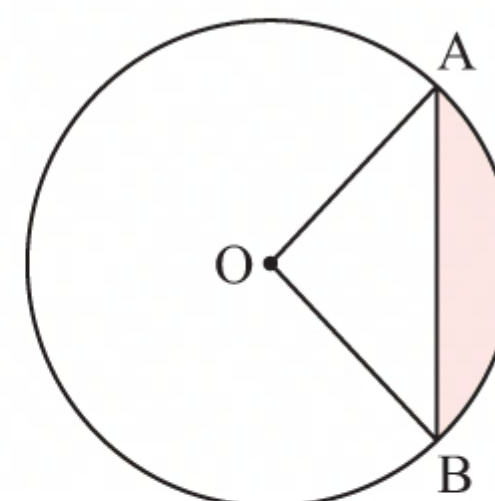
- 14** Find the value of the unknown:



- 15** O is the centre of a circle of radius 32 cm. Chord [AB] is 50 cm long.

Find:

- the measure of \widehat{AOB} , in radians
- the area of the shaded segment.



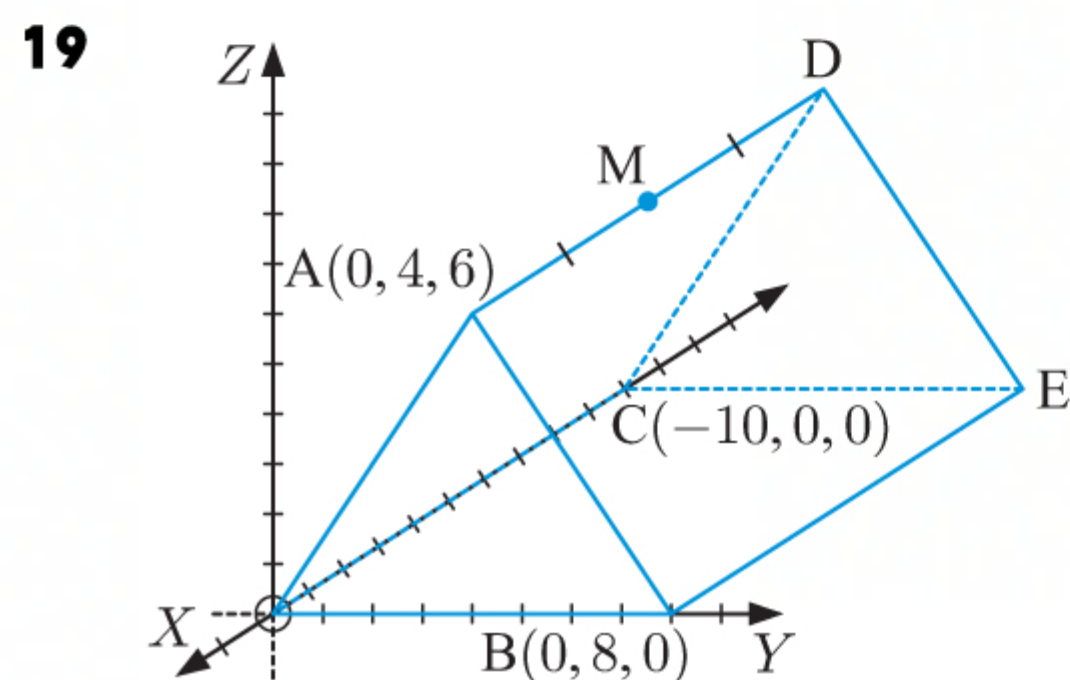
- 16** At 2:35 pm Fari sees an airplane directly overhead. At 2:38 pm he estimates that the angle of elevation to the plane is 15° . The plane is travelling in a straight line at 110 m s^{-1} . Calculate:

- the height of the plane above the ground
- the angle of elevation to the plane at 2:42 pm.

- 17** A helicopter lands 5 km east and 7 km south of its starting point.

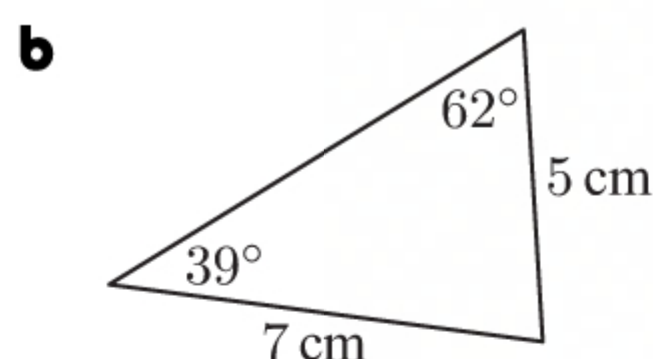
- Find the helicopter's distance from its starting point.
- Find the helicopter's bearing from its starting point.

- a** [AE] **b** [BD] **c** [BE] **d** [AM]



- a** State the coordinates of M.
- b** Find the measure of \widehat{CMD} .
- c** Find the angle between the following line segments and the base plane BECO:
 - i** [OD]
 - ii** [EM]

- a**
-
- A triangle with two sides labeled 2 m and 3 m, and an included angle of 82° .



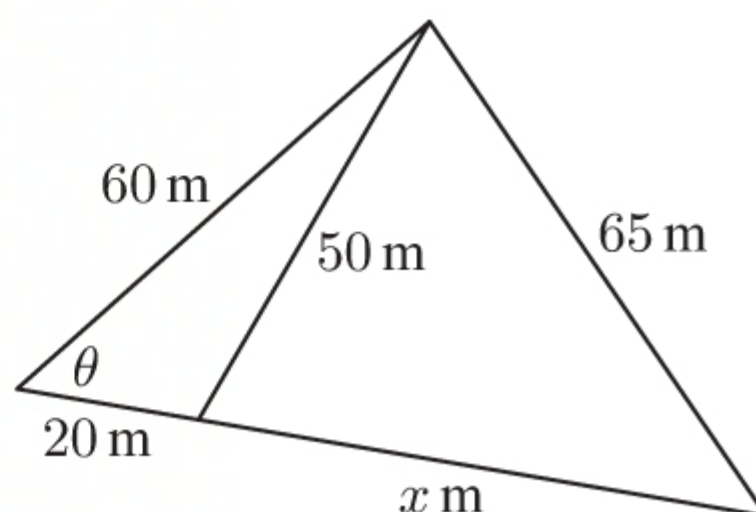
- Hence find the length of [CP].

- c** Find the area of triangle ABC.

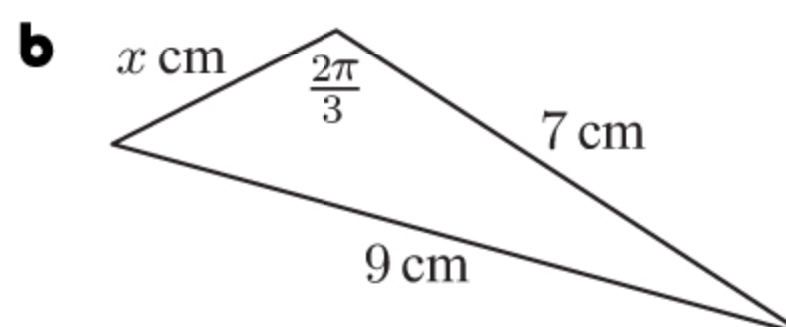
- b** Determine the measure of \widehat{BAC} .

- d** Find the volume of a triangular prism with triangular cross-section ABC, and with length 13.5 cm.

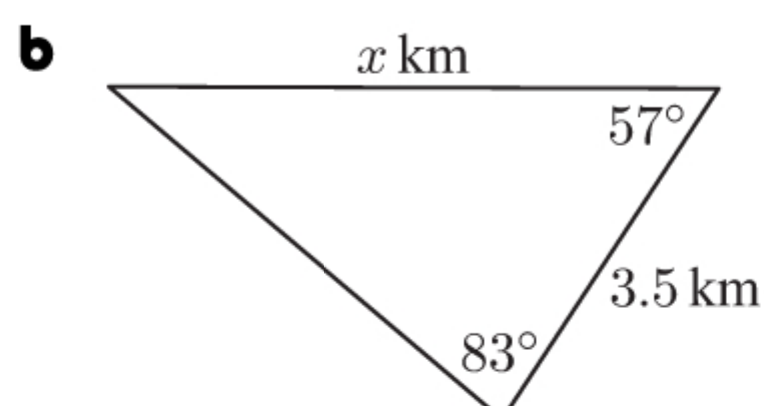
- b** Hence find the value of x .



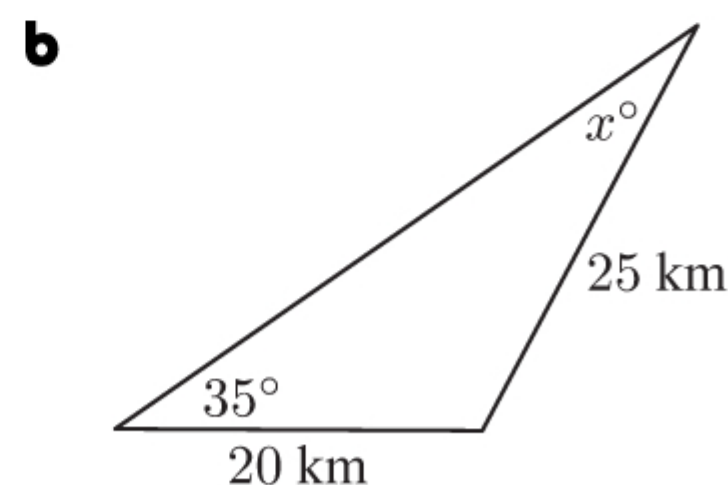
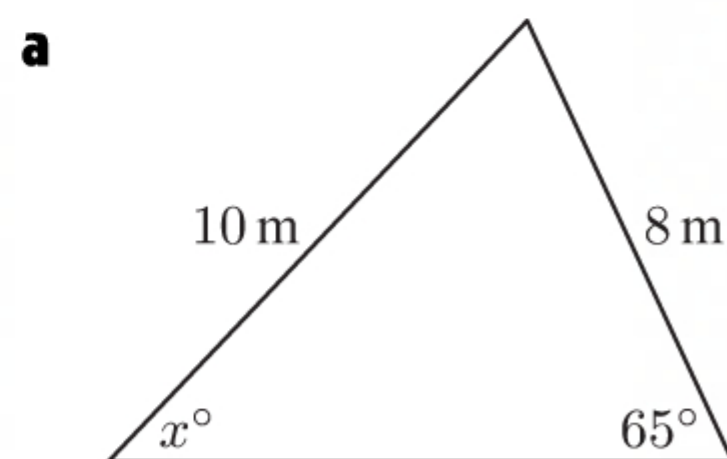
- a**
-
- A triangle is shown with a horizontal base of length 6 cm. The left side has a length of 4 cm, and the right side has a length of x cm. The angle between the base and the left side is labeled $\frac{\pi}{6}$.



- a**
-
- Diagram a shows a triangle with a left side of 16 m, a right side of x m, and an interior angle of 31° at the vertex between these two sides. The bottom-left angle is labeled 27° .



- 27** Find the value of x :

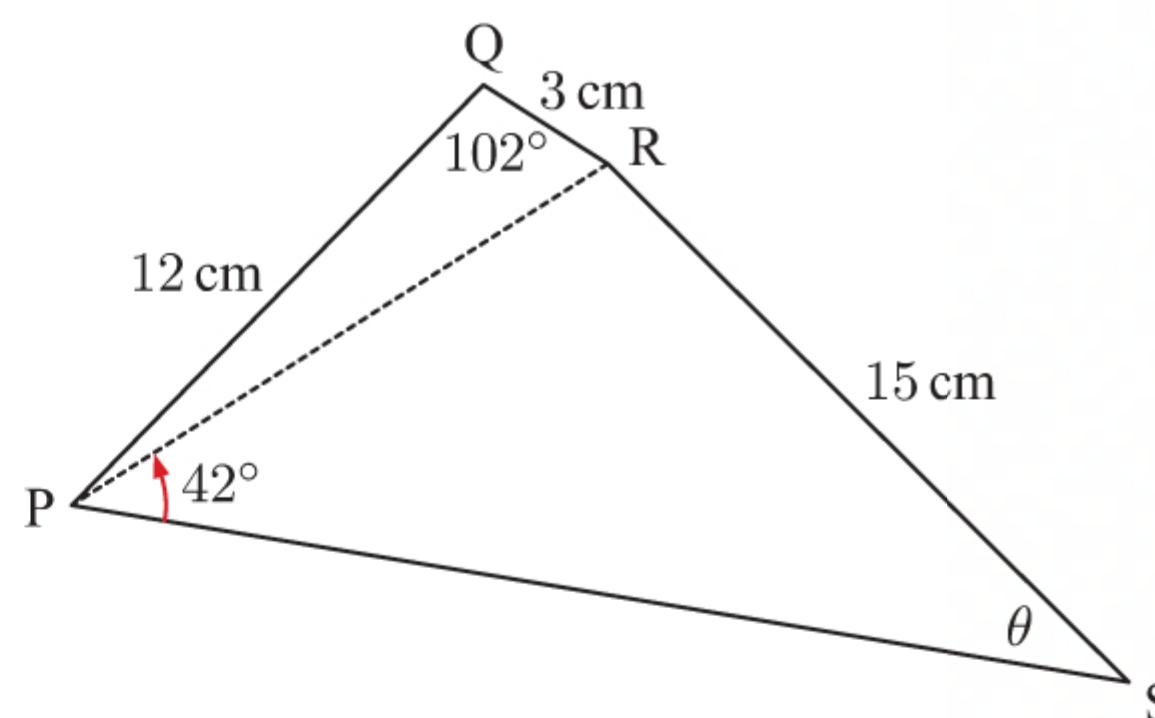


- 28** In triangle ABC, $AB = 15$ cm, $AC = 12$ cm, and \widehat{ABC} measures 30° .

- a** Find the two possible values of \widehat{ACB} .
b Given that \widehat{BAC} is acute, find its measure.

- 29** Quadrilateral PQRS has the measurements shown.

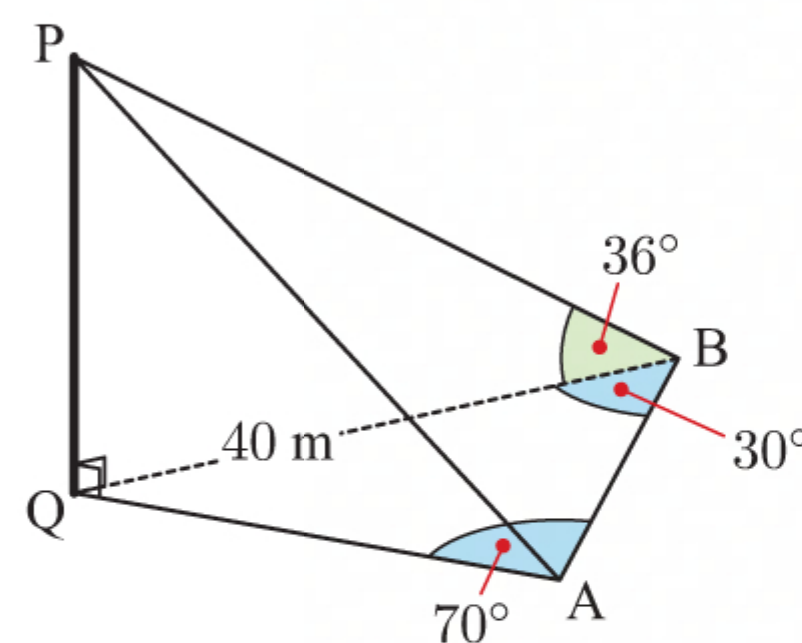
- a** Find the length of [PR].
b Determine the measure of the angle marked θ .



- 30** The diagram shows a vertical pole [PQ], which is supported by two wires fixed to the horizontal ground at A and B. $\widehat{PBQ} = 36^\circ$, $\widehat{BAQ} = 70^\circ$, $\widehat{ABQ} = 30^\circ$, and the distance BQ is 40 m.

Find:

- a** the height of the pole
b the distance between A and B.



- 31** Monument X is observed from two points B and C which are 330 m apart. \widehat{XBC} is 63° and \widehat{BCX} is 75° .

- a** Draw a neat, labelled diagram to illustrate this information.
b Find the distance between the monument and B.

- 32** Trains A and B are 10 km apart, and are approaching the same train station.

Train A is 8 km from the train station on the bearing 071° . Train B is on the bearing 296° from the train station.

- a** Display this information on a diagram.
b Find the bearing of train B from train A.
c Train B is travelling at an average speed of 7 m s^{-1} . Find, to the nearest second, the time it will take for train B to reach the train station.

- 33** Let $a = \sin 20^\circ$ and $b = \tan 50^\circ$. Write, in terms of a and b , expressions for:

- a** $\sin 160^\circ$ **b** $\tan(-50^\circ)$ **c** $\cos 70^\circ$ **d** $\tan 20^\circ$

- 34** Given that $\sin \theta = -\frac{1}{2}$ and $\cos \theta = -\frac{\sqrt{3}}{2}$ where $0^\circ < \theta < 360^\circ$, find the exact value of:

- a** θ **b** $\tan \theta$ **c** $\tan 2\theta$

- 35** Find the exact value of:

- a** $\sin \frac{5\pi}{3}$ **b** $\cos \frac{3\pi}{4}$ **c** $\tan\left(-\frac{\pi}{3}\right)$

- 36** Without using a calculator, evaluate:

- a** $\sin \frac{\pi}{3} \cos \frac{\pi}{4}$ **b** $2 \tan^2\left(\frac{2\pi}{3}\right) + 1$ **c** $\frac{\cos \frac{5\pi}{6} \tan^2\left(\frac{3\pi}{4}\right)}{\sin\left(-\frac{\pi}{3}\right)}$

- 37** Find θ if $0 \leq \theta \leq 2\pi$ and:

- a** $\cos \theta = -\frac{1}{2}$ **b** $\sin \theta = \frac{1}{\sqrt{2}}$ **c** $\tan^2 \theta = \frac{1}{3}$

- 38** Find the possible exact values of:

- a** $\cos \theta$ if $\sin \theta = \frac{4}{5}$ **b** $\sin \theta$ if $\cos \theta = -\frac{2}{7}$.

39 Without using a calculator, find:

a $\sin \theta$ if $\cos \theta = -\frac{1}{4}$ and $\pi < \theta < \frac{3\pi}{2}$

c $\tan \theta$ if $\sin \theta = -\frac{5}{6}$ and $\frac{3\pi}{2} < \theta < 2\pi$

40 Find exact values for $\cos \theta$ and $\sin \theta$ given that:

a $\tan \theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

b $\cos \theta$ if $\sin \theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$

d $\tan \theta$ if $\cos \theta = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$.

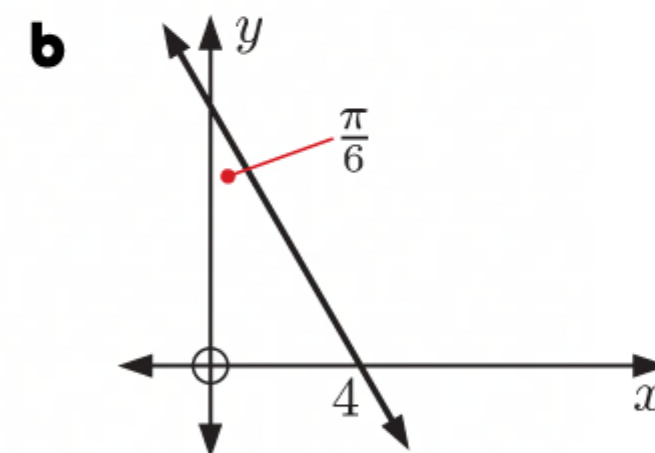
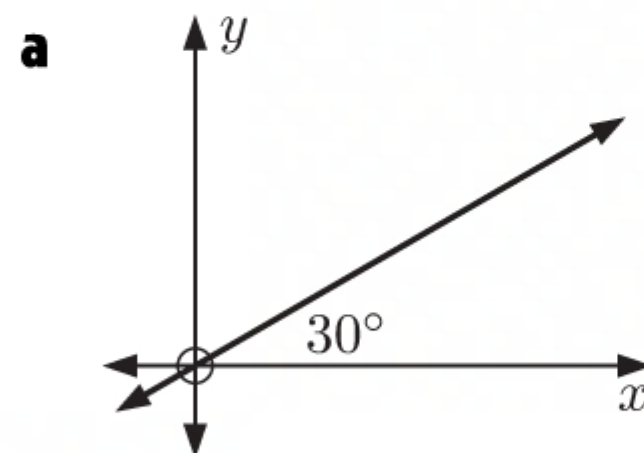
41 Find all θ such that $0^\circ \leq \theta \leq 360^\circ$ and:

a $\cos \theta = -0.3$

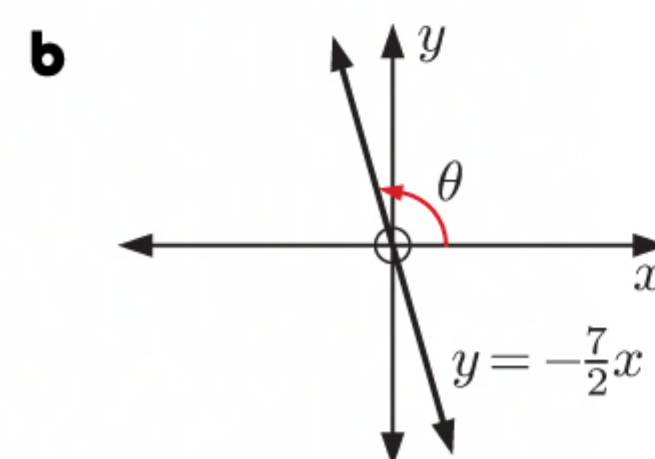
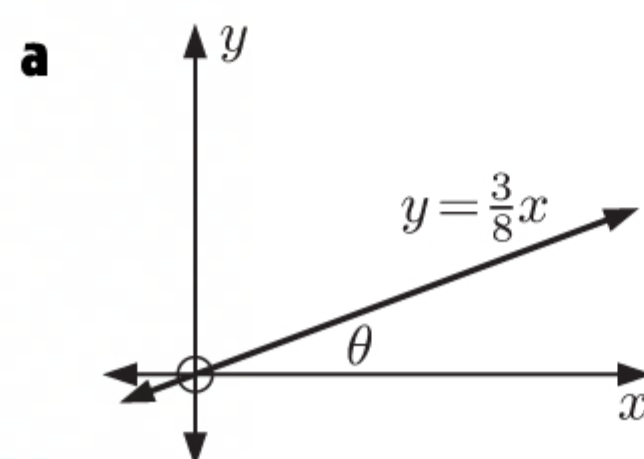
b $\sin \theta = -\frac{7}{9}$

c $\tan \theta = -\frac{3}{\sqrt{5}}$

42 Find the equation of the straight line illustrated:



43 Find, in radians, the measure of θ :



44 Find the amplitude, principal axis, and period of:

a $f(x) = \sin 4x$

b $f(x) = -2 \sin \frac{x}{2} - 1$.

45 For each of the following functions:

i State the amplitude.

iii State the period.

a $y = 2 \sin(x - \frac{\pi}{3})$ for $0 \leq x \leq 2\pi$

c $y = 3 \cos 2x$ for $0 \leq x \leq 2\pi$

e $y = \sin(2(x + \frac{\pi}{4}))$ for $0 \leq x \leq 2\pi$

ii State the principal axis.

iv Sketch the function.

b $y = \sin x + 2$ for $-\pi \leq x \leq \pi$

d $y = \cos \frac{x}{2} - 1$ for $0 \leq x \leq 2\pi$

f $y = 10 - 6 \sin 3x$ for $0 \leq x \leq 2\pi$

46 For each of the following functions:

i State the period.

ii Write the equations of the asymptotes.

iii Sketch the function.

a $y = \tan \frac{x}{2}$ for $-2\pi \leq x \leq 2\pi$

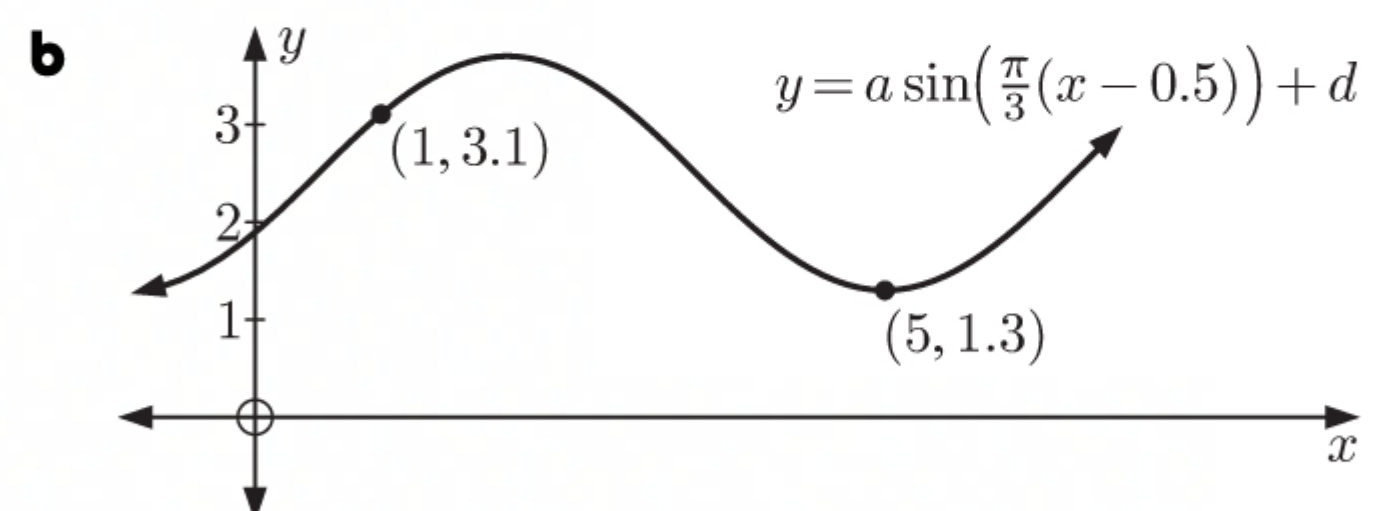
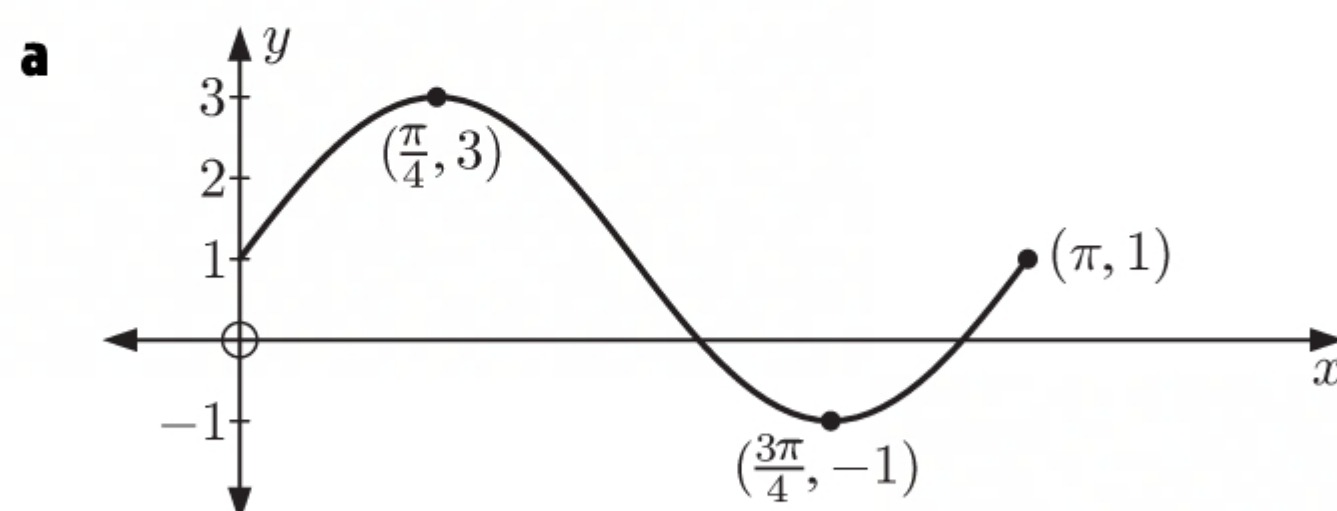
b $y = 5 \tan 3x$ for $0 \leq x \leq \pi$

47 State the transformations which map $y = \sin x$ onto:

a $y = 2 \sin \frac{x}{3}$

b $y = \sin(x + \frac{\pi}{3}) - 4$

48 Find the equation of each sine function.



49 a This graph shows $y = a \cos(b(x - c)) + d$ for $1 \leq x \leq 5$.

Use the graph to find the value of:

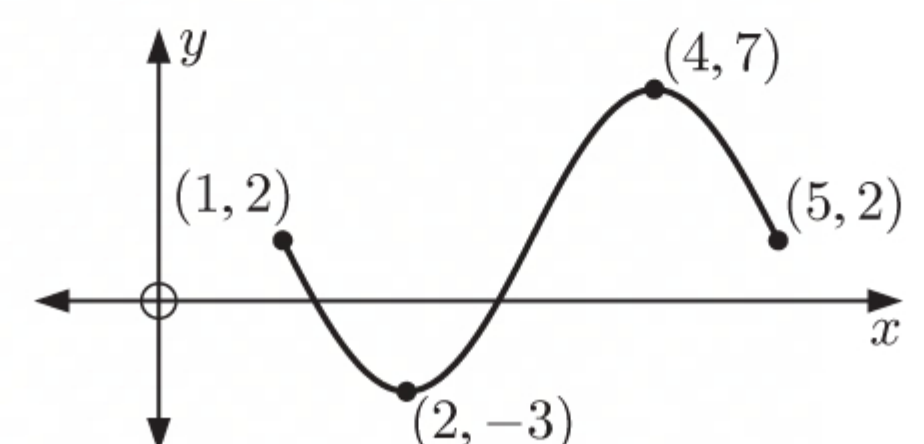
i a

ii b

iii d

iv c

b Write the function in **a** as a sine function.



- 50** The data in the table below shows the mean daily petrol price in a city for the past 28 days.

Day (t)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Price (P cents per L)	135	135	138	144	147	149	151	150	149	143	140	135	134	130

Day (t)	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Price (P cents per L)	129	128	131	135	136	143	144	148	148	150	151	145	142	137

We want to model the data with a trigonometric function of the form $P = a \sin(b(t - c)) + d$.

- a** Draw a scatter diagram of the data.
- b** Without using technology, estimate: **i** b **ii** a **iii** d **iv** c
- c** Check your answers to **b** using technology.
- 51** Suppose $f(x) = 2 \tan(3(x - 1)) + 4$, $-1 \leq x \leq 1$. Find:
- a** the period of $y = f(x)$ **b** the equations of any asymptotes
- c** the transformations that transform $y = \tan x$ into $y = f(x)$
- d** the domain and range of $y = f(x)$.
- 52** Solve for x if $0 \leq x \leq 2\pi$:
- a** $\sin x = 0.785$ **b** $2 \cos x = 5 \sin x$ **c** $\tan 3x = 0.9$ **d** $4 \sin^2 x = \cos^2 x$
- 53** Solve for x where $-\pi \leq x \leq 3\pi$, giving exact answers:
- a** $\sqrt{3} \tan \frac{x}{2} = -1$ **b** $\sqrt{3} + 2 \sin 2x = 0$ **c** $1 - \sqrt{2} \cos 3x = 0$ **d** $10 \sin \frac{x}{3} = 5\sqrt{3}$
- 54** Solve exactly:
- a** $\sin(x - \frac{\pi}{3}) = 0$, $-2\pi \leq x \leq \pi$ **b** $\cos(3x + \frac{\pi}{4}) = -\frac{1}{2}$, $0 \leq x \leq \pi$
- 55** Find the exact solutions of these equations for $0 \leq x \leq 2\pi$:
- a** $\sqrt{2} \cos x + 1 = 0$ **b** $\sin x = -\sqrt{3} \cos x$ **c** $\sin^2 x = \frac{3}{4}$
- d** $\tan^3 2x - 3 \tan 2x = 0$ **e** $4 \cos^2 x - 3 = 4 \cos x$ **f** $4 \cos^4 x + 1 = 5 \cos^2 x$
- 56** The population of butterflies after t years is $P(t) = 4500 + 500 \sin(\frac{2\pi}{7}(t - 3))$ for $0 \leq t \leq 10$.
- a** Find: **i** the initial population **ii** the population after 3 years.
- b** When is the population: **i** 4200 **ii** 4900?
- c** During what time interval(s) does the population drop below 4300?
- 57** Simplify:
- a** $2 \sin^2 \theta + 3 \sin^2 \theta$ **b** $\cos x \tan x - 2 \sin x$ **c** $\frac{-\cos(\frac{\pi}{2} - \theta) \sin(-\theta)}{\cos(-\theta) \sin(\pi - \theta)}$
- 58** Simplify:
- a** $-3 \sin^2 \theta - 3 \cos^2 \theta$ **b** $\sin \theta \cos^2 \theta + \sin^3 \theta$ **c** $\frac{\sin^2 \theta - 1}{\cos \theta}$
- 59** Expand and simplify, if possible:
- a** $(\sin^2 x + 3)^2$ **b** $(\tan \alpha + 1)^2$ **c** $(\sin x - 1)(\sin x + 1)$
- 60** Solve for x on $0 \leq x \leq 2\pi$, giving your answers as exact values:
- a** $2 \sin^2 x + 3 \cos x = 3$ **b** $\sin 2x + \sin x = 0$
- 61** Factorise:
- a** $1 - \cos^2 x$ **b** $2 \cos^2 \alpha - 7 \cos \alpha \sin \alpha - 4 \sin^2 \alpha$ **c** $2 \cos^2 \theta - 3 \sin \theta$
- 62** Solve for $0 \leq x \leq 2\pi$: $\sin^2 x - \cos^2 x = 0$
- 63** Suppose α is an obtuse angle and $\sin \alpha = \frac{2}{3}$. Find the value of: **a** $\cos \alpha$ **b** $\cos 2\alpha$
- 64** Suppose α is acute and $\cos 2\alpha = \frac{5}{13}$. Find the value of: **a** $\sin \alpha$ **b** $\cos \alpha$ **c** $\tan \alpha$
- 65** If $\cos 2x = \frac{5}{8}$, find the exact value of $\sin x$.
- 66** Solve for x where $-\pi \leq x \leq \pi$:
- a** $\sin 2x = \sin x$ **b** $-3 \cos 2x - 14 \sin x + 11 = 0$ **c** $\sin x + \cos x = 1$

TOPIC 4: STATISTICS AND PROBABILITY

SAMPLING

We obtain data from a **sample** of a population when it is impractical to obtain data from the entire population.

You should know the four main categories of **error** that can arise from sampling:

- **Sampling errors** occur when a characteristic of a sample differs from that of the population.
- **Measurement errors** are inaccuracies in measurement during data collection.
- **Coverage errors** occur when a sample does not truly reflect the population.
- **Non-response errors** occur when a large number of people selected for a survey choose not to respond.

SAMPLING METHODS

- In **simple random sampling**:
 - ▶ Each member of the population has the same chance of being selected in the sample.
 - ▶ Each set of n members of the population has the same chance of being selected as any other set of n members.
- In **systematic sampling**, the sample is created by selecting members of the population at regular intervals.
- In **convenience sampling**, members are chosen for the sample because they are easier to select or more likely to respond.
- In **stratified sampling** or **quota sampling**, the population is divided into subgroups, and the number of members sampled from each subgroup is proportional to the fraction of the population represented by that subgroup. If the members of each subgroup are randomly selected, the sample is a **stratified sample**. If the members are specifically chosen, the sample is a **quota sample**.

TYPES OF DATA AND ITS REPRESENTATION

Categorical data refers to data which describes a particular quality or characteristic.

Discrete data can take any of a set of exact number values $\{x_1, x_2, x_3, \dots\}$. It is normally **counted**.

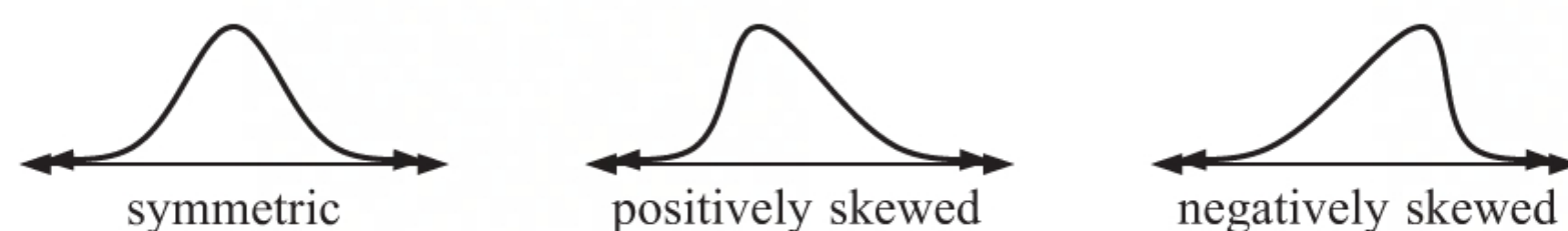
Continuous data can take any numerical value within a certain range. It is normally **measured**.

Grouped data is numerical data which is collected in groups or classes. The **modal class** is the class with the highest frequency.

A **column graph** is used to display discrete data and grouped data. The columns have spaces between them.

A **frequency histogram** is used to display continuous data. The classes are of equal width, and there are no spaces between the columns.

Data may be symmetric, positively skewed, or negatively skewed.



We use a **cumulative frequency graph** to display the cumulative frequency for each data value in a distribution. This enables us to read off the values at each percentile.

MEASURING THE CENTRE OF DATA

The **mean** of a set of scores is their arithmetic average.

For a large population, the **population mean** μ is generally unknown. The **sample mean** \bar{x} is used as an approximation for μ .

For ungrouped data, $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$

For data in a frequency table, $\bar{x} = \frac{\sum xf}{\sum f}$ where f is the frequency of each value.

For grouped data we can only estimate the mean. We use the **mid-interval value** within each group to represent all scores within that group.

The **median** is the middle value of an ordered data set.

- For an **odd number** of data, the median is one of the original data values.
- For an **even number** of data, the median is the average of the two middle values, and may not be in the original data set.

The **mode** is the most frequently occurring score. If there are two modes we say the data is **bimodal**. For continuous data we refer to a **modal class**.

PERCENTILES

The **k th percentile** is the score a such that $k\%$ of the scores are less than a .

The **lower quartile** (Q_1) is the 25th percentile.

The **median** (Q_2) is the 50th percentile.

The **upper quartile** (Q_3) is the 75th percentile.

You should know how to generate a **cumulative frequency graph** and use it to estimate Q_1 , Q_2 , and Q_3 .

MEASURING THE SPREAD OF DATA

The **range** is the difference between the maximum and the minimum data values.

The **interquartile range** $IQR = Q_3 - Q_1$.

The **variance** σ^2 is the average of the squares of the distances from the mean.

The **standard deviation** σ is the square root of the variance.

You should be able to use technology to calculate standard deviation.

OUTLIERS

Outliers are extraordinary data that are separated from the main body of the data. We test for outliers by calculating upper and lower boundaries:

- upper boundary $= Q_3 + 1.5 \times IQR$
- lower boundary $= Q_1 - 1.5 \times IQR$

Any data outside of these boundaries is considered an outlier.

BOX AND WHISKER DIAGRAMS

A **box and whisker diagram** or **box plot** illustrates the **five-number summary** of a data set:

- minimum value
- Q_1
- median
- Q_3
- maximum value

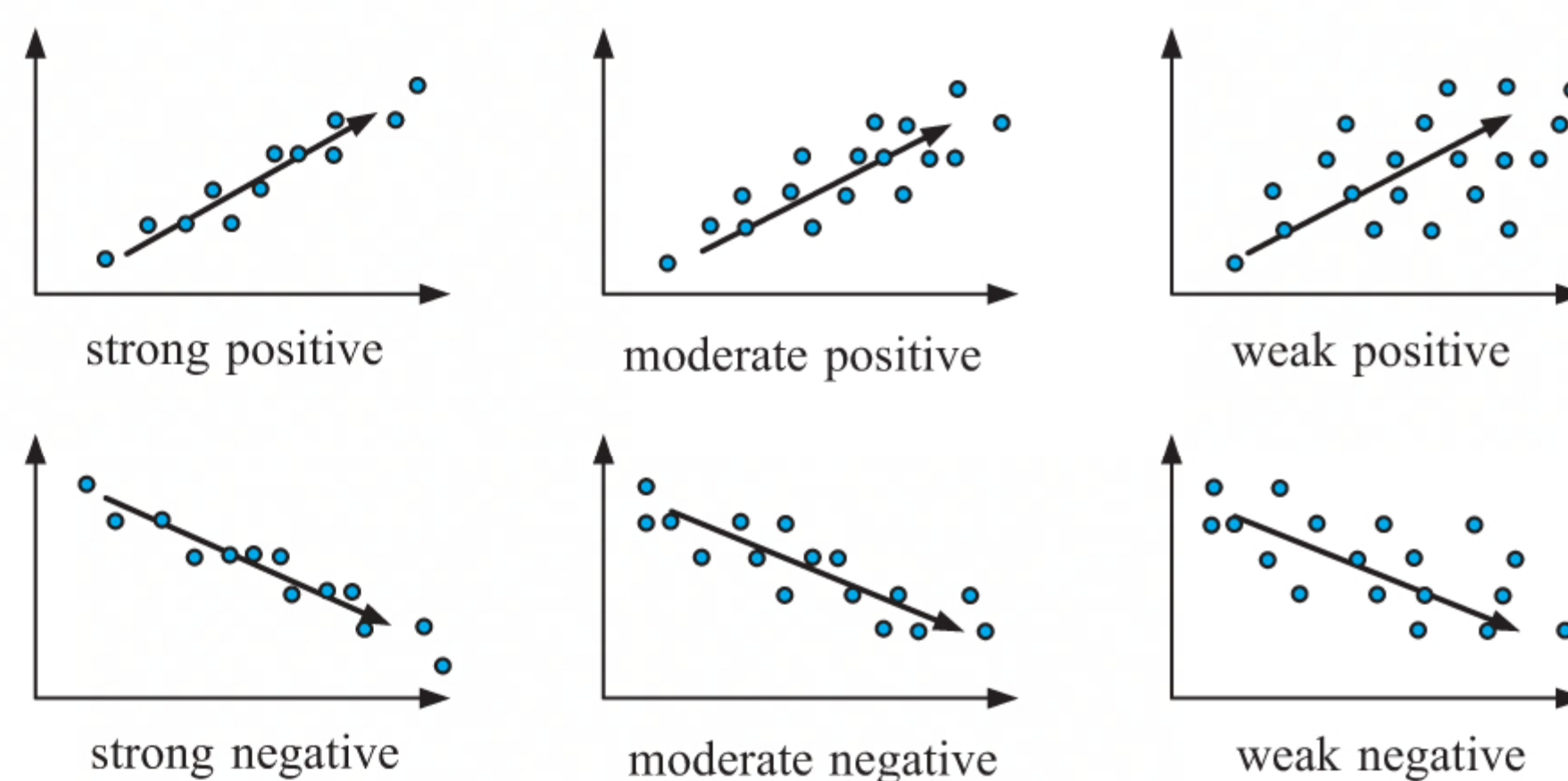


An outlier is indicated by an asterisk *.

BIVARIATE STATISTICS

Correlation refers to the relationship between two numerical variables.

We can use a **scatter diagram** to help identify **outliers** and to describe the correlation between variables. We consider **direction**, **strength**, and **linearity**.



If a change in one variable *causes* a change in the other variable then we say there is a **causal relationship** between them.

To measure the strength of the relationship between two variables, we use **Pearson's product-moment correlation coefficient** r .

The correlation coefficient lies in the range $-1 \leq r \leq 1$.

- The sign of r indicates the direction of correlation.
 - ▶ A positive value for r indicates the variables are positively correlated.
 - ▶ A negative value for r indicates the variables are negatively correlated.
- The size of r indicates the strength of correlation.
 - ▶ A value of r close to $+1$ or -1 indicates strong correlation between the variables.
 - ▶ A value of r close to zero indicates weak correlation between the variables.

Line of best fit

If two variables are linearly correlated, we can draw a line of best fit to illustrate their relationship.

We can draw a **line of best fit by eye**, which passes through the **mean point** (\bar{x}, \bar{y}) , and which fits the trend of the data.

To get a more accurate line of best fit, we use a method called **linear regression**. The line obtained is called the **least squares regression line**. You should be able to find this line using your calculator. In certain situations, it is more sensible to consider the regression line of x against y , rather than the regression line of y against x .

When using a line of best fit to estimate values, **interpolation** is usually reliable, whereas **extrapolation** may not be.

PROBABILITY

A **trial** occurs each time we perform an experiment.

The possible results from each trial of an experiment are called its **outcomes**.

The **sample space** U is the set of all possible outcomes of an experiment.

Experimental probability

In many situations, we can only measure the probability of an event by experimentation.

experimental probability = relative frequency of event

Theoretical probability

If all outcomes are equally likely, the probability of event A is $P(A) = \frac{n(A)}{n(U)}$.

For any event A , $0 \leq P(A) \leq 1$.

For any event A , A' is the event that A does not occur. A and A' are **complementary events**, and $P(A) + P(A') = 1$.

The event that both A **and** B occur is written $A \cap B$.

The event that A **or** B **or both** occur is written $A \cup B$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For **disjoint** or **mutually exclusive** events, $P(A \cap B) = 0$.

Making predictions using probability

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we *expect* the event to occur is np .

Independent events

Two events are **independent** if the occurrence of each of them does not affect the probability that the other occurs. An example of this is sampling **with replacement**.

For independent events A and B , $P(A \cap B) = P(A)P(B)$.

Dependent events

Two events are **dependent** if the occurrence of one of them *does* affect the probability that the other occurs. An example of this is sampling **without replacement**.

For dependent events A and B , $P(A \cap B) = P(A) \times P(B \text{ given that } A \text{ has occurred})$.

Conditional probability

For any two events A and B , “ $A \mid B$ ” represents the event “ A given that B has occurred”, and $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$.

For independent events, $P(A) = P(A \mid B) = P(A \mid B')$.

DISCRETE RANDOM VARIABLES

A **random variable** represents the possible numerical outcomes of an experiment.

A **discrete random variable** can take any of a set of distinct values.

If X is a discrete random variable with possible values $\{x_1, x_2, \dots, x_n\}$ and corresponding probabilities $\{p_1, p_2, \dots, p_n\}$, then:

- $0 \leq p_i \leq 1$ for all $i = 1, \dots, n$
- $\sum_{i=1}^n p_i = p_1 + p_2 + \dots + p_n = 1$
- $\{p_1, p_2, \dots, p_n\}$ describes the **probability distribution** of X .

We can also describe the probability distribution of X using a **probability mass function** $P(x) = P(X = x)$.

The **expectation** of a discrete random variable X is $E(X) = \mu = \sum_{i=1}^n x_i p_i$.

A game where X is the gain to the player is said to be **fair** if $E(X) = 0$.

The **mode** is the data value x_i whose probability p_i is the highest.

THE BINOMIAL DISTRIBUTION

In a **binomial experiment** there are two possible results: success and failure.

Suppose there are n independent trials of the same experiment with the probability of success being a constant p for each trial. If X represents the number of successes in the n trials, then X has a **binomial distribution**, and we write $X \sim B(n, p)$.

The **binomial probability mass function** is $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ where $x = 0, 1, 2, \dots, n$.

You should be able to use your calculator to find:

- $P(X = x)$ using the binomial probability distribution function
- $P(X \leq x)$ or $P(X \geq x)$ using the binomial cumulative distribution function.

If $X \sim B(n, p)$, then:

- $E(X) = \mu = np$
- $\text{Var}(X) = np(1 - p)$
- $\sigma = \sqrt{\text{Var}(X)} = \sqrt{np(1 - p)}$

THE NORMAL DISTRIBUTION

If the random variable X has a normal distribution with mean μ and variance σ^2 , we write $X \sim N(\mu, \sigma^2)$.

The probability density function is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $x \in \mathbb{R}$.

$f(x)$ is a bell-shaped curve which is symmetric about $x = \mu$.

It has the property that:

- $\approx 68\%$ of all scores lie between $\mu - \sigma$ and $\mu + \sigma$
- $\approx 95\%$ of all scores lie between $\mu - 2\sigma$ and $\mu + 2\sigma$
- $\approx 99.7\%$ of all scores lie between $\mu - 3\sigma$ and $\mu + 3\sigma$.

You should be able to use your calculator to find normal probabilities for the situations:

- $P(X \leq a)$
- $P(X \geq a)$
- $P(a \leq X \leq b)$

You should also be able to use your calculator to find the scores corresponding to particular probabilities. These scores are known as **quantiles**.

The standard normal distribution

Every normal X -distribution can be transformed into the **standard normal distribution** or **Z -distribution** using the transformation $Z = \frac{X - \mu}{\sigma}$.

The standard normal distribution has mean 0 and standard deviation 1, so $Z \sim N(0, 1^2)$.

We use Z -distributions when:

- we are looking for an unknown mean μ or variance σ^2
- we are comparing scores from two different normal distributions.

SKILL BUILDER QUESTIONS

1 Gerard wants to estimate the average height of the 500 students at his school. He randomly selects a sample of 10 students, and uses a tape measure to find the height of each student.

Explain why this approach may produce a: **a** coverage error **b** measurement error.

2 The students at Hoylebury Middle School are to be surveyed on their attitudes on wearing school uniform. The numbers of students in each year level are shown.

	Boys	Girls
Year 8	135	140
Year 9	130	145
Year 10	125	130

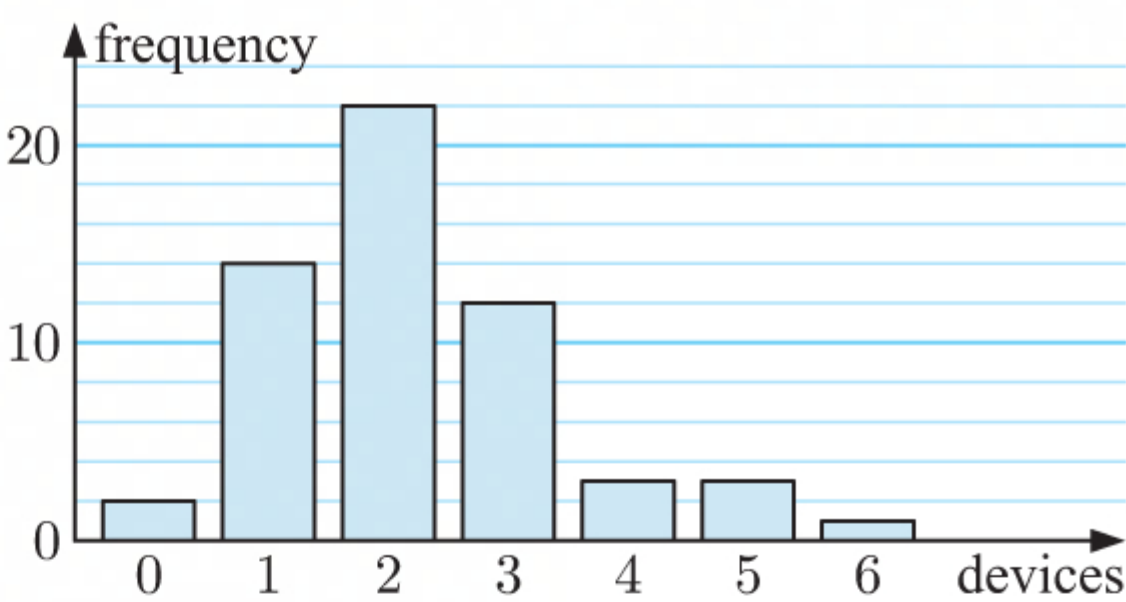
- a** **i** What are the advantages of surveying 50 students?
- ii** What are the disadvantages of surveying all students?
- b** A stratified sample system is used to select 50 students.
 - i** How many Year 8 boys will be selected?
 - ii** How many girls will be selected in total?
- c** Explain why a stratified sample is better than a random sample in this case.

3 Marie is organising a staff lunch in a large office building.

She asks the first 10 people to visit her office for their preferences, and then makes a decision.

- a** Explain why this is a convenience sample.
 - b** In what ways will Marie’s sample be biased?
 - c** Suggest a more appropriate sampling method that Marie should use.
- 4 A ticket inspector checks the tickets of every 20th passenger leaving a train terminal, starting from the 8th passenger.
- a** Identify the sampling method used. Explain your answer.
 - b** List the next six passengers to be checked.
 - c** Given that 5000 passengers left the terminal that day, find the number of passengers checked.
- 5 Classify each variable as categorical, discrete, or continuous:
- a** The number of houses on a particular street.
 - b** The number of hours spent travelling on an airplane.
 - c** The brand of laptop someone uses.

6 A random sample of people were asked “How many devices have you used to browse the internet in the last month?”. The results are displayed in the column graph.



- a** How many people were surveyed?
- b** Find the mode of the data.
- c** What percentage of people browsed the internet using 1 or 2 devices?
- d** Describe the distribution of the data.

7 The number of customers entering a convenience store each hour on a particular day were:

14, 23, 26, 34, 24, 18, 26, 16, 25

Without using technology, find the **a** mode **b** median **c** mean of the data.

8 An art gallery has added two new exhibits alongside their permanent collection. The number of tickets sold for each exhibit was counted every day for a month:

Exhibit A

42	49	55	48	62	81	91	50	60	59
47	73	84	89	55	59	35	42	51	83
75	28	30	19	39	45	69	65	27	32

Exhibit B

59	51	60	44	57	90	98	50	62	55
44	62	75	99	57	57	49	53	71	70
68	32	33	24	47	43	61	42	52	46

- a** Find the:
 - i** mean number of visitors for each exhibit
 - ii** median number of visitors for each exhibit.
- b** Which exhibit was more popular? Explain your answer.

- 9 a** The mean of 7 integers is 14. In ascending order, the integers are 9, 10, a , 13, b , 16, 21.
Find the values of a and b .
- b** In ascending order, a set of six numbers are: 1, 5, 9, 11, 16, p . The mean of the six numbers is the same as their median. Find p .

- 10** Miguel uses an application on his phone to find the amount of sleep he gets each night. The duration of his sleep, in hours, for the past 30 nights are:

7.5 6.8 7.8 6.3 8.6 9.1 7.1 5.8 7.7 7.3 7.7 7.4 11.5 7.1 7.4
8.0 7.6 7.1 9.1 8.0 7.5 7.4 7.5 8.1 8.6 8.7 6.8 7.4 7.7 8.5

- a** Calculate the mean and the median of the data.
- b** Identify the outlier in this data set.
- c** The outlier was the result of a recording error.
- i** Calculate the mean and the median of the data with the outlier removed.
- ii** Which measure of centre is most affected if the outlier is removed?

- 11** The frequency table alongside shows the number of cars owned by different families.

- a** Add a column to the table showing the *cumulative frequency* values.
- b** For this data set, calculate the:
- i** mean **ii** median **iii** mode.

Number of cars	Frequency
0	78
1	117
2	69
3	18
4	2
Total	284

- 12** After seven netball matches, Kai has averaged 11 goals per game.

- a** Find the value of a .
- b** How many goals will she need to score in the next game to improve her overall average to 12?

Score	7	9	a	13	16
Frequency	1	2	1	2	1

- 13** This table shows the weekly rent for a sample of studio apartments in Italy.

- a** Estimate the mean weekly rent.
- b** Find the probability that the weekly rent for a randomly chosen studio apartment will be €140 or greater.

Weekly rent (€ r)	Frequency
$80 \leq r < 100$	3
$100 \leq r < 120$	15
$120 \leq r < 140$	26
$140 \leq r < 160$	30
$160 \leq r < 180$	14
$180 \leq r < 200$	1

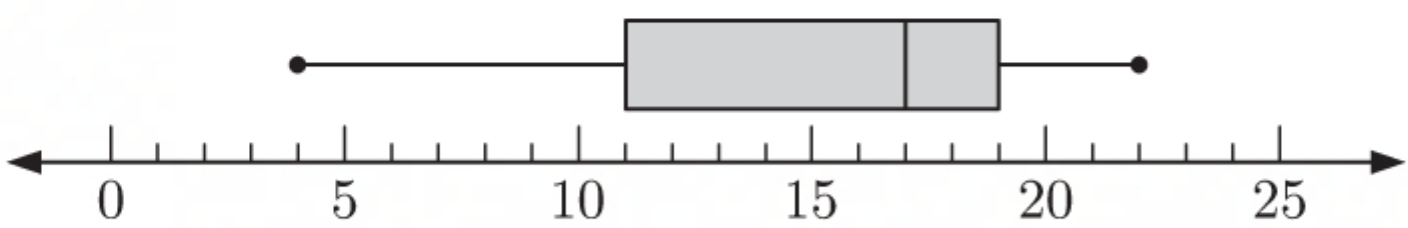
- 14** Cailan and Miles regularly play golf together, and have recorded their scores from their last 10 rounds:

Cailan: 84, 81, 86, 92, 85, 83, 80, 87, 90, 79

Miles: 87, 85, 83, 90, 88, 82, 84, 84, 91, 82

- a** Calculate the range and interquartile range for each data set.
- b** Which golfer had the lower: **i** range **ii** interquartile range?
- c** Which measure of spread is more appropriate for determining who is generally the more consistent golfer? Explain your answer.
- 15** Consider this data set: 16, 20, 10, 16, 4, 12, 23, 18, 17, 9, 18, 16, 31, 26, 18, 14, 12, 14, 15
- a** Write the data set in order, and construct a five-number summary.
- b** Calculate the interquartile range.
- c** Calculate the upper and lower boundaries, and hence identify any outliers in the data set.
- d** Draw a box plot to represent the data.

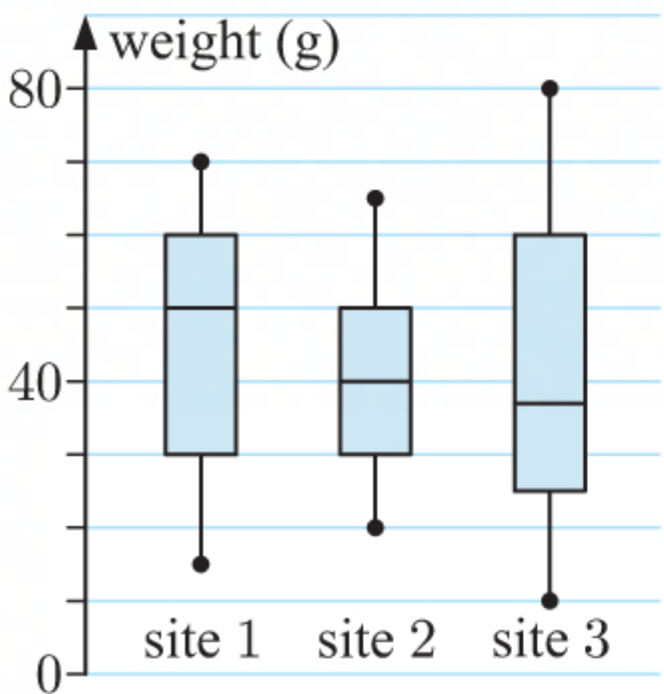
16 A box plot has been drawn to show the heights of some petunia seedlings, in centimetres.



State the:

- a minimum value
- b maximum value
- c median
- d upper quartile
- e lower quartile
- f range
- g interquartile range.

17 These parallel box plots show the weights of particular species of fungi collected from 3 different sites in a forest.



- a Write down the five-number summary for site 1.
- b Which site has the greatest range of weights?
- c At which site do the weights of fungi have the least variation?
- d Which site has the highest median weight of fungi?
- e Which site has the highest proportion of weights above 40 grams?

18 The heights of a random sample of trees in an apple orchard are summarised in the table alongside.

Height (h m)	Frequency
$7 \leq h < 8$	8
$8 \leq h < 9$	59
$9 \leq h < 10$	74
$10 \leq h < 11$	22
$11 \leq h < 12$	1

- a Construct a cumulative frequency graph for the data.
- b Estimate the median height.
- c Estimate the interquartile range.
- d Estimate the 90th percentile. Interpret your answer.

19 Anthony and Katherine are two musicians in an orchestra. They each recorded the number of hours they spent practising in the 10 days before a performance.

Anthony: $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 4, $4\frac{1}{2}$, 3, $3\frac{1}{2}$, 5, 6, 6
Katherine: 3, $3\frac{1}{2}$, 4, 3, 3, $3\frac{1}{2}$, 4, 4, $4\frac{1}{2}$, 4

- a Calculate the mean and standard deviation of each data set.
- b Which person generally practised for longer?
- c Which person practised more consistently?

20 This table shows the distribution of marks obtained on a logic test.
Use technology to find the mean and population standard deviation of the test scores.

Mark	3	4	5	6	7	8	9	10
Frequency	1	3	5	8	4	2	0	1

21 A journalist compares the scores given to two camera models by 6 online reviewers.

Camera A	8.5	8	9	7	8.5	7.5
Camera B	7	6	7.5	9	7.5	6

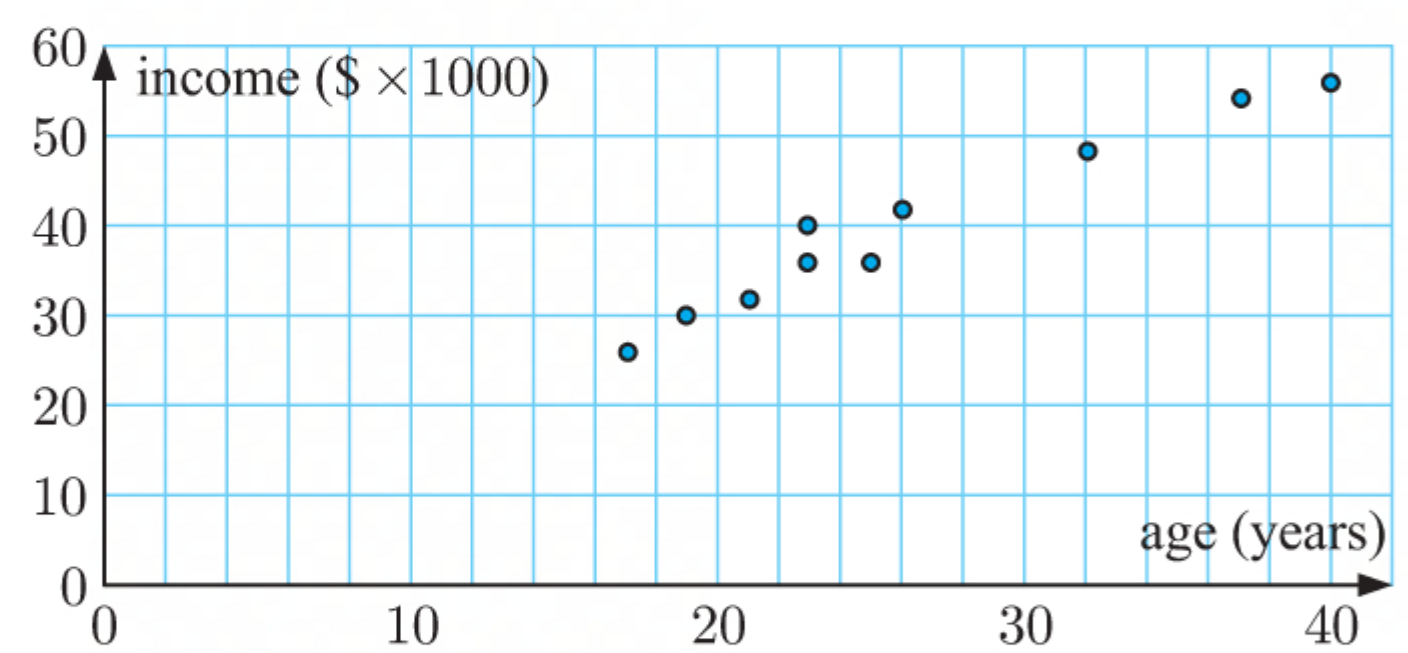
- a Draw a scatter diagram of the data.
- b Identify the outlier in the data.
- c It was found that the outlier was a recording error, and was removed.
 - i Describe the correlation between camera A’s scores and camera B’s scores.
 - ii Does an increase in camera A’s scores cause an increase in camera B’s scores? Explain your answer.

22 Ten students were given aptitude tests on language skills and mathematics. The table below shows the results:

Language (x)	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
Mathematics (y)	32	45	27	38	18	25	35	22	40	40

- a Plot the data on a scatter diagram.
- b Find the correlation coefficient r .
- c Use your results to comment on the statement: “Those who do well in languages also do well in mathematics.”

- 23** This scatter diagram shows the age and annual income of 10 randomly chosen individuals. The mean age is 27 and the mean income is \$40 000.



- Describe the relationship between the age and annual income for these individuals.
- Do you think there is a causal relationship between the variables?
Explain your answer.
- Draw a line of best fit by eye on the graph.



- Estimate the annual income for someone who is 30 years old. Comment on the reliability of your estimate.

- 24** A jeweller measured the volume and mass of some samples of silver. He suspects that one of the samples might be fake. The results are listed in the table.

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ($x \text{ cm}^3$)	3	6	4	7	16	8	5	12	9	6	10	11
Mass ($y \text{ g}$)	40	95	50	160	285	130	65	210	155	90	170	190

- Draw a scatter diagram for this data.
- Calculate Pearson's product-moment correlation coefficient r .
- Describe the relationship which appears to exist between the volume and mass of the samples of silver.
- Do you agree with the jeweller that there is a fake sample?
- Remove the suspect value from the data and find the equation of the regression line for the remaining data.
 - Use your equation to find the expected mass of the sample of silver with the same volume as the suspect sample.

- 25** 9 students sat a Mathematics examination. The number of hours that each of them studied and the results they obtained are shown in the table.

Study time ($x \text{ h}$)	7	6	3	16	15	11	18	32	20
Result ($y \%$)	56	42	25	80	65	60	85	96	90

- Write down the equation of the least squares regression line.
- Describe the correlation between the variables.
- Do you think there is a causal relationship between the variables? Explain your answer.
- Tony's score in the examination was 70%. Use the line of best fit to estimate how long he studied for.
- Interpret the y -intercept and the gradient of the equation of the line of best fit.

- 26** The average height h (in mm) of grass t days after being mowed, is shown in the table below.

Time ($t \text{ days}$)	0	1	2	3	4	5	6	7	8	9
Height ($h \text{ mm}$)	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3

- Calculate Pearson's product-moment correlation coefficient r .
- Explain the significance of the size and sign of r .
- The regression line for h against t is $h \approx 0.4879t + 4.9145$. Use this equation to estimate the:
 - height of the grass after 14 days
 - time required for the grass height to reach 20 mm.

- 27** A group of friends spend their holiday at a beach. Each day, the friends head out to sea for some fishing. Their distance from the shore and the number of fish caught each day is shown in the table.

Distance from shore ($x \text{ km}$)	3.7	1.3	4.3	2.8	0.9
Fish caught (y)	5	4	9	5	2

- Which regression line should be used to model the relationship between the variables? Explain your answer.
- Use an appropriate regression line to estimate the number of fish caught if the distance from shore is 7 km.
- Comment on the reliability of your estimate.

- 28** An annual squash tournament groups players into 5 divisions according to their skill level.

The table shows the number of players at the tournament over 3 years.

Find the probability that a player:

- in the 2017 tournament played in division 1
- in any of the past tournaments played in division 3
- in the 2019 tournament did *not* play in division 2 or 4.

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

- 29** A hospital recorded the age and gender of its 1020 melanoma patients over one year. The data is shown alongside.

- Complete the table.
- Find the probability that a randomly selected melanoma patient was:
 - male
 - female and younger than 40
 - 60 or older, given they were female
 - male, given they were 40 or older.

	< 40	40 - 59	≥ 60	Total
Male	56	127		
Female	75	113	230	
Total				1020

- 30** A die is rolled, and a square spinner with sectors 1, 2, 3, and 4 is spun.

- Draw a grid to illustrate the sample space of possible outcomes.
- Use your grid to find the probability of getting:
 - two 1s
 - two 5s
 - a sum of 6
 - a 2 and a 3
 - a 2 or a 3 (or both)
 - exactly one 4.

- 31** Suppose $P(A) = 0.37$, $P(B) = 0.41$, and $P(A \cup B) = 0.78$.

- Find $P(A \cap B)$.
- What can you say about A and B ?

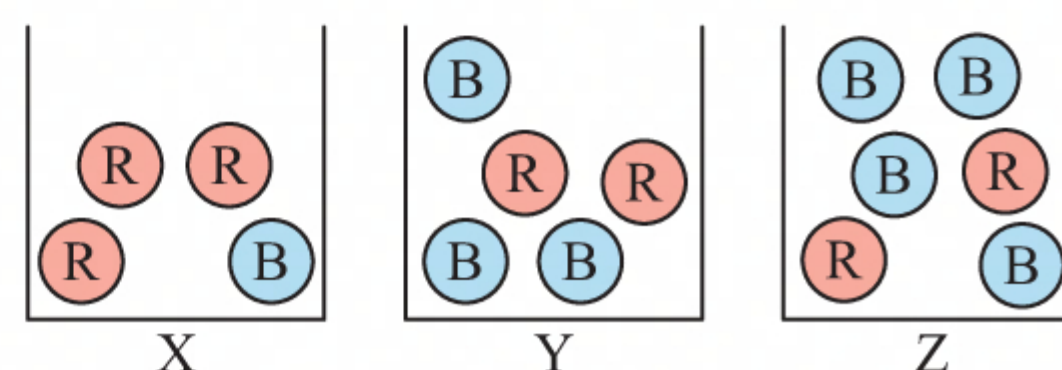
- 32** Given that $P(A) = \frac{23}{50}$, $P(B) = \frac{5}{7}$, and $P((A \cup B)') = \frac{1}{12}$, find $P(A \cap B)$.

- 33** One ball is drawn from each of the boxes shown.

- Draw a tree diagram to illustrate the situation.

- Find the probability that:

- exactly two red balls are drawn
- blue balls are drawn from boxes X and Z
- at most one blue ball is drawn.



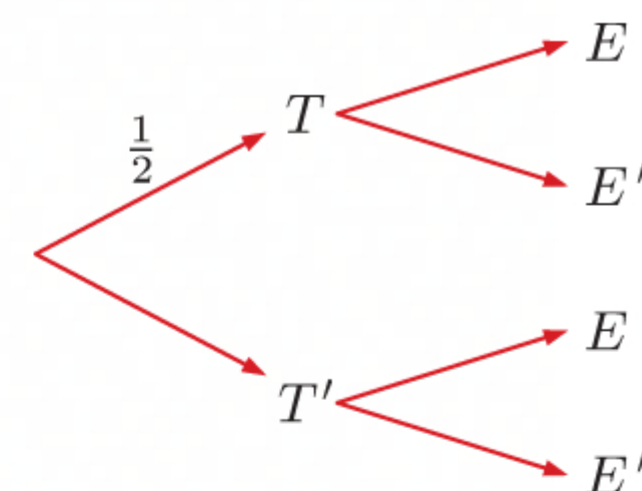
- Suppose an extra red ball is added to box Y. Which of the probabilities in **b** will be affected?

- 34** Suppose you toss a coin and roll a die simultaneously.

Let T represent a tail with the coin and E represent a 2 or a 5 with the die.

- Complete the tree diagram showing the probabilities of the different outcomes.

- Find:
 - $P(T \cap E')$
 - $P(T \cup E')$



- 35** Events A and B are independent. Given that $P(A \cup B) = 0.63$ and $P(B) = 0.36$, find $P(A)$.

- 36** A box of chocolates contains 6 dark brown, 4 light brown, and 2 white truffles. Two truffles are selected from the box without replacement.

Find the probability of selecting:

- 2 white truffles
- different coloured truffles.

- 37** A box contains 4 blue balls and n red balls. When two balls are drawn from the box without replacement, the probability that both are red is $\frac{1}{3}$. Find n .

- 38** 40% of students in a class own an orange highlighter, 20% own a blue highlighter, and 50% do not own either coloured highlighter.
- Draw a Venn diagram to describe the situation.
 - Find the probability that a randomly selected student:
 - owns a blue highlighter, given they own an orange highlighter
 - owns an orange highlighter, given they do not own a blue highlighter.
- 39** Suppose $P(X) = \frac{3}{7}$, $P(Y) = \frac{2}{9}$, and $P(X \cup Y) = \frac{3}{5}$.
- Find:
 - $P(X \cap Y)$
 - $P(X | Y)$
 - $P(Y | X)$
 - Are X and Y independent events? Explain your answer.
- 40** Suppose $P(A \cap B) = 0.2$ and $P(A' \cap B) = 0.3$. Given that A' and B are independent, find $P(A' \cup B)$.
- 41**
 - If 3 coins are tossed, find the probability that two fall heads and the other falls tails.
 - Suppose 3 coins are tossed 400 times. On how many occasions would you expect to see exactly one tail?
- 42** Two fair dice are rolled. Let X be the difference between the numbers rolled.
- Explain why X is a discrete random variable.
 - State the possible values of X .
 - Find $P(X = 3)$.
- 43** Find k for the following probability mass functions:
- $P(x) = k(x + 3)$ for $x = 0, 1, 2, 3, 4$
 - $P(x) = \frac{k^{x-3}}{x-1}$ for $x = 3, 4, 5$
- 44** A discrete random variable X has probability mass function $P(x) = \frac{a}{(x-3)^2}$ for $x = 0, 1, 2$.
- Find a .
 - Find $P(X = 2)$.
 - Find the mode and median of the distribution.
- 45** A bag contains 3 red tickets and 2 blue tickets. Tickets are selected from the bag, without replacement, until at least one ticket of each colour is selected. Let X be the total number of tickets selected.
- State the possible values of X .
 - Find the probability distribution of X .
 - Find the mode of X .
 - Find the expected value of X .
- 46** Josephine's number of safe hits at each softball match has the probability distribution alongside.
- | | | | | |
|--------|------|-----|-----|-----|
| x | 0 | 1 | 2 | 3 |
| $P(x)$ | 0.05 | k | 0.5 | 0.3 |
- Find:
- k
 - the mode of the distribution
 - Josephine's expected number of safe hits per game.
- 47** The table alongside shows the probability distribution for X .
- | | | | | |
|------------|-----|-----|-----|-----|
| x | 0 | 1 | 2 | 3 |
| $P(X = x)$ | 0.3 | 0.2 | m | n |
- If $E(X) = 1.55$, find m and n .
- 48** A random variable X has the probability mass function $P(x) = \frac{x^2 + kx}{50}$ for $x = 1, 2, 3, 4$.
- Find k .
 - Find the mean of the distribution of X .
 - Find $P(X \geq 2)$.
- 49** A bag contains 1 blue ticket, 3 red tickets, and 8 yellow tickets. A player randomly selects a ticket from the bag, and receives \$40 for a blue ticket, \$20 for a red ticket, and \$5 for a yellow ticket.
- Calculate the expected return for one trial of this game.
 - Given that the game costs \$15 to play, explain why it would not be advisable to play this game.
 - Find the number of extra red tickets that should be added to the bag to make the game fair.
- 50** 80% of residents in a particular suburb oppose the construction of traffic lights at a particular intersection. A survey of 20 randomly selected residents is conducted.
- Find the probability that:
- exactly 16 residents oppose the construction
 - 16 or more residents oppose the construction
 - between 10 and 15 residents oppose the construction
 - more than 8 residents support the construction.

- 51** 5% of all items coming off a production line are defective. The manufacturer packages the items in boxes of six, and guarantees a refund if more than two items in a box are defective.

- On what percentage of boxes will the manufacturer have to pay a refund?
- Patrick purchases 10 boxes. Find the probability that he will get a refund for exactly 1 box.

- 52** A hundred seeds are planted in ten rows of ten seeds per row. Assuming that each seed independently germinates with probability $\frac{1}{2}$, find the probability that the row with the maximum number of germinations contains at least 8 seedlings.

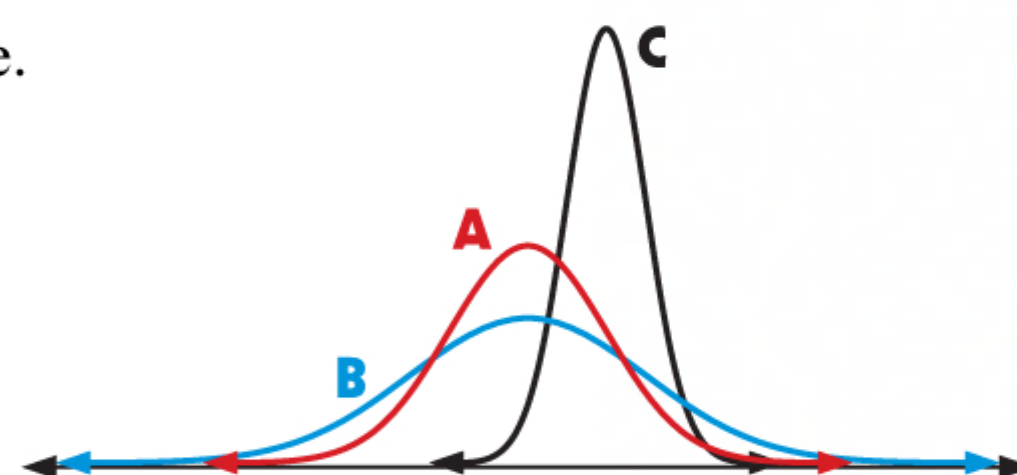
- 53** In a game, a player rolls a biased four-sided die. The probability of obtaining each possible score is shown in the table.

Score	1	2	3	4
Probability	$\frac{1}{12}$	k	$\frac{1}{4}$	$\frac{1}{3}$

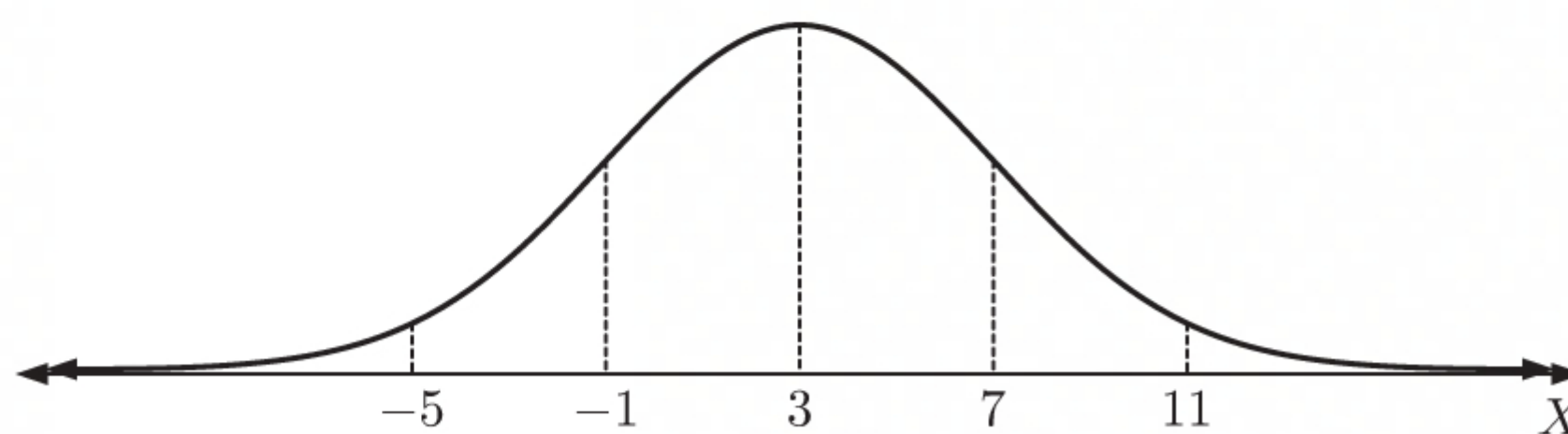
- Find the value of k .
 - Let the random variable X denote the number of 2s that occur when the die is rolled 2400 times. Calculate the exact mean and standard deviation of X .
- 54** A multiple choice test consists of 30 questions with 5 answers to choose from. For each question, only one choice is correct. Let Y be the number of correct answers chosen if each answer is randomly guessed.
- Find the mean μ and standard deviation σ of Y .
 - Find $P(Y = 20)$.
 - Find $P(Y \geq \mu + 2\sigma)$.

- 55** Suppose $X \sim N(\mu, \sigma^2)$. Match each pair of parameters with the correct curve.

- $\mu = 4, \sigma = 1$
- $\mu = 2, \sigma = 2$
- $\mu = 2, \sigma = 3$

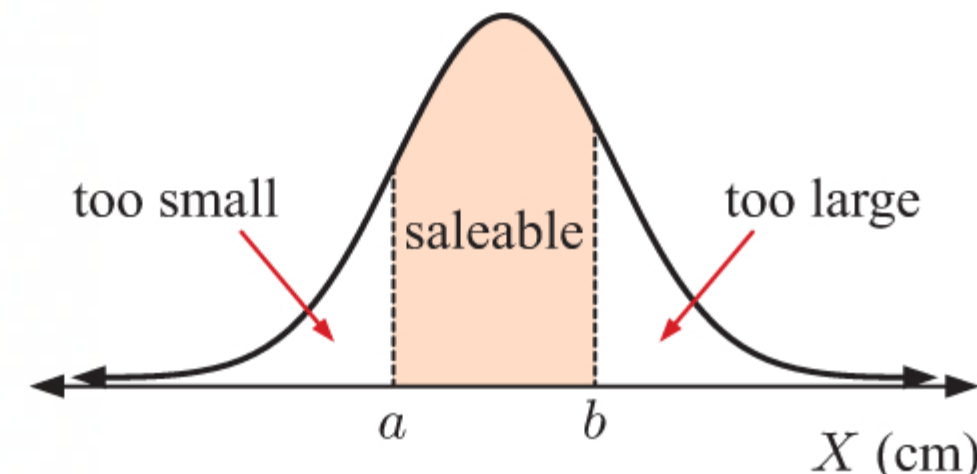


- 56** Consider the distribution curve of $X \sim N(3, 4^2)$ shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

- $N(1, 4^2)$
 - $N(3, 2^2)$
 - $N(2, 64)$
- 57** Suppose a population is normally distributed with mean $\mu = 30$ and standard deviation $\sigma = 5$. Copy and complete:
- Approximately 68% of the population lies between and 35.
 - Approximately 95% of the population lies between 20 and
 - Approximately of the population lies between 15 and 45.
- 58** Containers of a particular brand of ice cream have a capacity of 1050 mL. They are advertised as containing 1 litre of ice cream. The quantity of ice cream added to each container is normally distributed with mean 1020 mL and standard deviation 17 mL.
- Find the probability that the container has less than the advertised capacity.
 - Find the percentage of containers that overflow.
 - A sample of 75 containers are taken. Find the probability that at most three of the containers overflow.
- 59** The volume of drink dispensed by a coffee machine is normally distributed with mean 254 mL and standard deviation 2.3 mL.
- Find the probability that a randomly selected drink from the machine will have volume less than 254 mL.
 - Find the percentage of drinks dispensed by the machine which have volume between 252 mL and 256 mL.
 - A sample of 80 drinks is taken from the machine. Determine the number of drinks which will be expected to have volume at least two standard deviations above the mean.
 - The machine operator guarantees that at least 95% of drinks will have volume at least 250 mL.
 - Is the guarantee valid?
 - A technician adjusts the machine so the standard deviation is now 2.5 mL. What effect does this have on the operator's guarantee?

- 60** A machine fills bottles with tomato sauce. Each bottle is filled independently of all other bottles. The volume of sauce in each bottle is normally distributed with mean 500 mL and standard deviation 2.5 mL. Bottles are deemed to require extra sauce if the machine delivers less than 495 mL.
- Calculate the probability that a randomly selected bottle requires extra sauce.
 - From a sample of 200 bottles, calculate the probability that at least 8 bottles require extra sauce.
- 61** The time taken for a skier to complete a particular downhill run is normally distributed with mean 45 seconds and standard deviation 4 seconds.
- Find the probability that the skier completes:
 - one downhill run in under 40 seconds
 - two consecutive downhill runs in under 40 seconds each.
 - The skier completes a total of 60 independent runs. How many times would you expect the run to take between 44 seconds and 47 seconds?
- 62** The mean birth weight of babies in a population is normally distributed with mean 3.4 kg and standard deviation 300 grams.
- What proportion of babies in this population have birth weights:
 - in excess of 4 kg
 - between 3 kg and 4 kg?
 - A *low birth weight* corresponds to any newborn weighing in the lowest 10% of birth weights. State the weight below which a baby is classified as having a *low birth weight*.
- 63** The length of a zucchini is normally distributed with mean 24.3 cm and standard deviation 6.83 cm. A supermarket buying zucchinis in bulk finds that 15% of them are too small and 20% of them are too large for sale. The remainder, with lengths between a cm and b cm, are able to be sold.
- 
- Find a and b .
 - A zucchini is chosen at random. Find the probability that:
 - it is of saleable length
 - its length lies between 20 cm and 26 cm
 - its length is less than 24.3 cm.
- 64** The lengths of adult fish of a certain species are normally distributed with mean 40 cm and standard deviation 5 cm.
- Find the probability that a randomly chosen adult fish of this species is:
 - longer than 45 cm
 - between 35 cm and 50 cm long.
 - Determine the minimum length of the longest 10% of this species of fish.
 - A randomly selected fish is shorter than 48 cm. Find the probability that it is between 40 cm and 44 cm long.
- 65** The continuous random variable X is normally distributed with $P(X < 56) = 0.8$.
- How many standard deviations from the mean is a score of 56?
 - If the standard deviation of X is 4, find the mean of the distribution. Give your answer correct to one decimal place.
- 66** The lengths of steel rods cut by a machine are normally distributed with mean 13.8 cm. It is found that 1.5% of all rods are less than 13.2 cm long.
- Find the probability that a randomly selected rod has length between 13.2 cm and 13.8 cm.
 - Find the standard deviation of rod lengths produced by this machine.
- 67** Suppose X is normally distributed with $P(X \leq 24) = 0.035$ and $P(X \geq 33) = 0.262$. Find the mean and standard deviation of X correct to 3 significant figures.

TOPIC 5: CALCULUS

LIMITS

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to a , we say that $f(x)$ has a **limit** of A as x approaches a , and we write $\lim_{x \rightarrow a} f(x) = A$.

We say that as x approaches a , $f(x)$ **converges** to A .

RATES OF CHANGE

The **instantaneous rate of change** of a variable at a particular instant is given by the **gradient of the tangent** to the graph at that point.

$\frac{dy}{dx}$ gives the rate of change in y with respect to x .

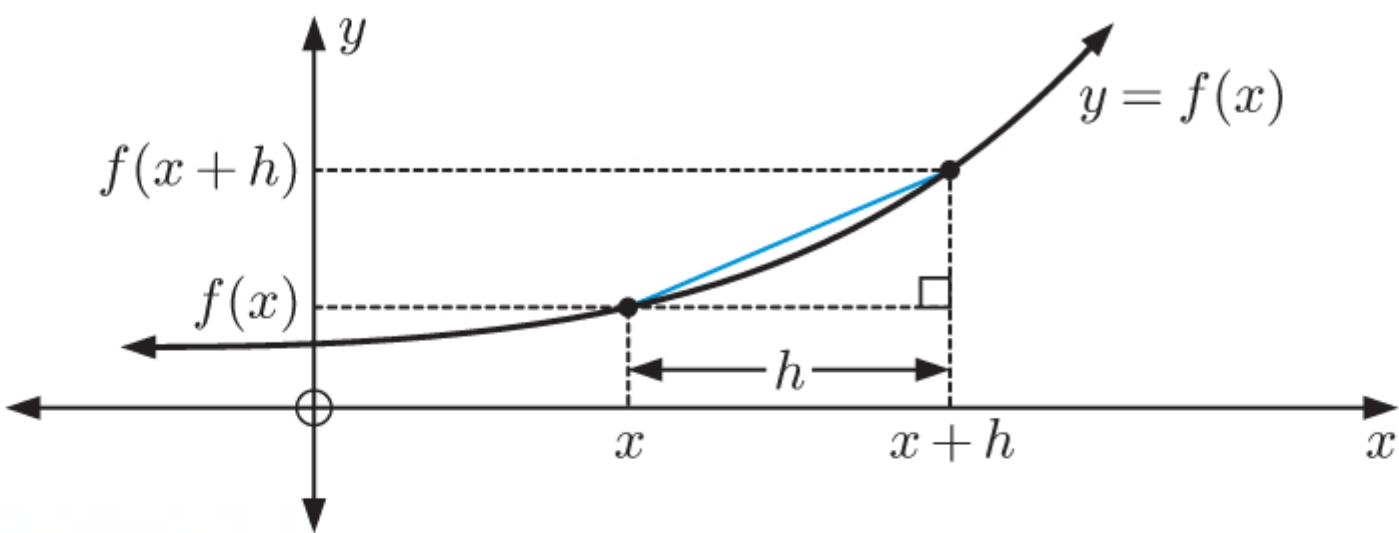
If $\frac{dy}{dx}$ is positive, then as x increases, y also increases.

If $\frac{dy}{dx}$ is negative, then as x increases, y decreases.

DIFFERENTIATION

The **gradient function** or **derivative function** $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ provides:

- the rate of change of f with respect to x
- the gradient of the tangent to $y = f(x)$ for any value of x .



RULES OF DIFFERENTIATION

$f(x)$	$f'(x)$	Name of rule
c	0	exponentials logarithms
x^n	nx^{n-1}	
$e^{f(x)}$	$e^{f(x)} f'(x)$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin x$	$\cos x$	trigonometric functions
$\cos x$	$-\sin x$	

$f(x)$	$f'(x)$	Name of rule
$cu(x)$	$cu'(x)$	addition rule product rule quotient rule
$u(x) + v(x)$	$u'(x) + v'(x)$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	

Chain rule

If $y = f(u)$ where $u = u(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

SECOND DERIVATIVES

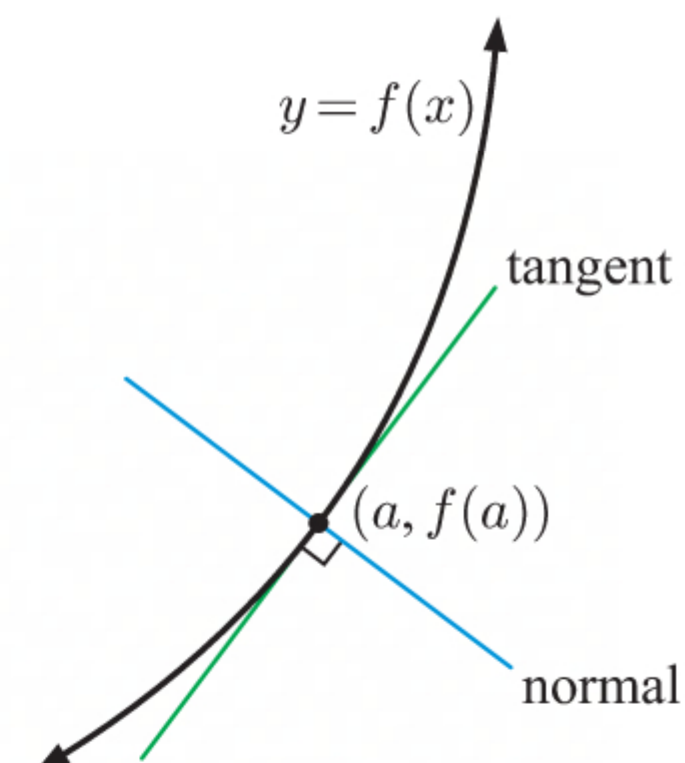
The second derivative of $y = f(x)$ is written $f''(x)$ or $\frac{d^2y}{dx^2}$.

PROPERTIES OF CURVES

Tangents and normals

For the curve $y = f(x)$:

- The gradient of the tangent at $x = a$ is $f'(a)$.
- The equation of the tangent at $x = a$ is $y = f'(a)(x - a) + f(a)$.
- The gradient of the normal at $x = a$ is $-\frac{1}{f'(a)}$.
- The equation of the normal at $x = a$ is $y = -\frac{1}{f'(a)}(x - a) + f(a)$.



Increasing and decreasing functions

$f(x)$ is **increasing** on an interval $S \Leftrightarrow f(a) \leq f(b)$ for all $a, b \in S$ such that $a < b$.

$f(x)$ is **decreasing** on $S \Leftrightarrow f(a) \geq f(b)$ for all $a, b \in S$ such that $a < b$.

For most functions:

- $f(x)$ is increasing on $S \Leftrightarrow f'(x) \geq 0$ for all x in S .
- $f(x)$ is decreasing on $S \Leftrightarrow f'(x) \leq 0$ for all x in S .

Stationary points

A **stationary point** of a function is a point such that $f'(x) = 0$.

You should be able to identify and explain the significance of local and global maxima and minima, as well as stationary and non-stationary inflections.

Stationary point where $f'(a) = 0$	Sign diagram of $f'(x)$ near $x = a$	Shape of curve near $x = a$
local maximum	$\begin{array}{c} + \quad \quad - \\ \leftarrow \quad a \quad \rightarrow \end{array} \quad f'(x)$	
local minimum	$\begin{array}{c} - \quad \quad + \\ \leftarrow \quad a \quad \rightarrow \end{array} \quad f'(x)$	
stationary inflection	$\begin{array}{c} + \quad \quad + \\ \leftarrow \quad a \quad \rightarrow \end{array} \quad f'(x)$ or $\begin{array}{c} - \quad \quad - \\ \leftarrow \quad a \quad \rightarrow \end{array} \quad f'(x)$	

Shape

If $f''(x) \leq 0$ for all $x \in S$, the curve is **concave down** on the interval S .



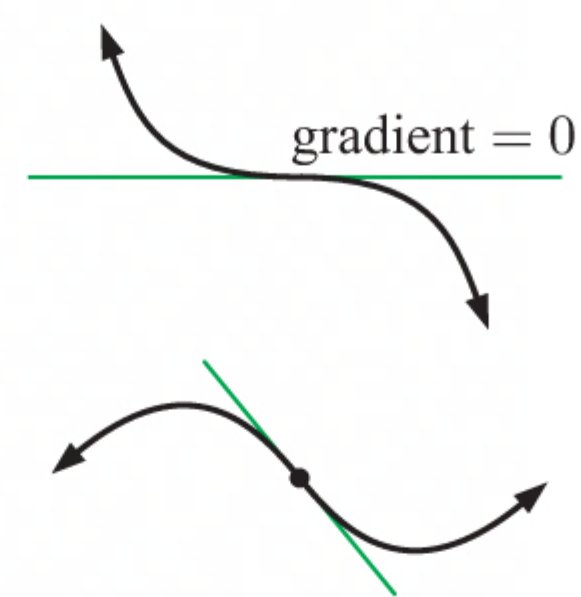
If $f''(x) \geq 0$ for all $x \in S$, the curve is **concave up** on the interval S .



There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes on either side of $x = a$. It corresponds to a change in shape of the curve.



If $f'(a) = 0$, the point of inflection is a **stationary inflection**: the tangent at $x = a$ is horizontal.



If $f'(a) \neq 0$, the point of inflection is a **non-stationary inflection**: the tangent at $x = a$ is *not* horizontal.

OPTIMISATION PROBLEMS

It is important to remember that a local minimum or maximum does not always give the minimum or maximum value of a function in a particular domain. You must check for other turning points in the domain, and the values of the function at the end points of the domain.

Optimisation problem solving method

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x . You should write down what domain restrictions there are on x .

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use the **sign diagram test** or **second derivative test** to determine whether you have a local maximum or local minimum.

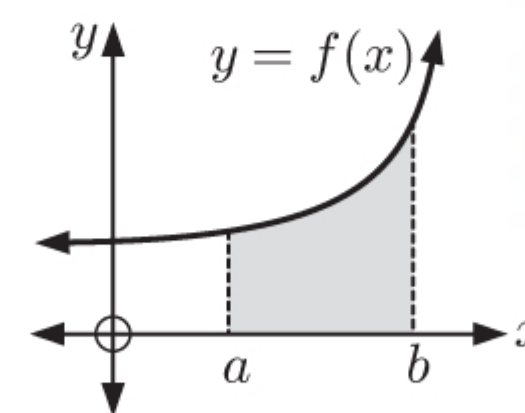
Step 5: Identify the optimal solution, also considering end points where appropriate.

Step 6: Write your answer in a sentence, making sure you specifically answer the question.

INTEGRATION

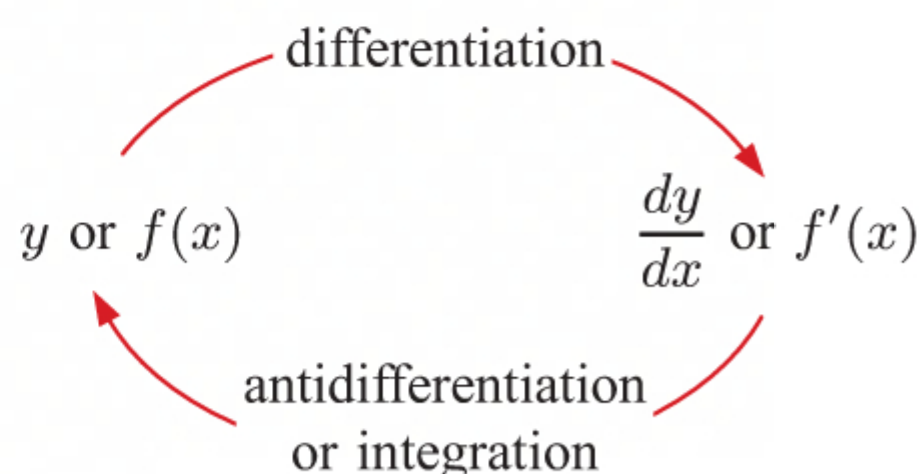
Area under a curve

If $f(x)$ is a continuous *positive* function on the interval $a \leq x \leq b$, then $\int_a^b f(x) dx$ is the area under the curve between $x = a$ and $x = b$.



Antidifferentiation

Antidifferentiation or **integration** is the reverse process of differentiation.



The **antiderivative** or **integral** of $f(x)$ is the simplest function $F(x)$ such that $F'(x) = f(x)$.

Fundamental Theorem of Calculus

For a continuous function $f(x)$ with antiderivative $F(x)$, $\int_a^b f(x) dx = F(b) - F(a)$.

Indefinite integrals

When performing an indefinite integral, we use the rules for differentiation in reverse. Do not forget to include the **constant of integration**.

Function	Integral
k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln x + c$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c, \quad a \neq 0$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad a \neq 0, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a} \ln ax+b , \quad a \neq 0$
$\cos(ax+b)$	$\frac{1}{a} \sin(ax+b) + c, \quad a \neq 0$
$\sin(ax+b)$	$-\frac{1}{a} \cos(ax+b) + c, \quad a \neq 0$

Trigonometric identities are often useful for integration, in particular:

- $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$
- $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

Integration by substitution

$$\int f(u) \frac{du}{dx} dx = \int f(u) du$$

Definite integrals

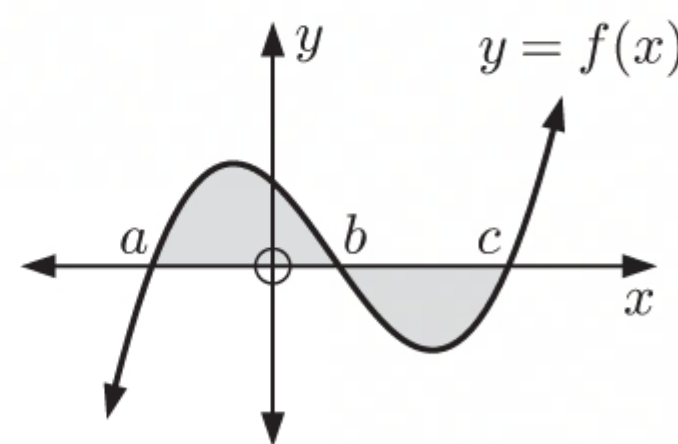
- $\int_a^a f(x) dx = 0$
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_b^a f(x) dx = -\int_a^b f(x) dx$
- $\int_a^b cf(x) dx = c \int_a^b f(x) dx$

To find the total area enclosed by $y = f(x)$ and the x -axis between $x = a$ and $x = b$, we need to be careful about where $f(x) < 0$.

On an interval $c \leq x \leq d$ where $f(x) < 0$, the area is $-\int_c^d f(x) dx$.

For example:

$$\begin{aligned} \text{The total shaded area} &= \int_a^b f(x) dx - \int_b^c f(x) dx \\ &\neq \int_a^c f(x) dx. \end{aligned}$$



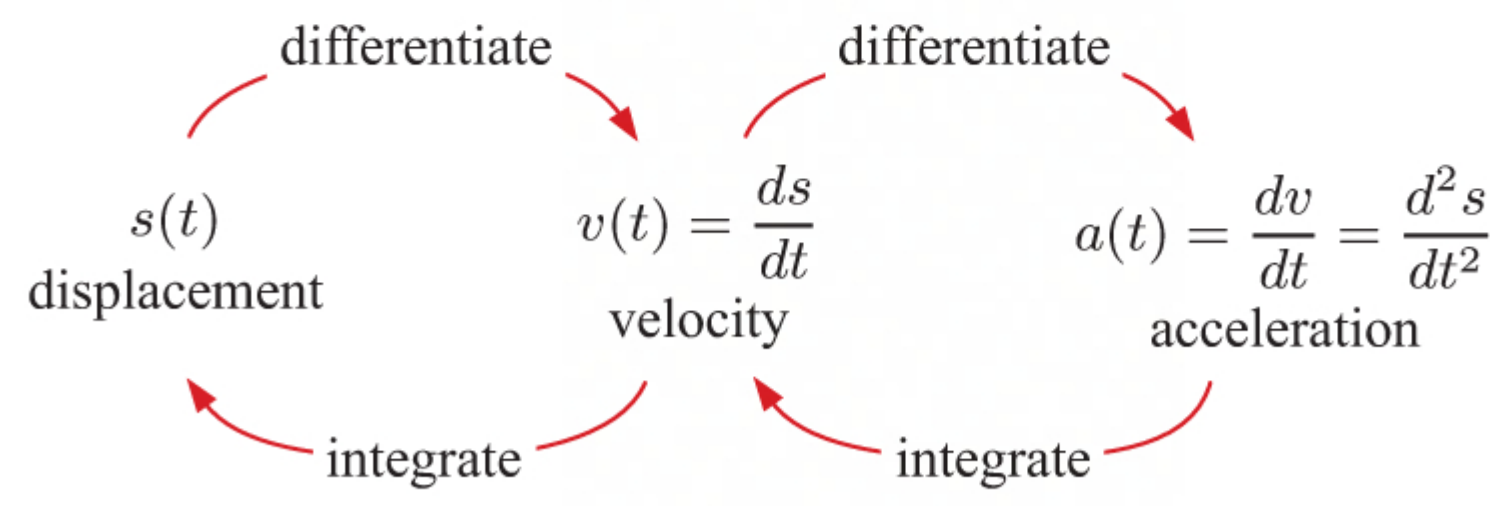
The area between two functions

The area *between* two functions $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ on $a \leq x \leq b$ is given by $A = \int_a^b [f(x) - g(x)] dx$.

KINEMATICS

Suppose an object moves along a straight line.

Its position relative to the origin at time t is given by a displacement function $s(t)$. Its instantaneous velocity is given by $v(t) = s'(t)$, and its instantaneous acceleration by $a(t) = v'(t) = s''(t)$.



You should understand the physical meaning of the different signs of displacement, velocity, and acceleration.

Displacement:

$s(t)$	Interpretation
$= 0$	The object is at O
> 0	The object is to the right of O
< 0	The object is to the left of O

Velocity:

$v(t)$	Interpretation
$= 0$	The object is instantaneously at rest
> 0	The object is moving to the right
< 0	The object is moving to the left

Acceleration:

$a(t)$	Interpretation
> 0	The velocity of the object is increasing
< 0	The velocity of the object is decreasing
$= 0$	The velocity of the object may be at a maximum or a minimum

Speed

The **speed** at any instant is the magnitude of the object’s velocity. If $S(t)$ represents the speed then $S = |v|$.

If the signs of $v(t)$ and $a(t)$ are the same then the speed of the object is increasing.

If the signs of $v(t)$ and $a(t)$ are different then the speed of the object is decreasing.

Displacement and distance travelled

For the time interval $t_1 \leq t \leq t_2$:

- displacement $= s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t) \, dt$
- total distance travelled $= \int_{t_1}^{t_2} |v(t)| \, dt.$

16 Find:

a $\frac{d}{dx} \left(\ln \left(\frac{x-4}{x^2+4} \right) \right)$

b $\frac{d}{dx} (\ln(x\sqrt{x^2+4}))$

c $\frac{d}{dx} \left(\ln \left(\frac{\sqrt{x^2+1}}{(x+3)(x-2)} \right) \right)$

17 Differentiate with respect to x :

a $3 \sin(x-4)$

b $12x - 2 \cos \frac{x}{3}$

c $x^2 \sin 3x$

d $(\sin x)e^{\cos x}$

18 Find $f'(x)$ for:

a $f(x) = \sqrt{\sin(2x+1)}$

b $f(x) = \cos \frac{x}{2} \sin \frac{x}{3}$

c $f(x) = \ln \left(\frac{\sin x}{x} \right)$

19 Find the gradient of the tangent to:

a $f(x) = \cos^4 x$ at the point where $x = \frac{3\pi}{4}$

b $f(x) = \frac{3 \sin^2 x}{\cos 2x}$ at the point where $x = -\frac{\pi}{3}$.

20 Find $\frac{d^2y}{dx^2}$ for:

a $y = \frac{3}{x^2}$

b $y = 2x^3 + 3x^2 + 2$

c $y = \frac{x+3}{6-x}$

21 Given $f(x) = \ln(\cos x)$, find:

a $f\left(\frac{\pi}{4}\right)$

b $f'\left(\frac{\pi}{4}\right)$

c $f''\left(\frac{\pi}{4}\right)$.

22 a Find the equation of the tangent to the curve $y = x^3 - 5$ at the point $(1, -4)$.

b Where does this tangent cut the x -axis?

23 Let $g(x) = -x \cos x$.

a Find $g'(x)$.

b Find the equation of the tangent to the graph $y = g(x)$ at the point where $x = \frac{\pi}{3}$.

24 Let $f(x) = -x^2 + 4x$.

a Find $f'(x)$.

b Find the equation of the tangent to $y = f(x)$ at the point where $x = k$.

c Suppose this tangent has positive gradient and passes through $(4, 9)$. Find the value of k .

25 Find the equation of the tangent to $f(x) = \ln(2x+3)$ at the point where $x = 2$.

26 Consider the curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$.

a Find the point at which these curves meet.

b Show that the tangents to the curves have the same gradient at this intersection point.

c Find the equation of the common tangent.

27 Consider the curve $y = \frac{a}{x} - x^2 + 1$ where $a \in \mathbb{R}$. The gradient of the tangent to the curve is -5 when $x = 2$.

a Find the value of a .

b Determine the equation of the tangent to the curve at $x = 2$.

28 Let $f(x) = \frac{x+2}{\sqrt{x-1}}$.

a State the domain of $f(x)$.

b Find the equation of the normal to $y = f(x)$ at $x = 10$.

29 Consider the cubic function $y = x^3 + ax^2 + bx + 3$.

a The tangent to the function at $(1, 8)$ has equation $y = 2x + 6$. Determine the values of a and b .

b Find the equation of the normal to the function at $x = -1$.

30 a Show that if $f(x) = \ln \left(\frac{1-2x}{x^2+2} \right)$, then $f'(x) = \frac{2(x-2)(x+1)}{(1-2x)(x^2+2)}$.

b On what intervals is $f(x)$ decreasing?

31 Let $f(x) = \frac{3x-4}{(x-1)(x+4)}$.

a Show that $f'(x) = \frac{-x(3x-8)}{(x-1)^2(x+4)^2}$, and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is: **i** increasing **ii** decreasing.

32 Find the exact coordinates and nature of the stationary points of:

a $y = xe^{-x}$

b $y = \frac{x-3}{x^2-5}$

33 Find the greatest and least values of:

a $f(x) = x^3 - 2x^2$ for $-1 \leq x \leq 1$

b $f(x) = x^2 - \frac{27}{x}$ for $-6 \leq x \leq -1$

c $f(x) = x^3 - 6x^2 + 12x - 10$ for $0 \leq x \leq 5$.

34 $f(t) = at^3e^{bt^3}$ has a minimum value of -2 when $t = -1$. Find constants a and b .

35 Consider the function $f(x) = \frac{e^{3x}}{kx}$, $k \neq 0$.

a Find the x -coordinate of the stationary point.

b For what values of k is the stationary point: **i** a local minimum **ii** a local maximum?

c Given that the stationary point has y -coordinate $-\frac{e}{2}$, find k and determine the nature of the stationary point.

d State the location and nature of the stationary point of $g(x) = -f(2x)$.

36 Consider $g(x) = 3 - 2\cos 2x$.

a Find $g'(x)$.

b Sketch $y = g'(x)$ for $-\pi \leq x \leq \pi$.

c Write down the number of solutions to $g'(x) = 0$ for $-\pi \leq x \leq \pi$.

d Mark a point M on the sketch in **b** where $g'(x) = 0$ and $g''(x) > 0$.

37 For each of the following functions, determine the intervals on which the function is:

i increasing

ii decreasing

iii concave upwards

iv concave downwards.

a $f(x) = x^2 + 3x + 5$

b $f(x) = e^{-x^2}$

c $f(x) = x \ln(x^2)$

38 Let $f(x) = xe^{1-2x^2}$.

a Find $f'(x)$ and $f''(x)$.

b Find the exact coordinates of the stationary points of the function and determine their nature.

c Find the exact x -coordinates of the inflection points of the function.

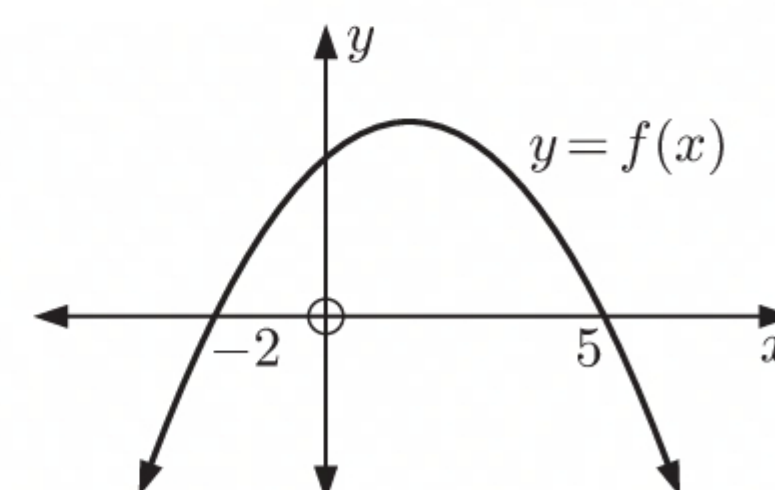
d Use technology to help sketch the function, showing the information you have found.

39 The function $f(x) = \frac{a \ln bx}{x}$ has an inflection point at $\left(\frac{e\sqrt{e}}{2}, \frac{9}{e\sqrt{e}}\right)$. Find the constants a and b .

40 For the function $y = f(x)$ with graph shown, sketch the graphs of:

a $y = f'(x)$

b $y = f''(x)$



41 The graph of $y = f'(x)$ is shown alongside.

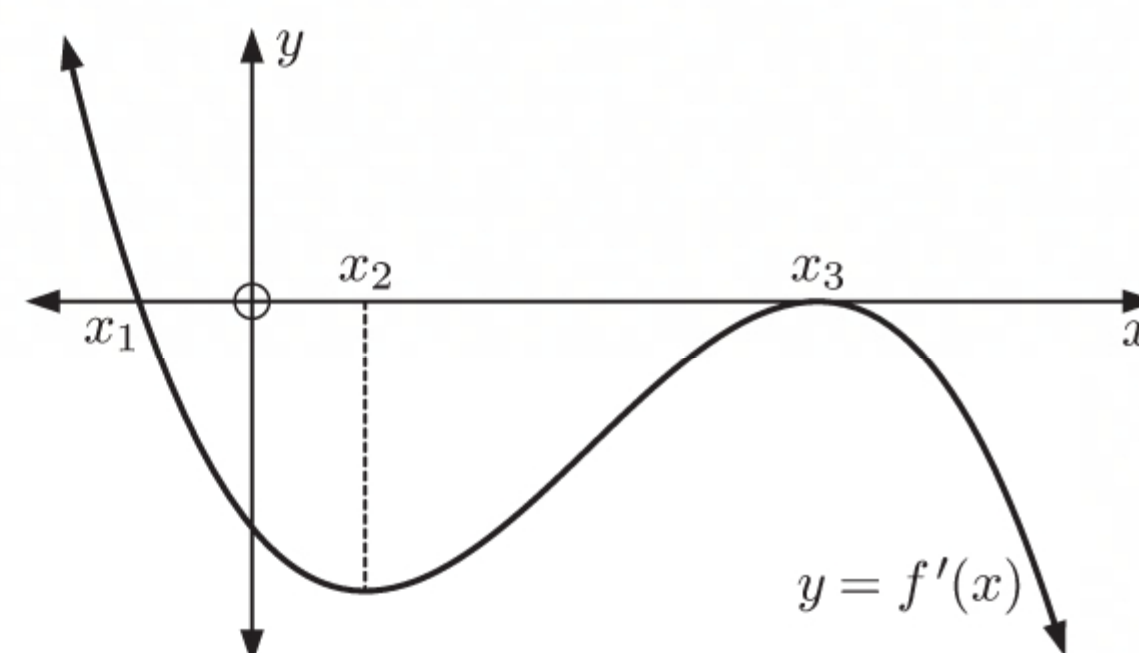
a For what values of x is $f(x)$:

i increasing

ii concave down?

b Given that $y = f(x)$ passes through the origin, sketch the graph of $y = f(x)$.

c Sketch the graph of $y = f''(x)$.



42 The volume of water in a tank is given by $V(t) = 10t^2 - \frac{1}{3}t^3$ litres, where t is the time in minutes and $0 \leq t \leq 30$.

a Find $V(5)$ and explain what this represents.

b Find $V'(t)$ in fully factorised form. Do not forget to include units.

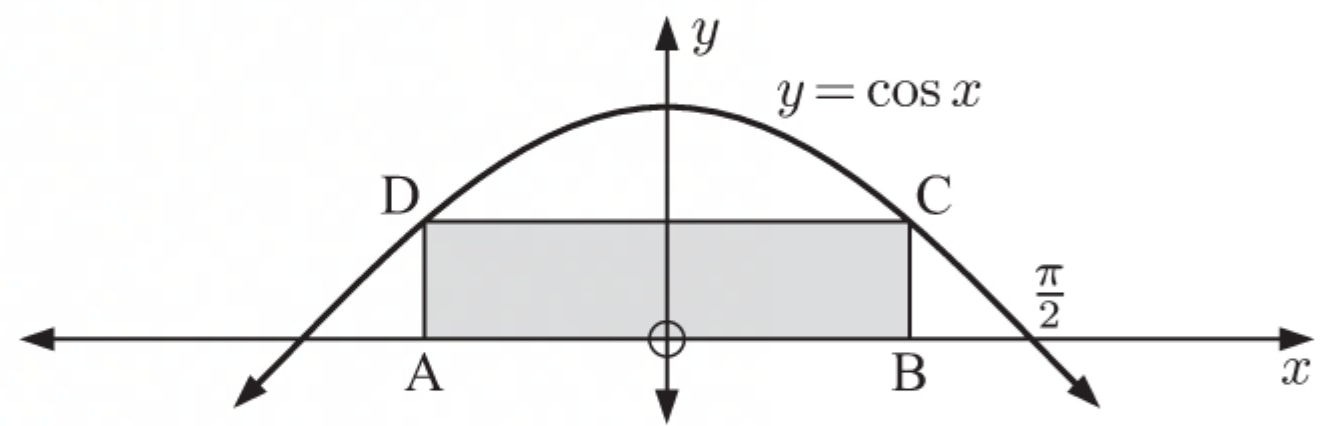
c Find t when $V'(t) = 0$.

d Find $V'(5)$ and $V'(25)$.

e Determine the time(s) at which the volume is increasing by 75 litres per minute.

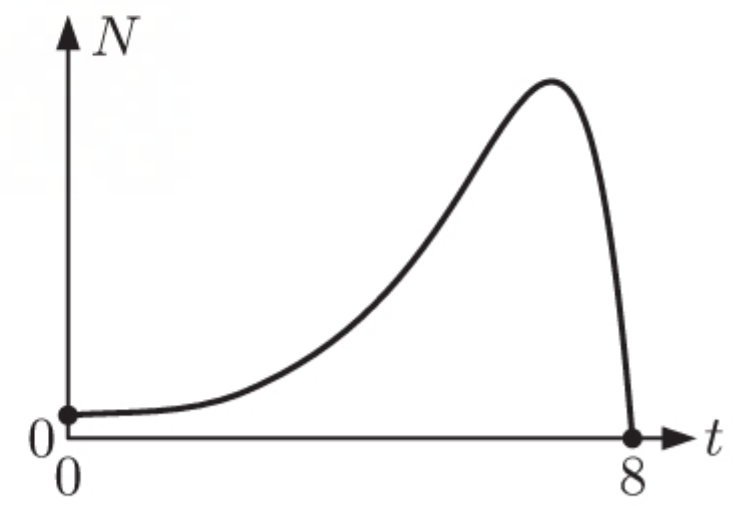
- 43** Rectangle ABCD is inscribed under one arch of $y = \cos x$. Suppose the point C has x -coordinate x .

Find the coordinates of C such that ABCD has maximum area.

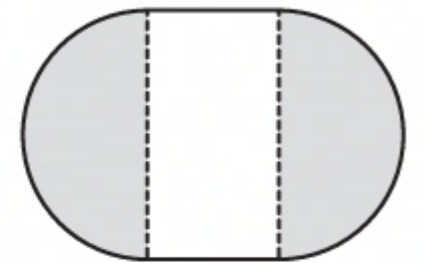


- 44** The number of bacteria found in a sample of human tissue t hours after infection occurred, is modelled by the function $N = (8 - t)e^{t-6}$ million, $0 \leq t \leq 8$.

The graph of this function is shown alongside.



- a** Show that $\frac{dN}{dt} = (7 - t)e^{t-6}$.
- b** Find the coordinates of the:
- turning point
 - point of inflection
 - t -intercept.
- c** Use these coordinates to state:
- the time when all the bacteria are dead
 - the maximum number of bacteria reached in the sample
 - the time at which the rate of increase of the bacteria is a maximum.
- d** Copy the graph and indicate clearly which points on the graph represent your answers to **c**.
- 45** The weight of a radioactive substance after t days is given by the function $W(t) = 100e^{-\frac{t}{20}}$ grams, $t \geq 0$.
- Find the initial amount of radioactive substance present.
 - Find the time necessary for half of the mass to decay.
 - Find $W'(t)$, and interpret its sign.
 - Find $W'(3)$, and interpret your answer.
 - Discuss W as t increases.
- 46** An ornamental pond of area A is to be built with straight sides and semi-circular ends as shown. The cost of tiling per unit length is 25% greater along the rounded ends than along the straight walls. Show that the total cost of tiling the walls is minimised when the shaded area is $\frac{2}{3}A$.

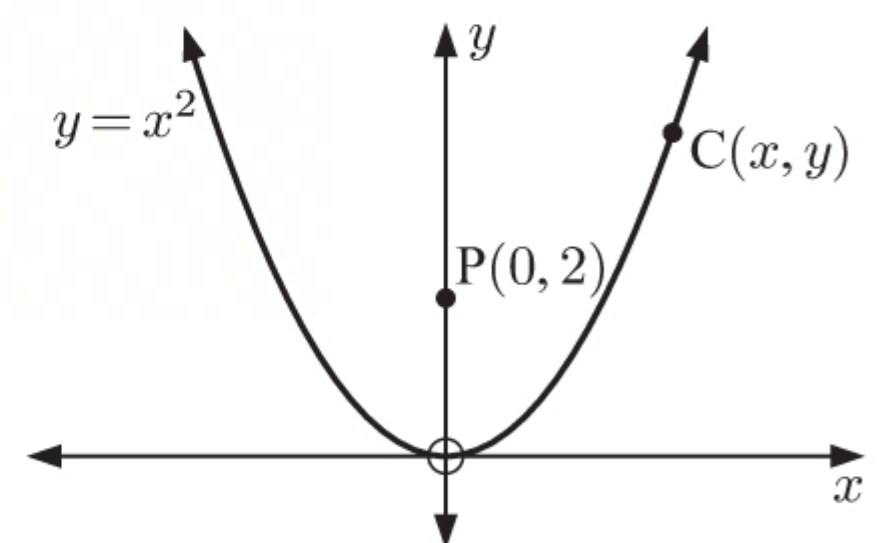


- 47** Terry wants to fence off a rectangular garden plot of area 48 m^2 . Three sides will be fenced with strong wire mesh costing \$18 per metre, and the remaining side will be fenced with corrugated iron costing \$30 per metre.

- By letting x be the length in metres of the side fenced with corrugated iron, show that the cost of fencing is $C = 48\left(\frac{36}{x} + x\right)$ dollars.
- Find the dimensions of the garden plot which will minimise the cost of fencing.

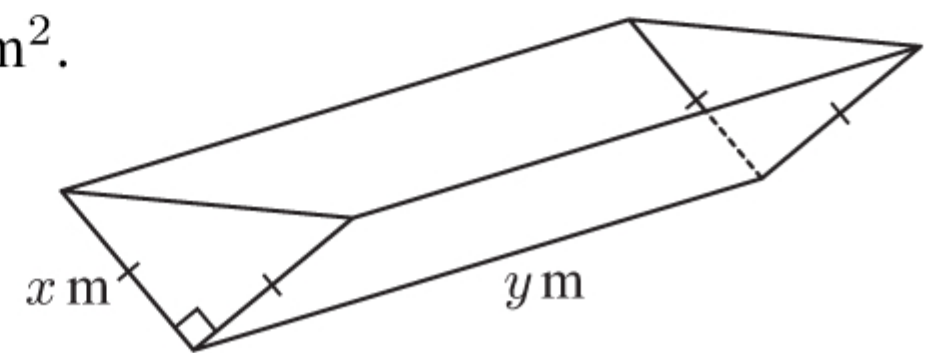
- 48** A comet travels in an orbit which can be described by the equation $y = x^2$ as shown in the diagram.

- Show that the distance of the comet at $C(x, y)$ from an observer at the point $P(0, 2)$ is given by $s(x) = \sqrt{x^4 - 3x^2 + 4}$.
- Find the shortest and the greatest distance between the comet and the observer for $-2 \leq x \leq 2$.



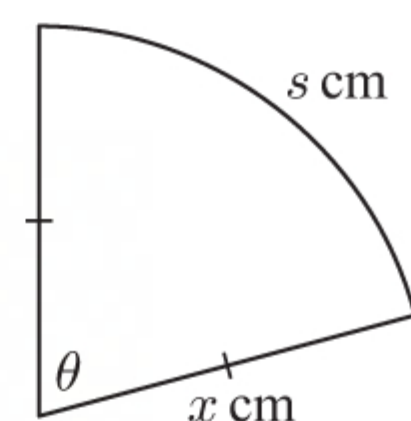
- 49** The diagram alongside shows an open trough. The total outside surface area is 27 m^2 .

- Show that $x^2 + 2xy = 27$.
- Find an expression for the volume V of the trough in terms of x only.
- Hence show that the volume of the trough is maximised if $x = y = 3$.



- 50** A 40 cm piece of wire is bent to form a sector of a circle with radius x cm.

- Write θ in terms of x .
- Show that the area of the sector is given by $A = 20x - x^2 \text{ cm}^2$.
- Find x and θ for which A is a maximum.



- 51** Integrate with respect to x :

- $3x^2 + 2x + 1$
- e^{4x}
- $\cos(2x + 1)$

52 a Given $f(x) = \sqrt{xe^x}$, find $f'(x)$.

b Hence find $\int \frac{2e^{\frac{x}{2}}(1+x)}{\sqrt{x}} dx$.

53 By considering $\frac{d}{dx}(x^2 \ln x)$, find $\int x \ln x dx$.

54 Integrate with respect to x :

a $x\sqrt{x} - 5 \cos x$

b $\sin x + \frac{1}{\sqrt[3]{x}}$

55 Find:

a $\int (x-3)^2 dx$

b $\int \frac{x^2 + 3x + 5}{\sqrt[3]{x}} dx$

56 Suppose $f'(x) = (x^2 + 2)^2$ and that $f(1) = \frac{8}{15}$. Find $f(x)$.

57 Suppose $f'(x) = \sqrt{4x+5}$ and that $f(0) = -\frac{\sqrt{5}}{6}$.

a For what values of x is $f'(x)$ defined?

b Find $f(x)$.

58 Find $f(x)$ given that:

a $f''(x) = e^x + 2x - 1$, $f'(0) = 4$, $f(0) = 1$

b $f''(x) = 2 + \sin x$, $f'(\pi) = 1$, $f(\frac{\pi}{2}) = \frac{\pi^2}{4}$

c $f''(x) = \frac{2}{\sqrt{x}} + 3x$, $f(1) = -\frac{19}{3}$, $f(4) = \frac{64}{3}$

59 Find:

a $\int (3x-5)^3 dx$

b $\int \frac{2}{\sqrt{4-x}} dx$

c $\int (e^{2x} + 3e^{-x+2}) dx$

60 Integrate with respect to x :

a $2 \sin(x-3) + e^{3x}$

b $\frac{2}{5x-1}$

c $\cos(5-7x)$

61 Find:

a $\int (2 \sin^2 x - 1) dx$

b $\int (\sin 2x - \cos 2x)^2 dx$

c $\int (\cos x + 2)^2 dx$

62 Find $f(x)$ given $f'(x) = \frac{4}{5-x}$ and $f(4) = 6$.

63 Find:

a $\int 3x^2(5+x^3)^4 dx$

b $\int xe^{x^2+2} dx$

c $\int \frac{e^{\frac{1}{x}}}{x^2} dx$

d $\int \frac{2(\ln x)^2}{x} dx$

64 Integrate with respect to x :

a $\sqrt{x^2+3x-1}(2x+3)$

b $\frac{e^x+2}{e^x+2x}$

c $\frac{6-8x}{2x^2-3x+2}$

65 Find:

a $\int \sin^4 x \cos x dx$

b $\int 3x^3 \sin(x^4) dx$

c $\int \frac{\sin 2x}{(3-\cos 2x)^3} dx$

66 a Show that $\cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$.

b Hence find $\int \cos^4 x dx$.

67 Find $\int_0^{\frac{\pi}{2}} (\sin 3x + 5 \cos x) dx$.

68 Find a given that $\int_a^{2a} \sqrt{x} dx = 2$.

69 If $y = x\sqrt{4-x}$, find $\frac{dy}{dx}$ and simplify your answer. Hence evaluate $\int_0^2 \frac{8-3x}{\sqrt{4-x}} dx$.

70 Find:

a $\int_1^5 \frac{2x^3+1}{x^2} dx$

b $\int_{-1}^1 e^x(2-3e^{-x})^2 dx$

c $\int_0^2 \frac{3}{5-2x} dx$

d $\int_{-2}^{\frac{\pi}{4}-2} \sin^2(x+2) dx$

71 a Find $\int (2x+3)(x^2+3x+4)^3 dx$ using an appropriate substitution.

b Hence find $\int_0^1 (2x+3)(x^2+3x+4)^3 dx$.

72 a Find $\int 8xe^{x^2+1} dx$ using an appropriate substitution.

b Hence find $\int_{-2}^2 8xe^{x^2+1} dx$.

73 Suppose $\int_0^2 f(x) dx = 2$. Determine the value of:

a $\int_2^0 f(x) dx$

b $\int_0^2 (3 + 2f(x)) dx$

c k such that $\int_0^2 (kf(x) + 2) dx = 10$.

74 Find a if $\int_0^a \frac{x}{x^2+1} dx = 3$ and $a > 0$.

75 Evaluate using technology:

a $\int_0^2 \sqrt{x}e^{x^2} dx$

b $\int_1^3 e^x \sin(x^2) dx$

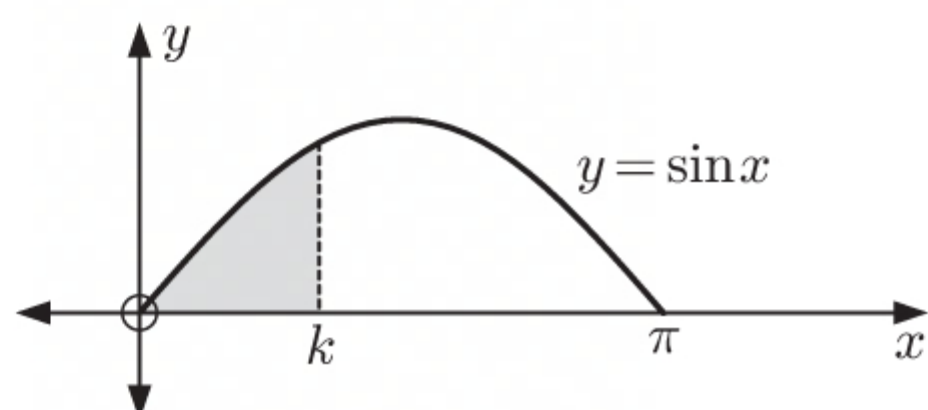
c $\int_{-1}^1 \sqrt{x^2 + \cos x} dx$

76 For a continuous function $f(x)$ defined on the interval $a \leq x \leq b$, the length of the curve can be found using $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$. Find the length of:

a $f(x) = x^2$ on the interval $0 \leq x \leq 1$

b $f(x) = \sin x$ on the interval $0 \leq x \leq \pi$.

77 The shaded region has area 0.42 units². Find k , correct to 2 decimal places.



78 a Sketch the curve $y = 3x^2 + 1$ for $0 \leq x \leq 3$.

b Find the area between the curve and the x -axis for $0 \leq x \leq 3$.

c Find k such that the area between the curve and the x -axis for $0 \leq x \leq k$ is 10 units².

79 Find the area of the region bounded by:

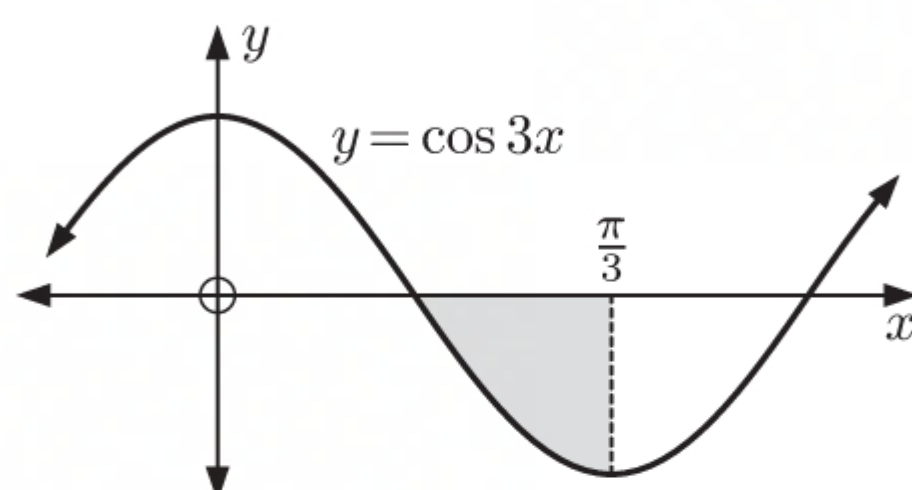
a $y = x^2 + x$, the x -axis, $x = 2$, and $x = 4$

b $y = \sin 2x$, the x -axis, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$

c $y = \frac{1}{\sqrt{x+2}}$, the x -axis, $x = 2$, and $x = 7$

d $y = e^{-3x}$, the x -axis, $x = 0$, and $x = 1$.

80 Find the shaded area:



81 The area of the region bounded by $f(x) = -\frac{10}{x+5}$, the x -axis, $x = 0$, and $x = k$, is $10 \ln 3$ units².

Find the possible values of k .

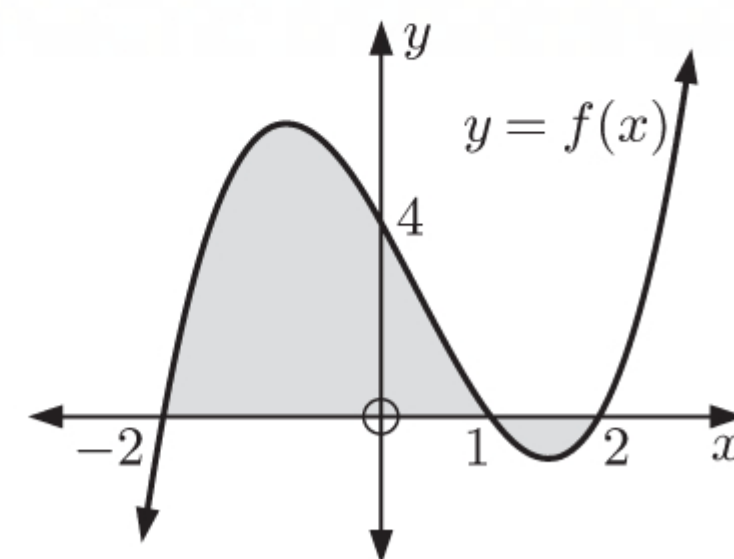
82 A parabola passes through the points $(-\pi, 0)$, $(\pi, 0)$, and $(0, \alpha)$. The area between the parabola and the x -axis is 4 units². Calculate the possible values of α .

83 Consider the graph of $f(x) = x^3 - x^2 - 4x + 4$.

a Find $\int_{-2}^2 f(x) dx$.

b Explain why the value obtained in **a** does not represent the shaded area.

c Find the shaded area.



84 Find the area of the region enclosed by:

a $y = x^2 - 3x$ and $y = x$

b $y = 4 - x^2$ and $y = -2x - 4$

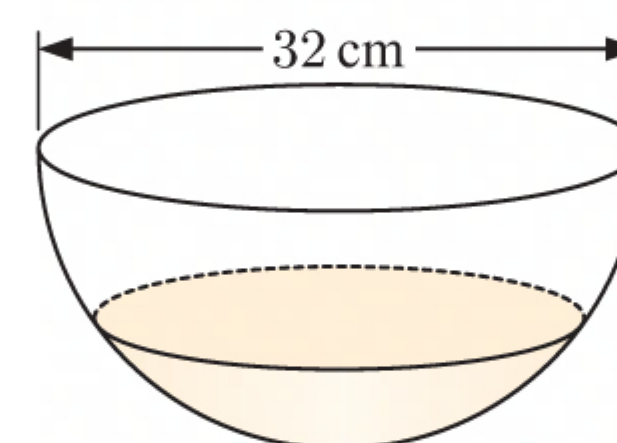
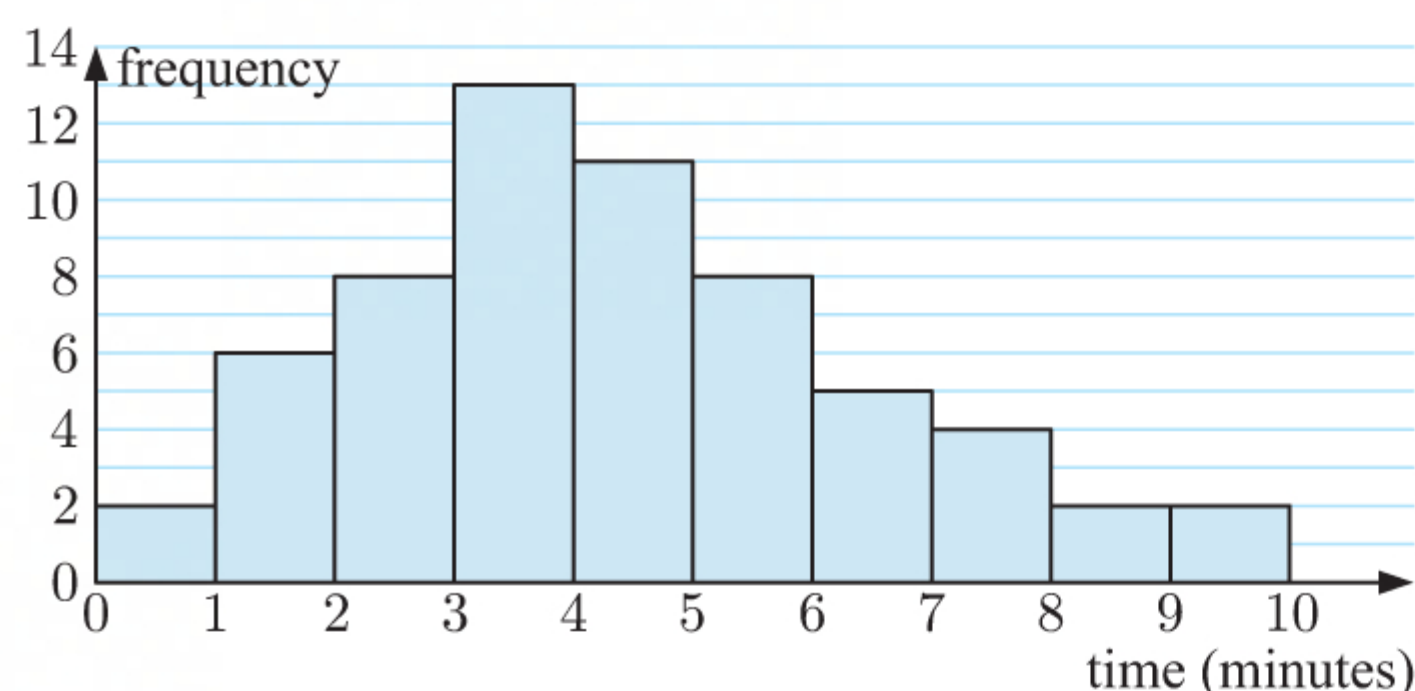
c $y = x^2 + 2x - 3$ and $y = x - 1$

- 85 a** Sketch the graph of $f(x) = \sin\left(x + \frac{\pi}{6}\right)$ for $-2\pi \leq x \leq 2\pi$.
- b** Find the area that lies above the line $y = \frac{1}{2}$ and below the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$.
- 86** Consider the function $f(x) = \left(2 - \frac{1}{x}\right)e^{-x}$, $x > 0$.
- a** Find the zero of $f(x)$. **b** Discuss the behaviour of $f(x)$ near $x = 0$ and as $x \rightarrow \infty$.
- c** Find the position and nature of the stationary point.
- d** Sketch the graph of $y = f(x)$, showing all the above information.
- e** Find, correct to 3 decimal places, the area enclosed by $y = f(x)$ and the line $y = x - 1$.
- 87 a** Write down the derivative with respect to x of: **i** $f(x) = \ln x$ **ii** $F(x) = x \ln x - x$.
- b** What is the relationship between $f(x)$ and $F(x)$? **c** Sketch the graph of $y = f(x)$.
- d** Let O be the origin, and let P be the point on $y = f(x)$ with x -coordinate t , $1 < t \leq e$.
- i** Show that the area bounded by [OP], $y = f(x)$, and the x -axis, is $\left(t - \frac{1}{2}t \ln t - 1\right)$ units².
- ii** For what value of t will this area be maximised?
- e i** Write down the equation of the tangent to $y = f(x)$ at the point $P(t, \ln t)$.
- ii** Find t such that this tangent passes through the origin.
- 88** Find the area enclosed by:
- a** $y = x^3 - 2x^2 - 3x$ and $y = 5x - 4x^2$ **b** $y = 2x^3 - 5x + 4$ and $y = x^3 + 2x^2 - 2$
- 89** The rate at which a tree grows t years after planting is given by $G(t) = \frac{2.5}{t+1}$ metres per year.
- a** Explain why the tree is always growing taller.
- b** Evaluate the following integrals, and interpret their meaning: **i** $\int_0^5 G(t) dt$ **ii** $\int_5^{10} G(t) dt$
- c** After 15 years, the tree is struck by lightning and is cut down. How much did the tree grow over its lifetime?
- 90** A particle is moving in a straight line with velocity given by $v(t) = t^3 - 3t^2e^{0.05t}$, where $t \geq 0$ is in seconds, and distance units are in metres.
- Use technology to find:
- a** the greatest speed reached by the particle in the first 4 seconds of motion
- b** the total distance travelled by the particle in the first 4 seconds of motion.
- 91** For the first 6 seconds of its motion, a particle moving in a straight line has velocity given by $v = t^3 - 9t^2 + 24t$ m s⁻¹, where t is the time in seconds.
- a** Find the acceleration function for the particle.
- b** Find the greatest velocity of the particle in the first 6 seconds.
- c** At what times in the first 6 seconds is the speed of the particle decreasing?
- 92** A particle moves on a straight line with acceleration given by $2 - 3t$ m s⁻². Initially the particle has displacement 3 m. When $t = 1$ s, the particle is momentarily at rest.
- a** Find the velocity function of the particle. **b** At what other time is the particle momentarily at rest?
- c** Find the displacement function of the particle.
- 93** A particle moves in a straight line with displacement function $s(t) = 12t - 3t^3 + 1$ cm, where $t \geq 0$ is in seconds.
- a** Find the velocity and acceleration functions for the particle's movement.
- b** Find the speed of the particle after: **i** 1 second **ii** 2 seconds.
- c** When is the particle's: **i** velocity decreasing **ii** speed decreasing?
- 94** A particle moves from rest along a straight line. Its velocity is given by $v = 2\sqrt{t} - t$ m s⁻¹, where $t \geq 0$ is the time in seconds.
- a** Find the speed of the particle after 5 seconds. **b** Find the acceleration function of the particle.
- c** Show that the particle changes direction after 4 seconds. **d** Find the total distance travelled in the first 9 seconds.

Mixed questions

MIXED QUESTIONS SET 1

- 1 Consider the quadratic $y = 2x^2 - 9x + 3$.
 - a Find the equation of the axis of symmetry.
 - b Find the coordinates of the vertex.
 - c Find the axes intercepts.
 - d Sketch the function.
- 2 The temperature inside Pam's caravan t hours after 6 am is given by the function $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right) ^\circ\text{C}$.
 - a Sketch the graph of T against t for $0 \leq t \leq 24$.
 - b Find the temperature inside Pam's caravan at: i 2 pm ii 9 pm.
 - c Find the maximum temperature inside Pam's caravan, and the time at which it occurs.
- 3 Let $f(x) = \ln(x\sqrt{1-2x})$.
 - a State the domain of the function.
 - b Show that $f'(x) = \frac{1-3x}{x(1-2x)}$.
 - c At what point(s) on the graph of $y = f(x)$ does the normal have gradient $-\frac{6}{5}$?
- 4 Consider the functions $f(x) = 5^x$ and $g(x) = 2x + 1$.
 - a Find $(f \circ g)(x)$.
 - b Find $(f \circ g)^{-1}(0.2)$.
 - c When $f(x)$ is horizontally stretched with scale factor k , the resulting graph passes through $(\frac{1}{6}, \sqrt{5})$. Find k .
- 5 The value of a car decreases by 10% each year. After 3 years its value is \$26 244.
 - a Find the original value u_0 of the car.
 - b Write a geometric sequence to describe the value of the car u_n after n years.
 - c In what year will the value of the car fall below \$10 000?
- 6 Before selecting a new mobile phone plan, George reviews the duration of calls he made over the last 3 months. George produced the histogram alongside to illustrate the data he collected.
 - a Write down the modal class.
 - b Organise the data into a frequency table.
 - c Estimate the mean length of a phone call.
 - d Estimate the probability that George's next call will last 6 minutes or longer.
- 7
 - a Suppose $h(x) = \sin x - \cos x$. Write the antiderivative $H(x)$ of $h(x)$.
 - b Sketch the graphs of $f(x) = \sin x + x$ and $g(x) = \cos x + x$ on the same set of axes for $x \in [0, 2\pi]$.
 - c Hence find the area enclosed between the curves for $0 \leq x \leq 2\pi$.
- 8
 - a Show that $\log_4(x^2 - x + 3) = \log_2 \sqrt{x^2 - x + 3}$.
 - b Hence solve $\log_2(x + 2) = \log_4(x^2 - x + 3)$
- 9 A hemispherical mixing bowl has dimensions shown.
 - a Find the capacity of the bowl.
 - b Suppose the bowl is 20% full with cake batter.
 - i How many litres of cake batter does it contain?
 - ii The cake batter is poured into a cylindrical cake tin with diameter 25 cm. How high will it reach up the tin?
- 10 Charlie and Charlotte are on a road trip in Australia. They travel 36 km north-west from Wollongong to Picton, then 210 km south-west from Picton to Canberra.
 - a How far is Canberra from Wollongong?
 - b Find the bearing of Wollongong from Canberra.



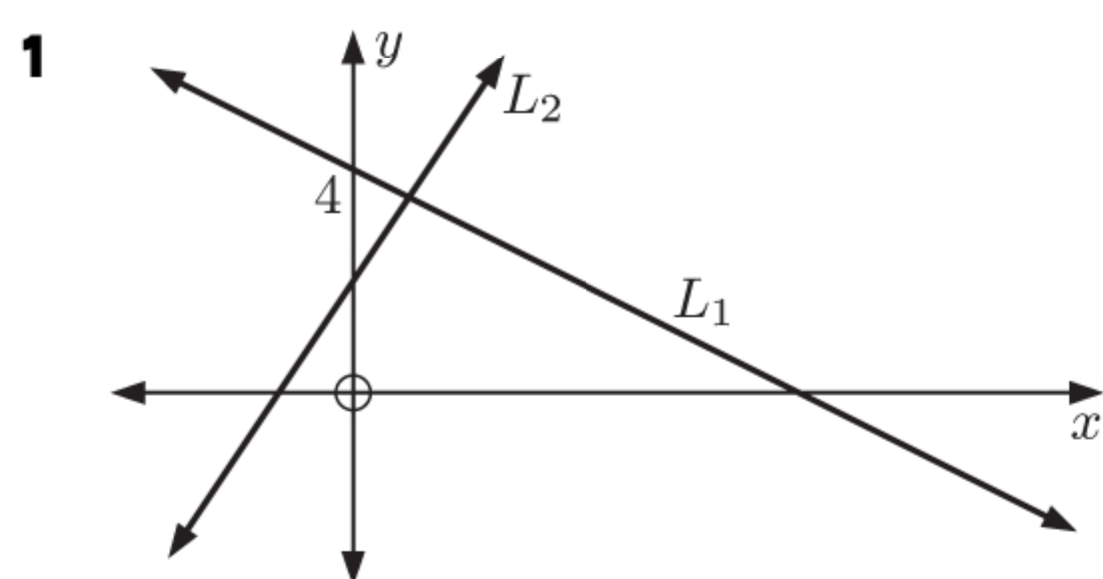
- 11** The velocity of a boat travelling in a straight line after t seconds is given by $v(t) = 30 - 20e^{-0.2t}$ m s⁻¹.
- Find the boat's:
 - initial velocity
 - velocity after 2 seconds.
 - How long does it take for the boat's velocity to reach 20 m s⁻¹? Give your answer correct to two decimal places.
 - What happens to $v(t)$ as $t \rightarrow \infty$?
 - Calculate $v'(t)$ and show that the acceleration is always positive.
 - Graph $v(t)$ against t , showing the information from **a** to **c**.
 - How far did the boat travel before its velocity reached 20 m s⁻¹?

- 12** A random variable X has the following distribution table:

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

- Is the random variable X discrete or continuous?
- Find k .
- Find the mode and median of X .
- Find $E(X)$.

MIXED QUESTIONS SET 2



- L_1 has gradient $-\frac{1}{2}$ and passes through $(0, 4)$.
Find the equation of L_1 , giving your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.
- L_2 passes through $(-2, -1)$ and $(4, 8)$.
Find the point of intersection of L_1 and L_2 .

- 2** An infinite geometric series has terms $u_1 = 27$ and $u_4 = 8$.

- Find the common ratio r .
- Find the 6th term of the series.
- Using summation notation, write an expression for the sum S of the infinite series.
- Evaluate S .

- 3** At an athletics competition, Carl ran the 100 m and 200 m events. His times are summarised in the table, along with the event means and standard deviations.

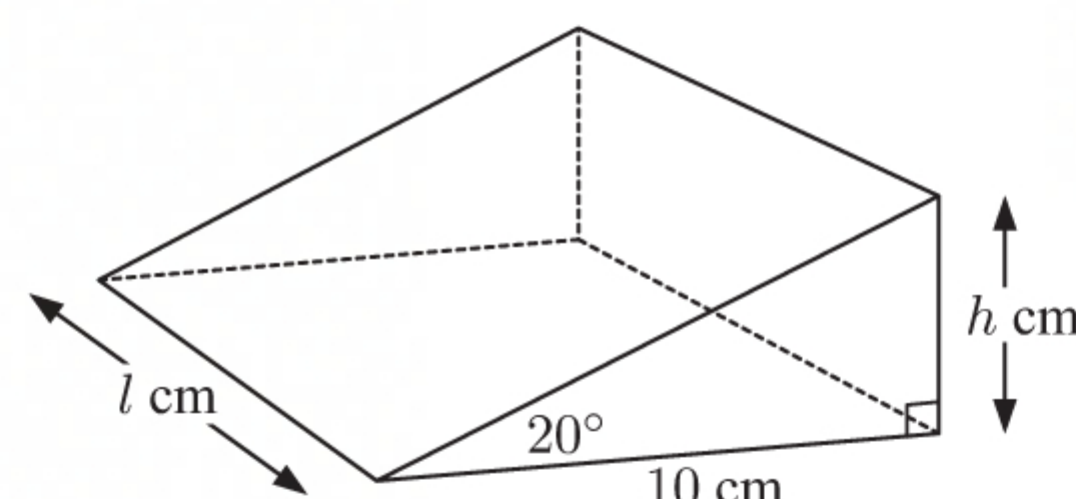
Event	Time (seconds)	μ (seconds)	σ (seconds)
100 m	9.99	10.20	0.113
200 m	17.30	18.50	0.706

- Assuming the times for each event are normally distributed, calculate Carl's z -scores for each event.
 - Based on the results of **a**, in which event did he perform better?
- 4** Consider the function $f(x) = ax^3 - bx^2$. The line $y = x - 6$ is a tangent to $y = f(x)$ at $x = 3$.
- Find the constants a and b .
 - Find the point where the tangent meets $y = f(x)$ again.
 - Graph $y = f(x)$ and $y = x - 6$ on the same set of axes.

- 5**
- Show that $\frac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos \theta$ for all θ such that $\cos \theta \neq -1$.
 - Hence solve $\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1}{2}$ for $-\pi < \theta < \pi$.

- 6** A manufacturer produces wooden door-stops with the shape of the triangular prism shown.

- Calculate the height h correct to 4 significant figures.
- Determine the area of the triangular end of the prism.
- The volume of the door-stop is 60 cm³. Determine its length l .



- Calculate the total surface area of each door-stop. Give your answer correct to 3 significant figures.

- 7 The management of a large shopping centre chain sent a survey team to one of its suburban shopping centres. Between 10 am and 3 pm on a very busy Thursday, 100 people in the main mall were asked the following multiple choice question:

“At which type of shopping centre do you prefer to shop?”

A suburban **B** central city **C** equally preferred **D** neither **E** no opinion

- a** Give *two* reasons why this survey is likely to contain a coverage error.
b The results were: suburban 33%, central city 8%, equally preferred 51%, neither 4%, no opinion 4%

Management concluded that “*more than four times as many people prefer suburban shopping to the central city*”. Explain why this conclusion is unreasonable.

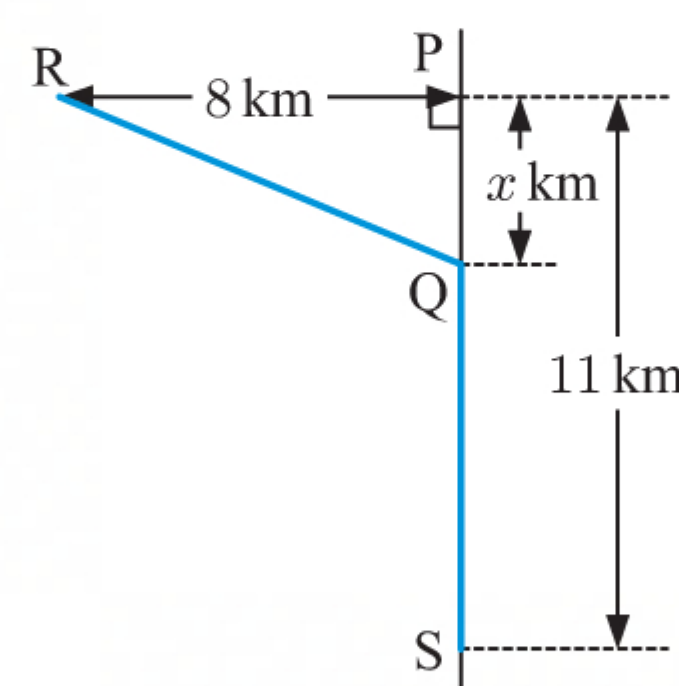
- 8 The current in an electrical circuit t milliseconds after it is switched off is given by $I(t) = 40e^{-0.1t}$ amps.

- a** What current was flowing in the circuit initially?
b What current was flowing in the circuit after 100 milliseconds?
c Sketch $I(t)$ and $I = 1$ on the same set of axes.
d How long did it take for the current to fall to 1 amp?

- 9 An offshore oil rig is at point R, 8 km from a straight shore. The point P is on the shore directly opposite the rig. A refinery is on the shore at S which is 11 km from P.

A pipeline is to be constructed under the sea from R to reach the shore at the point Q. From Q a pipeline is to be taken overland to S. The cost of the pipeline is \$5 million per km under the sea and \$3 million per km overland.

- a** If Q is x km from P, show that the cost to construct the pipeline from R to S is $C(x) = 5\sqrt{x^2 + 64} + 33 - 3x$ million dollars.
b Find the minimum cost of the pipeline.



- 10 Suppose $a > b > c > 0$.

- a** Show that: **i** $a^2 - b^2 > 0$ **ii** $b^2 - c^2 > 0$
b Hence show that $(ab)^2 + (bc)^2 - (ac)^2 > b^4$.

- 11 A tinned food company examined a sample of its tins of corn and tins of pineapple for defects. The results are summarised in the table alongside.

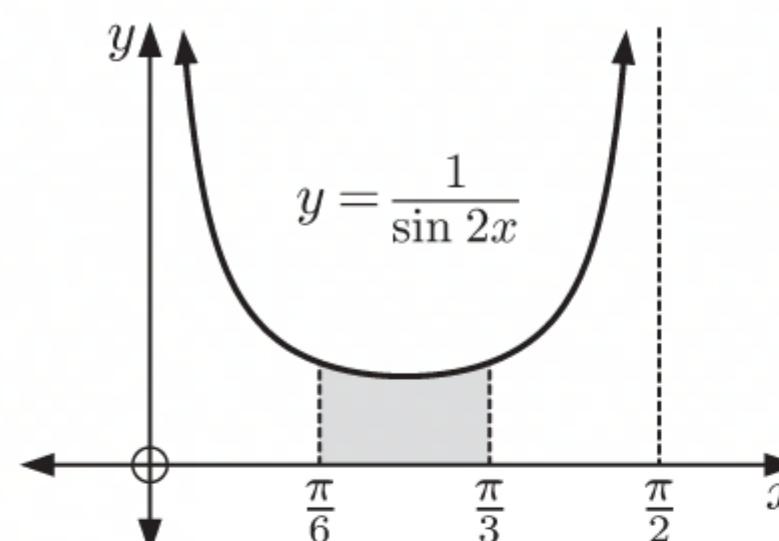
	Defective	Not defective
Corn	37	581
Pineapple	24	617

- a** How many tins were included in the sample?
b Estimate the probability that the next randomly selected tin:
i is not defective **ii** is a defective tin of pineapple
iii is defective, given it is a tin of corn.

- 12 **a** If $y = \ln(\tan x)$, $0 < x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \frac{k}{\sin 2x}$ for some constant k .

- b** Alongside is a graph of $y = \frac{1}{\sin 2x}$.

Show that the shaded area is $\frac{1}{2} \ln 3$ units².



MIXED QUESTIONS SET 3

- 1 **a** Determine the discriminant of $x^2 + 8x + k = 0$.

- b** Hence find the values of k for which the equation has:

i no real roots **ii** two distinct real roots.

- 2 Let $f(x) = \frac{x-3}{2-x}$.

- a** State the domain and range of f .

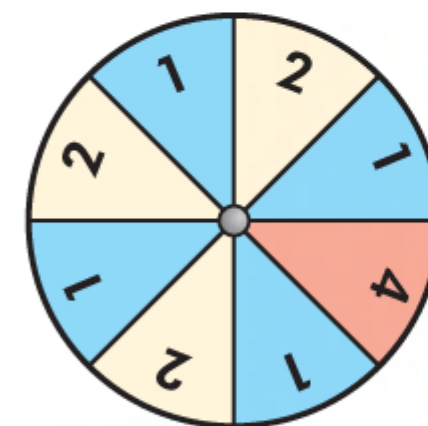
- b** Write down the equations of the asymptotes of $y = f(x)$.

- c** Find the axes intercepts of $y = f(x)$.

- d** Sketch $y = f(x)$, showing the features you have found.

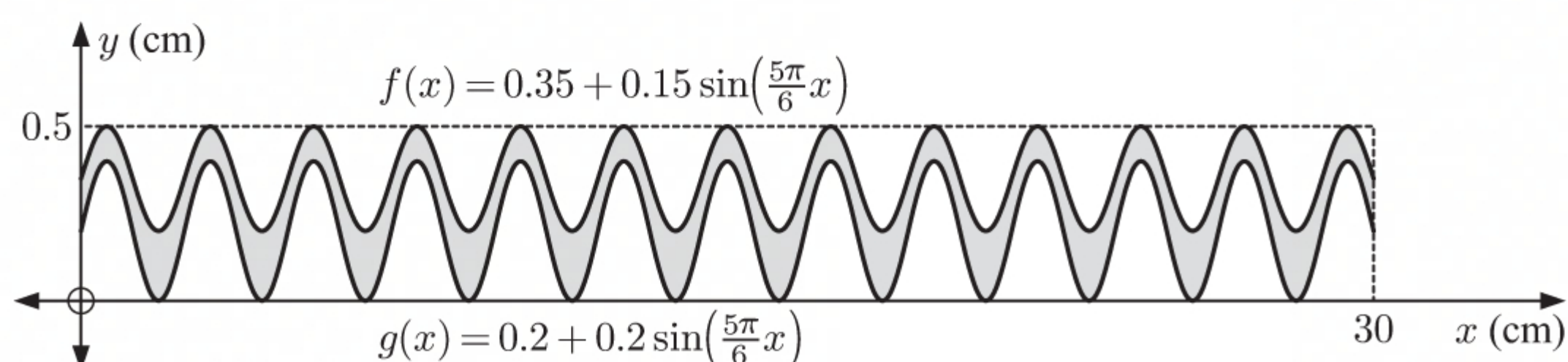
3 Solve for x exactly: $\sqrt{2} \sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 1$, $-\pi \leq x \leq 2\pi$.

- 4 A game is played in which the wheel shown is first spun by the player, and then by the game operator. The player wins \$ a if their spin is higher than the operator's. It costs \$ k to play the game.



Find the relationship between a and k so that the game is fair.

- 5 The cross-section of a 1 m long strip of cardboard is shown below.

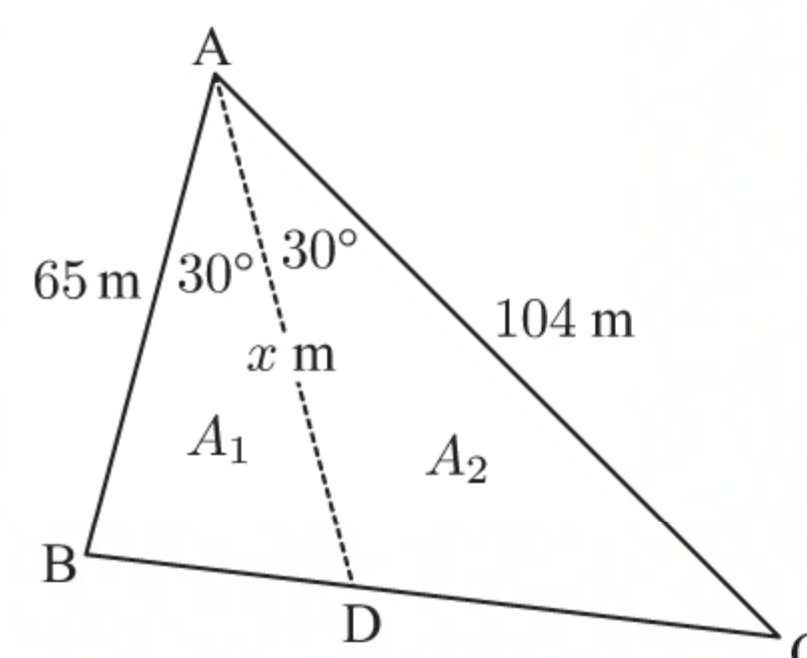


Find the volume of the cardboard.

- 6 The population of a hive of bees is given by $P(t) = 120 \times (2.25)^{\frac{t}{3}}$, where t represents time in weeks.
- Sketch $P(t)$ for $0 \leq t \leq 20$.
 - Find the population of bees in the hive after 10 weeks.
 - Write a function for t in terms of P .
 - How long will it take for the population to reach 5000?
- 7 The table shows the amount of petrol remaining in a motorbike's fuel tank and the number of kilometres travelled. The capacity of the tank is 10 litres.

Remaining fuel (x litres)	10	8	6	4	2	1
Distance (y km)	0	90	190	260	330	370

- Plot this data on a scatter diagram.
 - Find the equation of the regression line for y against x .
 - Interpret the y -intercept of the regression line.
 - The motorbike has travelled 220 km since its tank was refilled.
 - Use your equation to estimate the amount of fuel left in the tank.
 - Find the average distance travelled per litre over the 220 km.
- 8 A particle is initially at rest. It moves in a straight line with acceleration $a(t) = 2 - 6t \text{ m s}^{-2}$, where t is the time in seconds, $t \geq 0$.
- Find the velocity function.
 - Find the change in *displacement* of the particle in the first second.
 - Find the *total distance* travelled by the particle in the first second.
- 9 The probability of Mark waking up early is 0.8. If he wakes up early, he will pack lunch with probability 0.6. If he does not wake up early, he will pack lunch with probability 0.15.
- Display the sample space of possible outcomes on a tree diagram.
 - Hence determine the probability that Mark will pack lunch today.
- 10 A farmer owns a triangular field ABC.
- D is the point on [BC] such that [AD] bisects \widehat{BAC} . The farmer divides the field into two parts A_1 and A_2 by constructing a straight fence [AD] of length x m.
- Use the cosine rule to calculate the length of [BC].
 - Find the total area of the field.
 - Find, in terms of x , the area of:
 - A_1
 - A_2 .
 - Hence find x .



- 11 Suppose that p , q , and r are consecutive odd integers, $p < q < r$. Show that $2q(p + r)$ is a perfect square.

- 12** Consider the curve $y = xe^{2x}$.
- Find the exact value of $k \in \mathbb{R}$ such that $y = k$ is a horizontal tangent to the curve.
 - For which values of $k \in \mathbb{R}$ does the line $y = k$ meet the curve at:
 - exactly one point
 - two distinct points
 - no points?
 - Now consider the family of curves $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$.
 - Show that $y = x$ is a tangent to all such curves and find the point of contact.
 - Find the equation of the normal to $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$, when $x = 0$, and find the acute angle this normal makes with the x -axis.

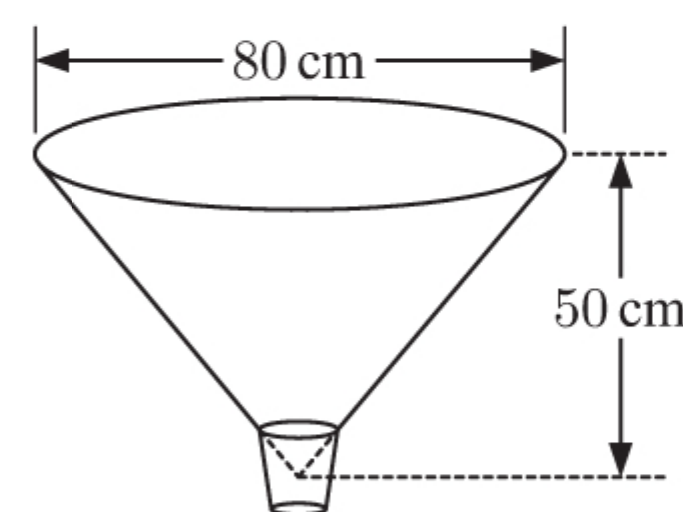
MIXED QUESTIONS SET 4

- Suppose $f(x) = 4x - 3$ and $g(x) = x + 2$. Find the value of x such that $(f \circ g^{-1})(x) = f^{-1}(x)$.
- Suppose $f'(x) = a\sqrt{x} + bx$ where a and b are constants. Find $f(x)$ given that $f(0) = -4$, $f(1) = -1$, $f(2) = 4\sqrt{2}$.
- The data below are the recent sale prices, in thousands of dollars, of houses in two neighbourhoods.

<i>Neighbourhood A:</i>	275	281	320	265	305	258	310	430	285
	290	297	345	195	230	269	300	258	273
<i>Neighbourhood B:</i>	325	300	412	370	297	505	340	333	290
	428	305	520	360	410	275	320	431	410

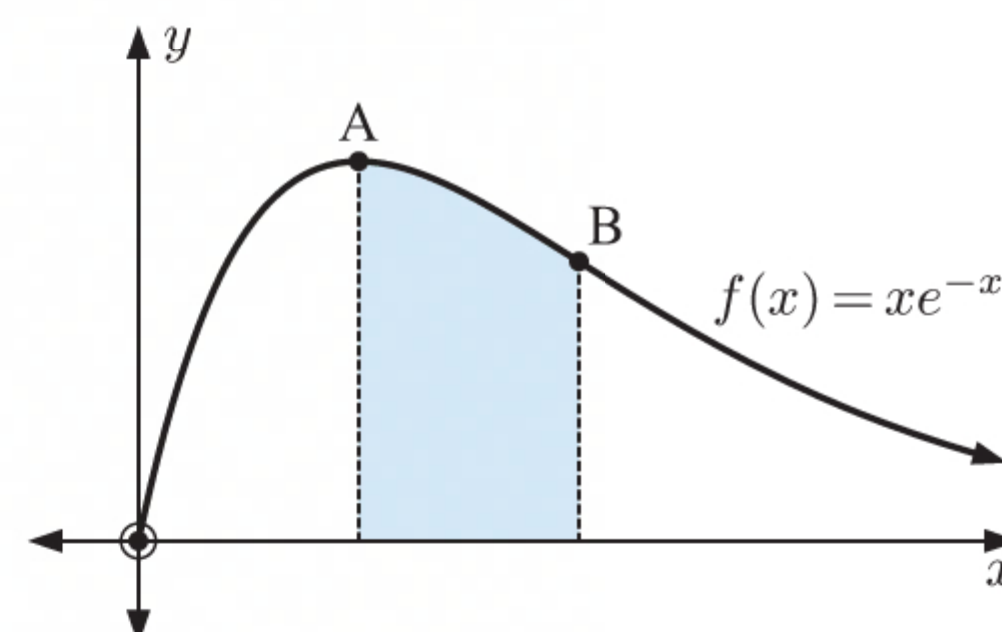
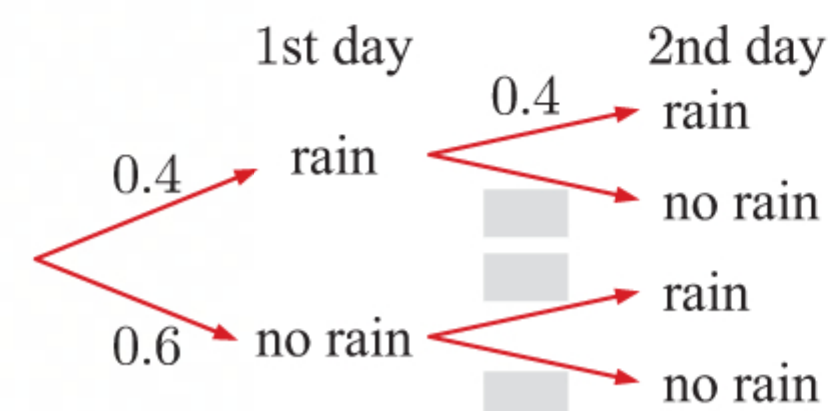
- Is the data discrete or continuous?
 - Use technology to find the five-number summary for each data set.
 - Display the data in a parallel box plot.
 - Compare and comment on the distributions of each data set.
- 4** In triangle PQR, $\widehat{PRQ} = 40^\circ$, $PR = 12$ cm, and $PQ = 8$ cm.
- Show that there are two possible measures of \widehat{PQR} .
 - Sketch triangle PQR for each case.
 - In each case, find:
 - the measure of \widehat{QPR}
 - the perimeter of the triangle.
- 5** Consider the graphs with equations $y = \frac{2}{x}$ and $y = x - 1$.
- Solve $\frac{2}{x} = x - 1$ using algebra.
 - Use technology to plot the graphs on the same set of axes.
 - Hence solve for x : $\frac{2}{x} < x - 1$
- 6** The coefficients of the first three terms of $(x + a)^3$ form an arithmetic sequence. Find the constant a .
- 7** The heights X of maize plants two months after planting are normally distributed with mean μ cm and standard deviation 6.8 cm. 75% of a crop of maize plants are less than 45 cm high.
- Find:
 - μ
 - $P(X < 25)$
 - a such that $P(X < 25) = P(X > a)$.
 - Six maize plants are randomly chosen. Find the probability that exactly four of them are more than 35 cm high.
- 8** A conical funnel is 80 cm wide and 50 cm high.

- Estimate the capacity of the funnel in mL. Write your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.
- The funnel is half full with liquid, and its contents are poured into a cylindrical tube 20 cm wide. How high up the tube will the liquid reach?



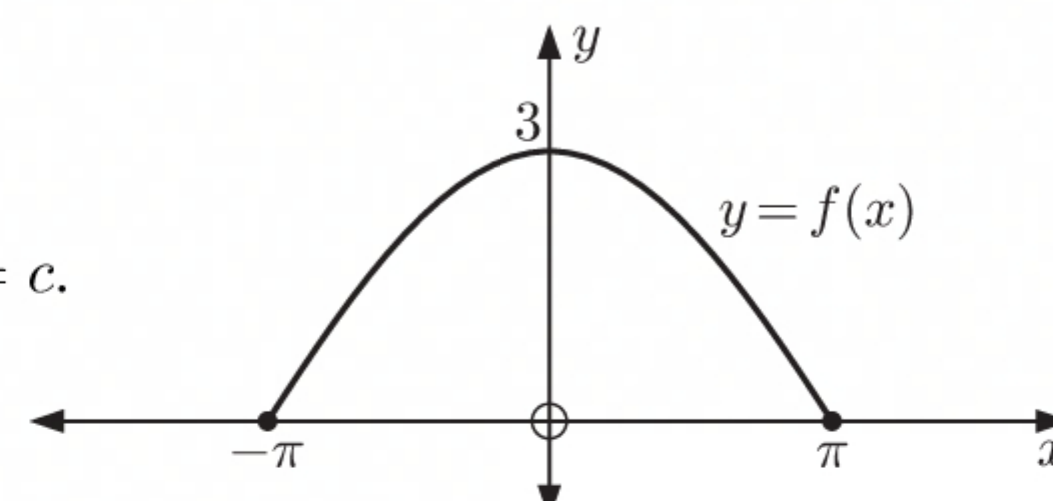
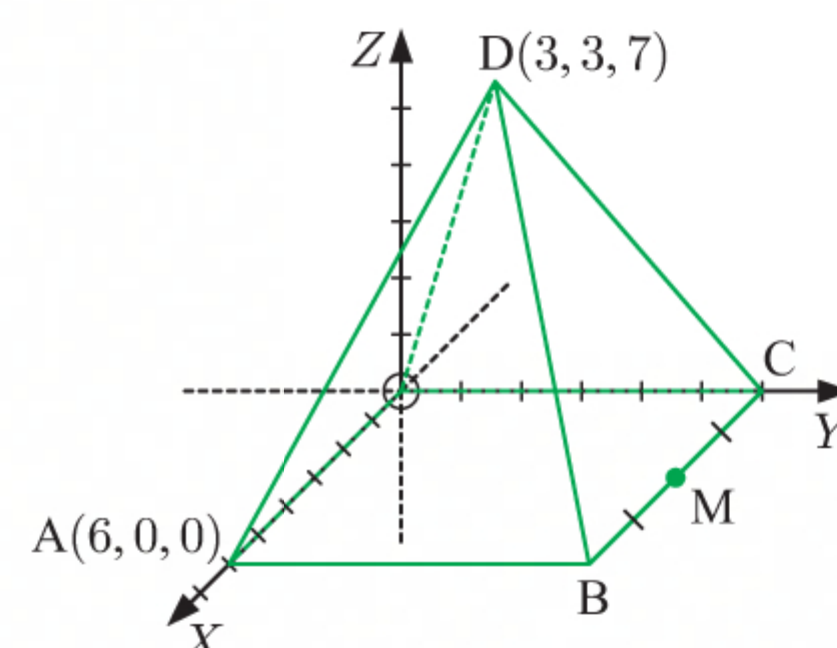
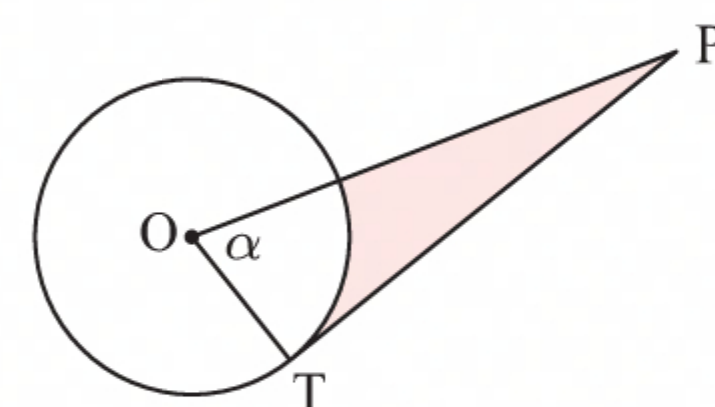
- 9** At time t seconds, the tip of a pendulum has acceleration $6 \cos 2t \text{ cm s}^{-2}$. At $t = 0$, the pendulum is stationary.
- Find the speed of the tip of the pendulum after 4 seconds.
 - Find the total distance travelled by the tip of the pendulum in the first 5 seconds to two decimal places.

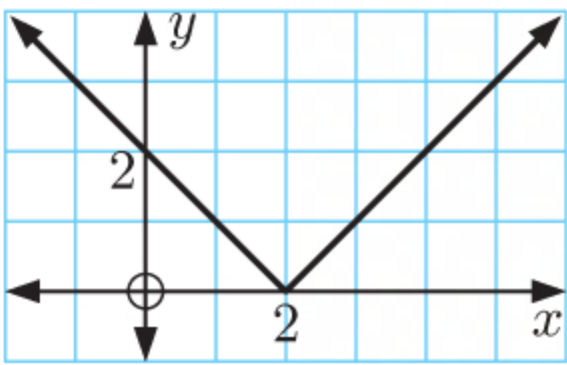
- 10** In a busy harbour, the time difference between successive high tides is 12.3 hours. The water level varies by 2.4 metres between high and low tide. The first high tide of the day is 4.7 metres, occurring at 1 am.
- Find a cosine model for the height H of the tide t hours after midnight.
 - Sketch a graph of the water level in the harbour for $0 \leq t \leq 24$.
- 11** The probability of rain falling on any day in Dunedin is 0.4. The tree diagram shows the possible outcomes when two consecutive days are considered.
- Complete the tree diagram by filling in the missing probabilities.
 - Hence determine the probability of:
 - rain on both days
 - no rain on exactly one day.
 - Given that rain fell on at least one day, find the probability of rain on the second day.
- 12** The graph of $f(x) = xe^{-x}$, $x \geq 0$ is shown.
- Find the y -intercept.
 - Find $f'(x)$ and hence find the coordinates of A.
 - Find the exact x -coordinate of the point of inflection B.
 - Find the area of the shaded region.



MIXED QUESTIONS SET 5

- Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x -axis at -4 , passes through $(1, 5)$, and has axis of symmetry $x = -1$.
- [PT] is a tangent to the given circle. The circle has radius 9 cm and $OP = 30$ cm. Find:
 - α
 - the area of the shaded region.
- Twins Pierre and Francesca were each given \$100 on their 10th birthday. They immediately put their money into their individual money boxes. Each week throughout the next year they added a portion of their weekly pocket money. Pierre added \$10 each week. Francesca added 50 cents the first week, \$1 the next, \$1.50 the next, and so on, adding an extra 50 cents each subsequent week.
 - How much did Francesca add to her money box in the last week before her 11th birthday?
 - Find the total amount that each child had added to his or her money box after 8 weeks.
 - Who had more money in their money box after one year? Explain your answer.
- Solve for $-\pi \leq x \leq \pi$: $2 \sin^2 x = 3 \cos x + 2$
- Consider the square-based pyramid alongside. Find:
 - the coordinates of B and C
 - the volume of the pyramid
 - the coordinates of M
 - the surface area of the pyramid.
- Solve for x : $4^x + 4 = 17(2^{x-1})$
- The function f has the form $f(x) = a \cos bx$, $-\pi \leq x \leq \pi$.
 - State the values of a and b .
 - Find the equation of the normal to $y = f(x)$ at the point where $x = c$.
 - Find the values of c such that the normal passes through the origin.
 - Sketch $y = f(x)$ and the normals found in **b ii** on the same set of axes.





- 8 The graph alongside shows a relation between x and y .
- a Is the relation a function? Explain your answer.
 - b State the domain and range of the relation.
 - c Sketch the result when the graph is translated through $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, then reflected in the x -axis.
- 9 Yiren is filling her swimming pool with water. She suddenly realises it is overflowing and turns the tap off. The water continues to overflow at the rate $R(t) = \frac{12}{\sqrt{t+1}}e^{-\sqrt{t+1}} \text{ L s}^{-1}$ where t is in seconds, $t \geq 0$.
- a At what rate is the water still overflowing after 10 seconds?
 - b Find $\int R(t) dt$ using an appropriate substitution.
 - c Hence find $\int_0^{60} R(t) dt$.
 - d Interpret the meaning of the integral found in c.
- 10 A university club committee holds weekly meetings. Each committee member has a 70% chance of attending a given meeting. A meeting can only go ahead if at least 10 committee members are present.
- a If the club has 15 committee members, what percentage of meetings will go ahead?
 - b Find the smallest number of committee members required to ensure that at least 90% of the meetings will go ahead.
- 11 The displacement of an object after t seconds is $s = \sin\left(\frac{\pi}{(t+1)^2}\right) \text{ cm}$, $t \geq 0$.
- a Find the displacement of the object after 1 second.
 - b Find the first time that the object has displacement 0.5 cm.
 - c Show that the velocity of the object is $v = -\frac{2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right) \text{ cm s}^{-1}$.
 - d Find the first time that the object is stationary.
- 12 The distance travelled by two similar toy cars after rolling down a slope was measured 40 times each. The measurements were rounded to the nearest tenth of a metre.

Red car	3.6	4.6	5.6	6.4	4.2	5.3	6.1	4.5
	5.4	4.6	3.9	6.2	5.8	4.5	5.4	6.1
	4.5	5.6	5.7	4.8	3.9	5.6	6.1	5.9
	4.1	5.3	4.2	6.2	7.4	5.4	5.8	4.5
	3.9	5.4	5.7	4.8	5.4	5.7	6.1	6.4

Blue car	Number of rolls	40
	Median distance	4.8 m
	Shortest distance	3.2 m
	Longest distance	6.7 m
	Q_1 Lower quartile	4.1 m
	Q_3 Upper quartile	5.4 m

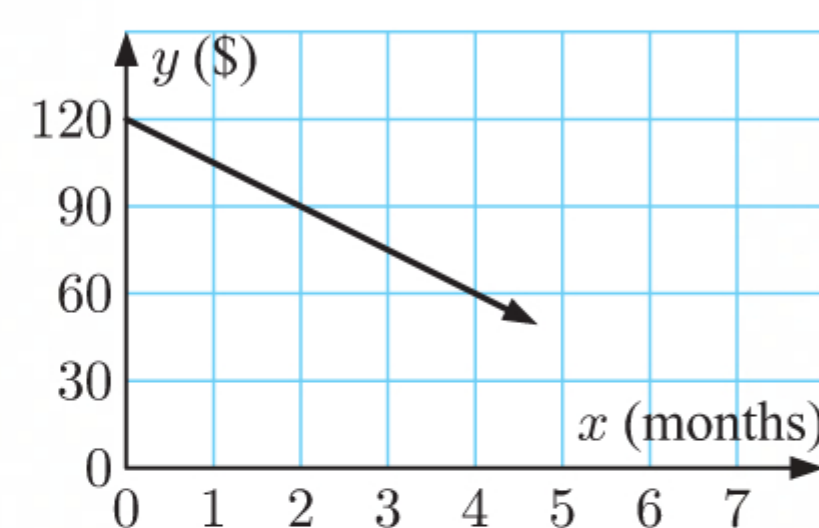
- a Complete this table of cumulative frequencies for the red car data.
- b Draw the cumulative frequency graph for the distance travelled by the red car.
- c Use the graph to find the following statistics for the red car:
 - i median distance
 - ii lower quartile
 - iii upper quartile
- d Draw a parallel box and whisker diagram to display the data for both cars.
- e Compare the statistics for distance travelled by the two toy cars. Is it reasonable to assume that the same machine manufactured these two toys? Explain your answer.

Distance (m)	Cumulative frequency
$3.5 \leq d < 4$	
$4 \leq d < 4.5$	
$4.5 \leq d < 5$	
$5 \leq d < 5.5$	
$5.5 \leq d < 6$	
$6 \leq d < 6.5$	
$6.5 \leq d < 7$	
$7 \leq d < 7.5$	

MIXED QUESTIONS SET 6

- 1 Let $f(x) = 3 - 4^{-x}$.
- a Points $A(2, p)$ and $B(-2, q)$ lie on $y = f(x)$. Determine p and q .
 - b For the graph of $y = f(x)$, determine the:
 - i y -intercept
 - ii equation of the horizontal asymptote.
 - c Sketch the graph of $y = f(x)$, showing all details from above.
 - d Write down the range of $f(x)$.

- 2** Michael purchases a music subscription at the start of the year. The graph shows the amount of money left in the subscription account after x months.

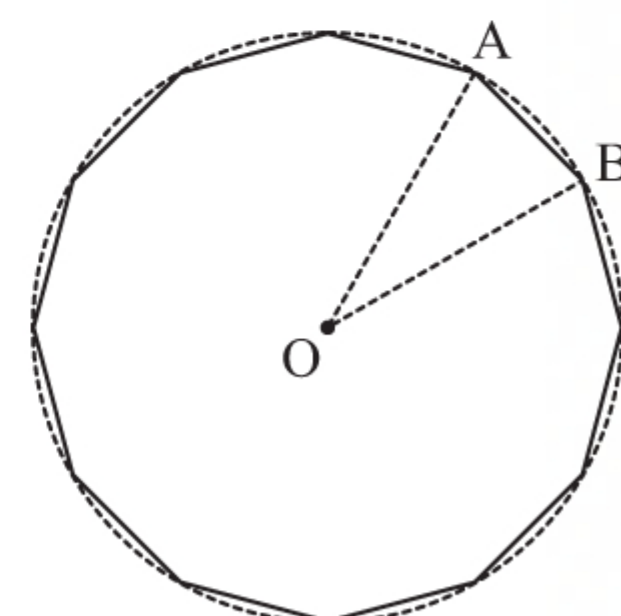


- Find the gradient and y -intercept of the line, and interpret your answers.
- Find the equation of the line.
- How long will it take for the account to run out of money?

- 3** A particle moves in a straight line with velocity $v(t) = e^{2t} - 3e^t \text{ m s}^{-1}$ at time t seconds, $t \geq 0$.

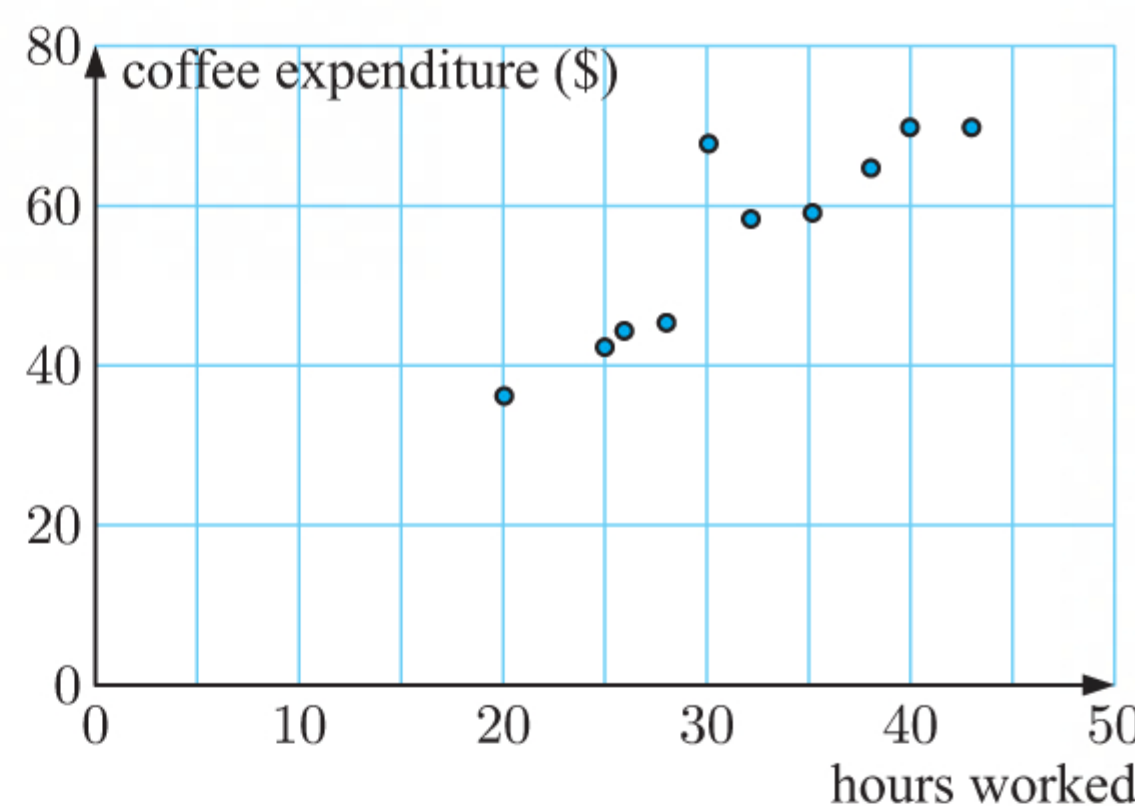
- Find the initial velocity.
- Show that the particle is stationary when $t = \ln 3$ seconds.
- The particle is initially 1 m right of the origin. Show that its position after $\ln 5$ seconds is also 1 m right of the origin.

- 4** A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 6 cm. Points A and B are adjacent vertices of the dodecagon, and both lie on the circle.



- Deduce that $\angle AOB = 30^\circ$.
- Show that the area of triangle AOB = 9 cm^2 .
- Hence determine the area of the dodecagon.

- 5** This scatter diagram displays the amount James spends on coffee in the cafeteria against the number of hours he works in the week.



- James worked an average of 32 hours, and his average expenditure was \$56 per week. Plot the mean point $P(32, 56)$ on the graph.
- Draw a line of best fit by eye which passes through P.
- Use this line to predict the amount James will spend on coffee if he works a 35 hour week.
- Describe the nature and strength of the linear relationship between the variables. Comment on whether the prediction in **c** is reliable.

- 6** Consider the binomial expansion of $\left(kx + \frac{1}{\sqrt{x}}\right)^9$.

- Write down a formula for the general term.
- Given that the constant term is $-10\frac{1}{2}$, find k .

- 7** Prove that $\tan t = \frac{\sqrt{1 - \cos^2 t}}{\cos t}$, $0 < t < \pi$.

- 8** A manufacturer states that the contents of its cereal boxes weigh an average of 320 g. A random sample of 24 boxes was weighed, with the following results recorded in grams:

312	320	326	330	326	322	326	330	331	315	323	316
315	325	311	320	308	325	320	332	316	309	314	324

- Organise the data using a frequency table, with the class intervals $305 \leq w < 310$, $310 \leq w < 315$, and so on.
- Draw a frequency histogram to display the data.
- Describe the distribution of the data.
- Find the modal class of this data.
- Calculate the mean of the data. How does it compare to the manufacturer's claim?

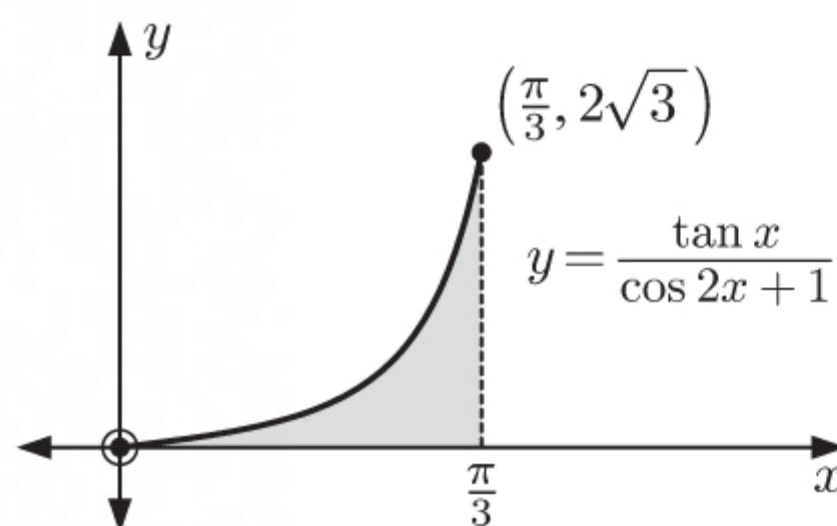
- 9** Let $y = x(x^2 - 12x + 45)$.

- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
- Find the turning points of the function.
- Find the inflection point of the function.
- Sketch the graph of $y = x(x^2 - 12x + 45)$.
- For what values of a does the equation $x^3 - 12x^2 + 45x - a = 0$ have three distinct real roots?

- 10** Suppose $f(x) = 25 - x^2$ and $g(x) = \frac{2}{\sqrt{x}}$.

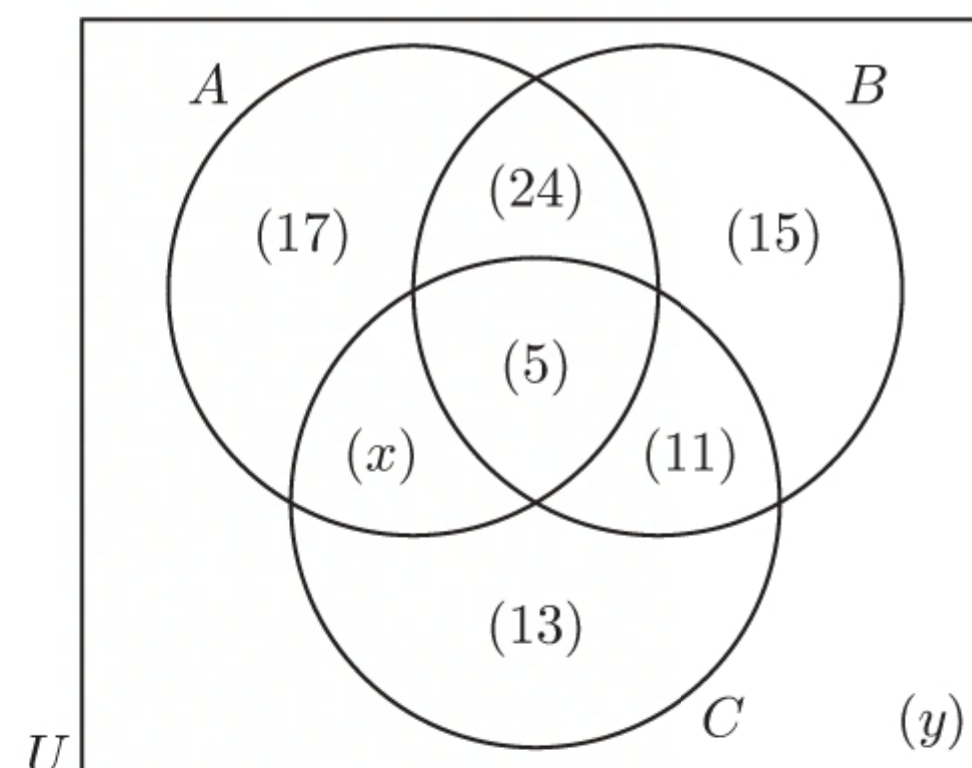
- Find $(g \circ f)(x)$, and state its domain.
- Solve $(g \circ f)(x) = 1$.
- Find the asymptotes of $y = (g \circ f)(x)$.

- 11 a** Show that $\frac{\tan x}{\cos 2x + 1} = \frac{\sin x}{2 \cos^3 x}$.
- b** Hence find the shaded area.



- 12** 100 diners at a restaurant were given a set three-course meal. After the meal, the diners were asked whether they liked each of the courses. The results are summarised alongside.

- a** Given that 48 people liked course A , find x and y .
- b** Which course was the most popular?
- c** Find the probability that a randomly selected diner liked:
- all of the courses
 - course B , but not course C
 - exactly two courses, given that the diner liked course C
 - none of the courses, given that the diner disliked course B .



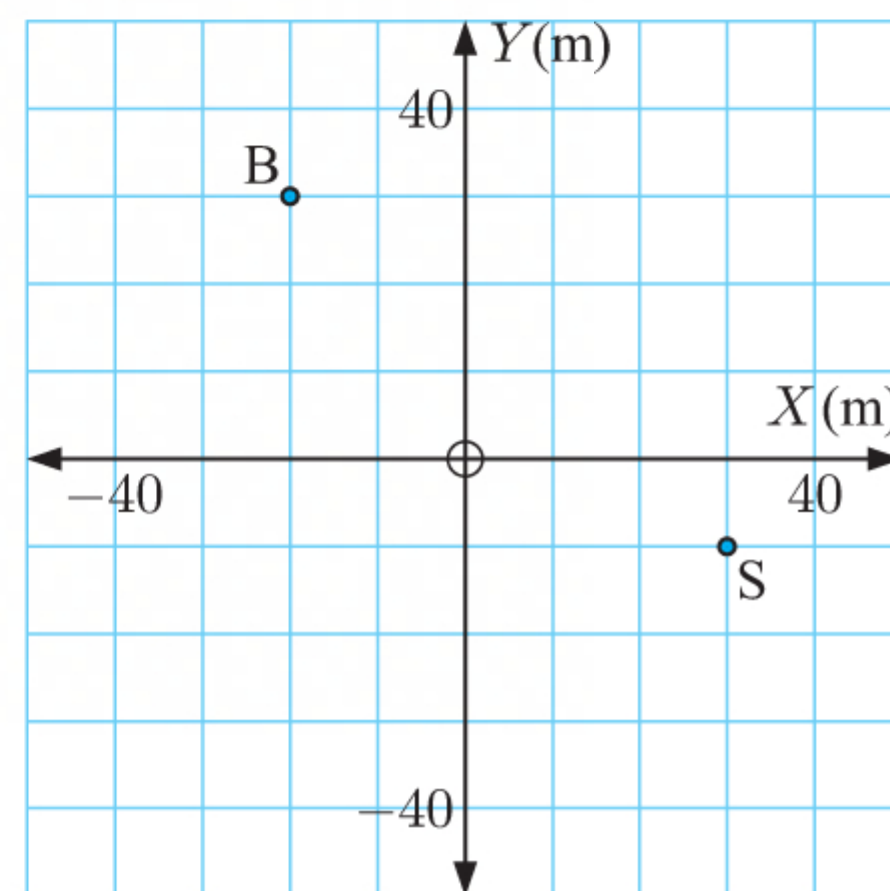
MIXED QUESTIONS SET 7

- 1** If k is an odd integer, show that $k^3 - k$ is divisible by 4.
- 2** Cynthia invested \$2000 in an account that pays 4.4% p.a. interest compounded quarterly for 5 years.
- Find the final value of the investment.
 - How much interest did Cynthia earn?
 - Given that inflation averages 2.5% p.a. over the investment period, find the real value of the investment.

- 3** This grid shows the position of a boat B and a shipwreck S .

The boat's anchor is directly below the boat, 50 m below sea level. The shipwreck is 40 m below sea level. Suppose sea level has Z -coordinate 0.

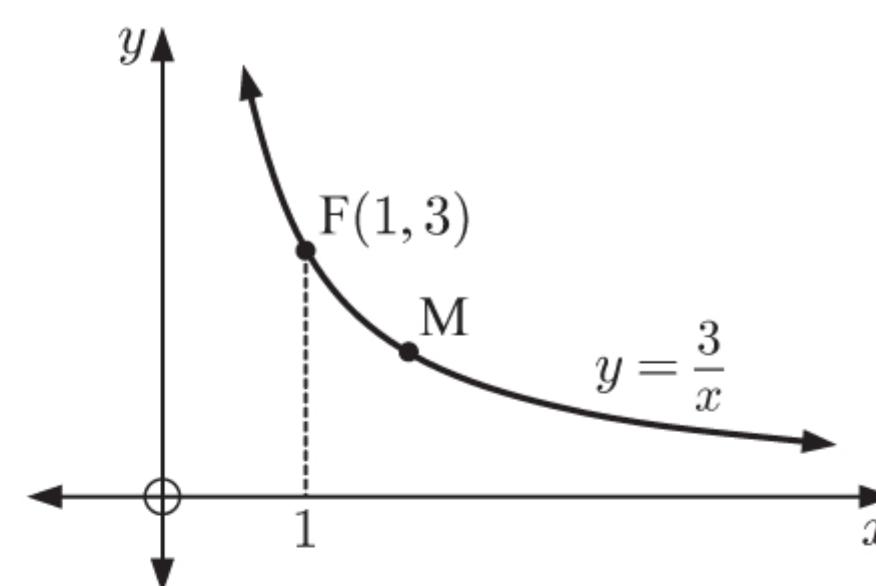
- a** Find the 3-dimensional coordinates of:
- the anchor
 - the shipwreck.
- b** A diver swims from the boat to the shipwreck. How far does the diver swim?
- c** Find:
- the angle of depression from the boat to the shipwreck
 - the angle of elevation from the anchor to the shipwreck.



- 4** Consider the graph of $y = \frac{3}{x}$.

The point F is on the curve. Let M be a point close to F with x -coordinate $1 + h$.

- a** What is the y -coordinate of M ? **b** Show that $3 - \frac{3}{h+1} = \frac{3h}{h+1}$.
- c** Find the gradient of $[FM]$ in terms of h .
- d** Using this expression, state the gradient of the *tangent* at F .



- 5** 160 m of fence is used to enclose a rectangular field.

- a** Given that one side of the field has length x m, find the area of the field in terms of x .
- b** Find the dimensions of the field which would maximise the area.
- c** Suppose the actual area of the field is 1200 m^2 .
- Find the dimensions of the field.
 - The average production yield for this field is 6.5 kg m^{-2} . Determine the amount of production lost by not using the dimensions which maximise the area.

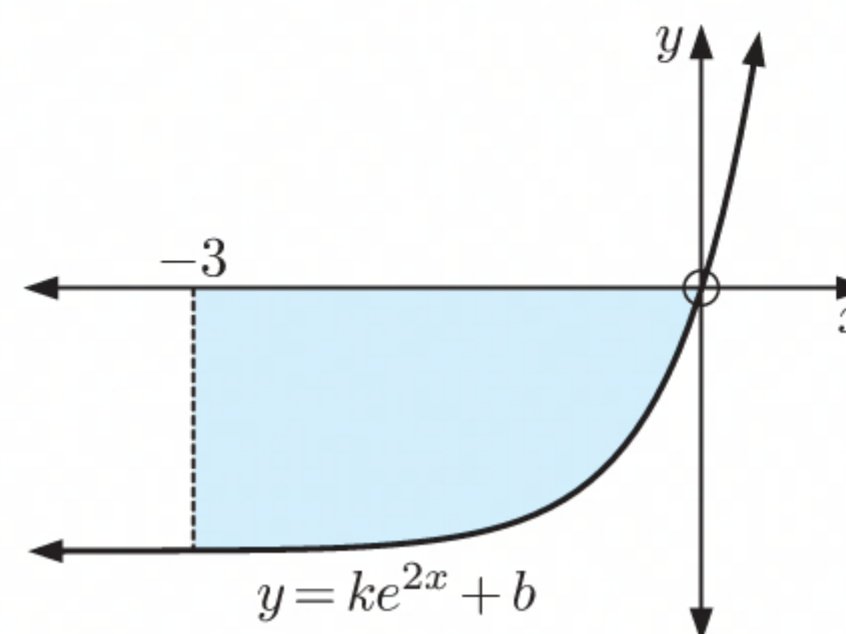
- 6 The heights of a sample of 80 children from a junior school were measured. The results are shown in the table alongside.

Estimate the: **a** mean **b** standard deviation.

Height (h cm)	Number of students
$80 \leq h < 90$	8
$90 \leq h < 100$	12
$100 \leq h < 110$	17
$110 \leq h < 120$	30
$120 \leq h < 130$	13

- 7 The area of the shaded region is $\frac{3}{e^6}$ units².

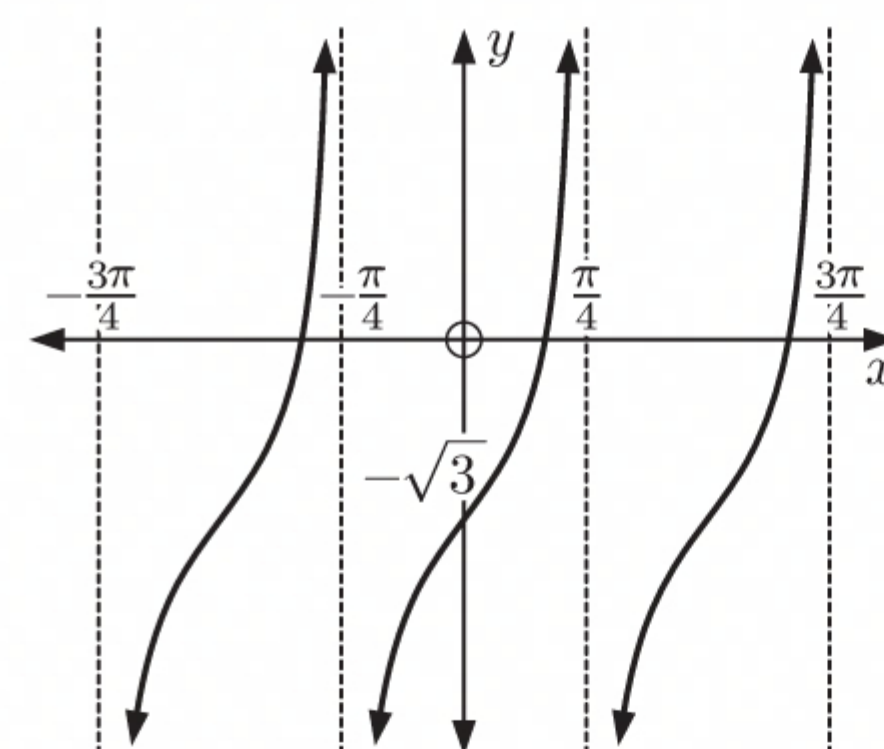
Find b and k .



- 8 Consider the graph of $y = \tan ax + b$ shown.

a Find the values of a and b .

b Hence find the x -intercepts of the function, for $-\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}$.

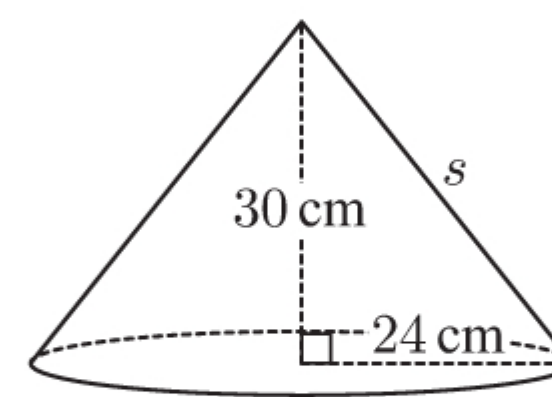


- 9 A and B are mutually exclusive events. If $P(B) = 0.3$ and $P(A \cup B) = 0.55$, find $P(A)$.
- 10 $y = 3x^2 + 2x$ is stretched vertically with scale factor 2 and then translated by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find the equation of the image.
- 11 Let $f(\theta) = \frac{2 - \cos \theta}{\sin \theta}$, $0 < \theta \leq \frac{\pi}{2}$.
- a** Show that $f'(\theta) = \frac{1 - 2 \cos \theta}{\sin^2 \theta}$.
- b** Find the minimum value of $f(\theta)$.
- c** Sketch the graph of $f(\theta)$.
- 12 A bag contains 6 red balls and 4 white balls. A game is played in which the player draws 3 balls from the bag without replacement. The player wins if 3 red balls are drawn.
- a** Find the probability of the player winning a single game.
- b** Let X be the number of wins when the game is played 60 times.
- i** Find the mean μ and standard deviation σ of X .
- ii** Find $P(X = \mu)$.
- iii** Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$.

MIXED QUESTIONS SET 8

- 1 The solution of $2^{x-1} = 3^{2-x}$ is $x = \log_a b$ where $a, b \in \mathbb{Z}^+$. Find a and b .
- 2 Two fair dice are rolled, and the difference between the scores is noted.
- a** Display the possible results on a 2-dimensional grid.
- b** Hence find the probability that the difference between the scores is 4.
- 3 Consider the function $f(x) = x^3 - 3x^2 - x + 3$, where f is defined on the domain $-2 \leq x \leq 3$, $x \in \mathbb{R}$.
- a** Use technology to help sketch the graph of $y = f(x)$, showing any axes intercepts and turning points.
- b** Determine the range of f .

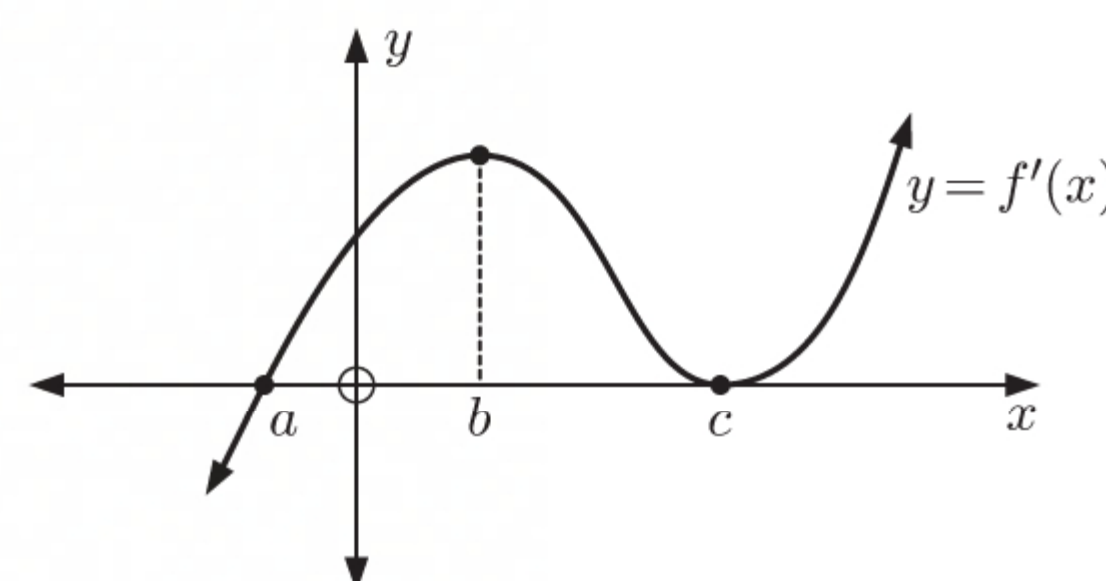
- 4 A solid right-circular cone has base radius 24 cm and vertical height 30 cm.
- Show that the slant height s is 38.4 cm, correct to 3 significant figures.
 - Determine the total surface area of the cone. Give your answer in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.



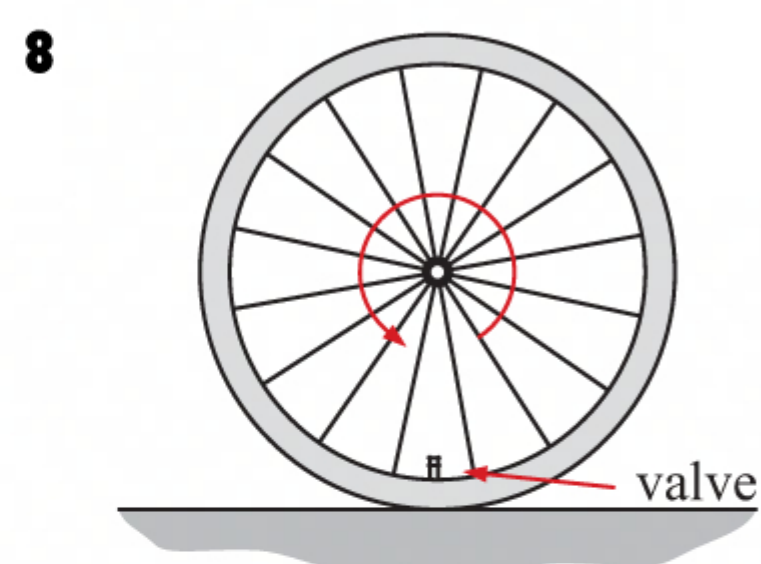
- 5 A winemaker wants to examine the effect of weed spray in his vineyard. He randomly selects 50 sample spots, each of area 1 m^2 , and counts the number of weeds in each spot. The results are shown in the table alongside.
- Determine the value of p .
 - Estimate the mean number of weeds per spot.
 - What percentage of sample spots had fewer than 10 weeds?

Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	p
20 - 24	5
25 - 29	2

- 6 Alongside is a sketch of the gradient function $y = f'(x)$.
- Sketch a possible curve for $y = f(x)$.
 - Let $y = f_1(x)$ be a curve such that $f_1'(x) = f'(x)$ for all x . Write down the form of all possible functions $f(x)$ in terms of $f_1(x)$.



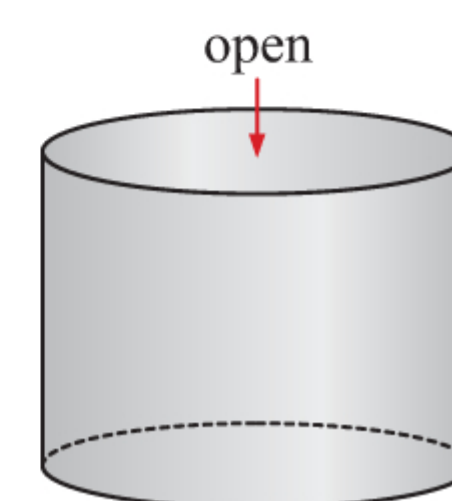
- 7 The weight of a radioactive substance after t years is given by $W(t) = 5 \times (0.965)^t$ grams, $t \geq 0$.
- Find the percentage decrease in weight of the substance each year.
 - Find the weight of the substance after 300 years. Write your answer in the form $a \times 10^k$ where $1 \leq a < 10$, $k \in \mathbb{Z}$.
 - How long will it take for the weight to fall below 1 g?



A bicycle wheel sits on the road so its valve is at the bottom. The tyre has inner radius 35 cm and outer radius 40 cm.

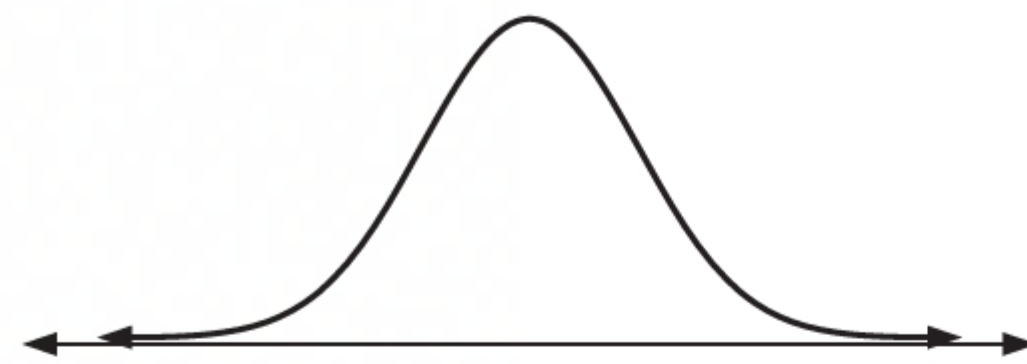
The wheel begins to rotate at a constant speed of 4 revolutions per second.

- Find the height of the valve above the road after:
 - 0 seconds
 - $\frac{1}{12}$ second.
 - The height of the valve above the road after t seconds can be modelled by the function $H(t) = a \sin(b(t - c)) + d$ cm.
Find: **i** a **ii** d **iii** b **iv** c
 - How long does it take the valve to rise to 60 cm above the road?
- 9 An open cylindrical bin is to be made from PVC plastic and is to have capacity 500 litres. Find the dimensions of the bin which minimises the amount of PVC plastic used.



- 10 Suppose $f(x) = x^2 + 2x$, $x \leq -1$.
- Find $f^{-1}(x)$, and state its domain and range.
 - Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

- 11** The random variable X is normally distributed with mean μ and standard deviation σ . Let k be such that $P(X < k) = 0.7$.
- Illustrate μ and k on the normal distribution curve.
 - Find:
 - $P(X > k)$
 - $P(\mu < X < k)$
 - $P(\mu - \sigma < X < k)$
 - If $P(X \geq t) = 0.2$, find $P(k \leq X \leq t)$.
- 12** Let $f(x) = \frac{9}{2} - x^2$. The normal L at the point $P(a, f(a))$, $a > 0$, passes through the origin.
- Sketch $y = f(x)$, showing the vertex and axes intercepts.
 - Show that the equation of L is $y = \frac{1}{2a}x + 4 - a^2$.
 - Hence find the area of the region enclosed by L and $y = f(x)$.

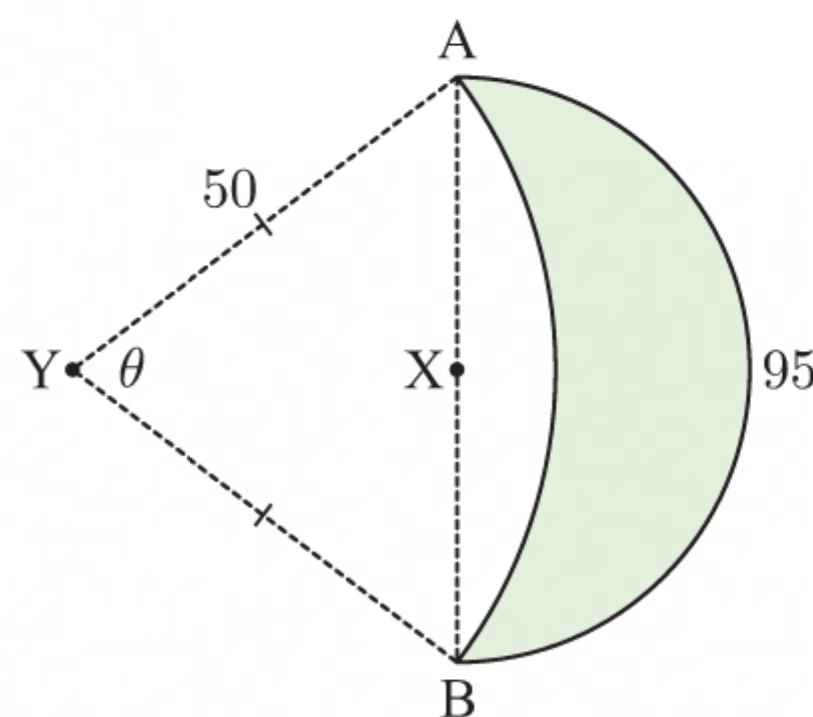


MIXED QUESTIONS SET 9

- A quadratic function has the form $f(x) = ax^2 + bx + 7$. It is known that $f(2) = 7$ and $f(4) = 23$.
 - Construct a set of simultaneous equations involving a and b .
 - Find a and b .
 - Hence calculate $f(-1)$.
- Find the equation of the tangent to $f(x) = (x^2 + 1)e^{-x}$ at the point where $x = 1$.
- Hayley and Patrick were training for a road cycling race. During the first week they both cycled 60 km. Hayley cycled an additional 20 km each subsequent week, whereas Patrick increased his distance by 20% each subsequent week.
 - How far did each of them cycle in the 5th week of training?
 - Who was the first to cycle 210 km in one week?
 - Who cycled a greater total distance in the first 12 weeks? Explain your answer.
- X and Y are the centres of the two arcs AB shown.

Find:

- the length AX
- the angle θ
- the shaded area.



- Suppose $f(x) = \log_3(x + 1) + 2$.
 - State the transformation which maps $y = \log_3 x$ to $y = f(x)$.
 - Find the domain and range of f .
 - Find the axes intercepts of f .
 - Sketch the graph of $y = f(x)$.
 - Find the inverse function f^{-1} .
- Find the coefficient of x^7 in the expansion of $(x - 1)(2 - x)^9$.
- The following data shows Craig's weekly grocery bills, in dollars, for the last 5 months.

181,	155,	163,	200,	149,	185,	160,	159,	164,	171,
173,	212,	303,	191,	169,	161,	207,	140,	132,	165

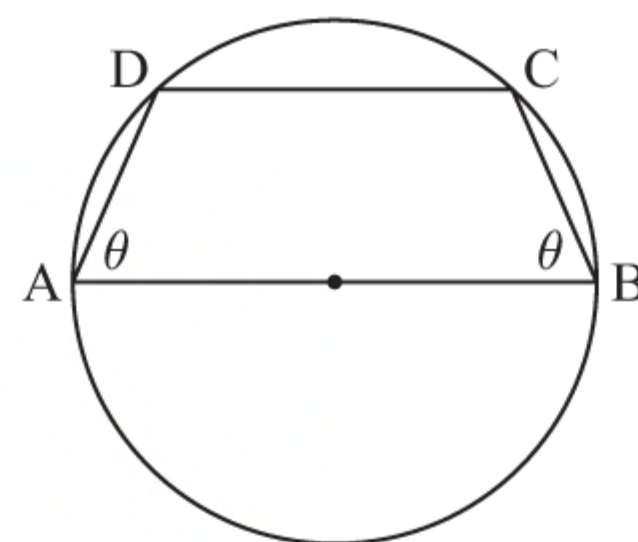
 - Find the median, lower quartile, and upper quartile of the data set.
 - Find the interquartile range of the data set.
 - The bill of \$303 occurred when Craig bought groceries for a large Christmas lunch. Show that this value is an outlier.
 - Draw a box plot of the data set.

- 8 Consider the function $f(x) = \frac{1}{x} - \frac{4}{x-2}$.
- Find the value of x for which $f(x) = 0$.
 - Find and classify all stationary points.
 - Find the coordinates of the point of inflection. Give your answer to two decimal places.
 - Draw the graph of $y = f(x)$, showing all of the above information.
 - Find the exact area enclosed by the graph, the x -axis, and the lines $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

- 9 Suppose $P(A) = 0.35$, $P(B) = 0.7$, and $P(A \cup B) = 0.8$.
- Calculate $P(A \cap B)$.
 - Represent this information on a Venn diagram.
 - Find:
 - $P(A' \cap B')$
 - $P(A | B)$
 - State, with a reason, whether events A and B are independent.

- 10 $[AB]$ is the diameter of a circle with radius 6 cm.

- Show that the area of trapezium $ABCD$ is given by $A = 18(2 \sin 2\theta - \sin 4\theta)$.
- Find the angle θ which maximises the area of $ABCD$.



- 11 Consider $f(x) = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$, $0 \leq x \leq \frac{\pi}{2}$.
- Show that $f(x) = \frac{2}{\sin 2x}$.
 - Solve the equation $\sin 2x = 0$, and hence state the equations of the asymptotes of $y = f(x)$ on $0 \leq x \leq \frac{\pi}{2}$.
 - Without using calculus, find the least value of $f(x)$ and the corresponding value of x .
 - If $\sin a = \frac{1}{3}$, find $f(2a)$ correct to 4 significant figures.
- 12 Eleven students participate in a basketball game. Their time spent training the previous week, and the number of points they scored in the game are shown in the table below.

<i>Time spent training (t hours)</i>	2.5	1	2.5	3.5	4	2.5	2	3	3	2	1.5
<i>Points scored (y)</i>	2	0	5	16	9	8	2	6	10	0	2

- Explain why it would be appropriate to use the regression line of t against y in this case.
- Find the regression line of t against y .
- Use the regression line to estimate:
 - the time spent training by a player who scored 7 points
 - the number of points scored by a player who spent 5 hours training.
- Comment on the reliability of your estimates in **c**.

MIXED QUESTIONS SET 10

- The graph of $y = 3 - \frac{k}{x-1}$ has x -intercept $\frac{5}{3}$.
 - Find the value of k .
 - Find the y -intercept.
 - State the equations of the asymptotes.
 - Sketch the curve, showing the features you have found.
- Write $0.\overline{34}$ as an infinite geometric series.
 - Hence write $0.\overline{34}$ as a rational number.
- Two year 7 students are selected each week to hoist the flag before the start of class. Year 7 has been divided into 2 classes: class A has 30 students, and class B has 27 students.
 - Find the probability that, in any given week, the two selected students selected are in the same class.
 - Over the course of 20 weeks, how many times do you expect that the two selected students are in the same class?

4 Find a and b given that $2^a 8^b = \frac{1}{2}$ and $\frac{3^{-a}}{3^{b+1}} = 9$.

5 Consider the lines $L_1: y = \frac{3}{4}x + 1$ and $L_2: y = -x - 1$.

a Find the angle that each line makes with the positive x -axis.

b Hence find the acute angle between L_1 and L_2 .

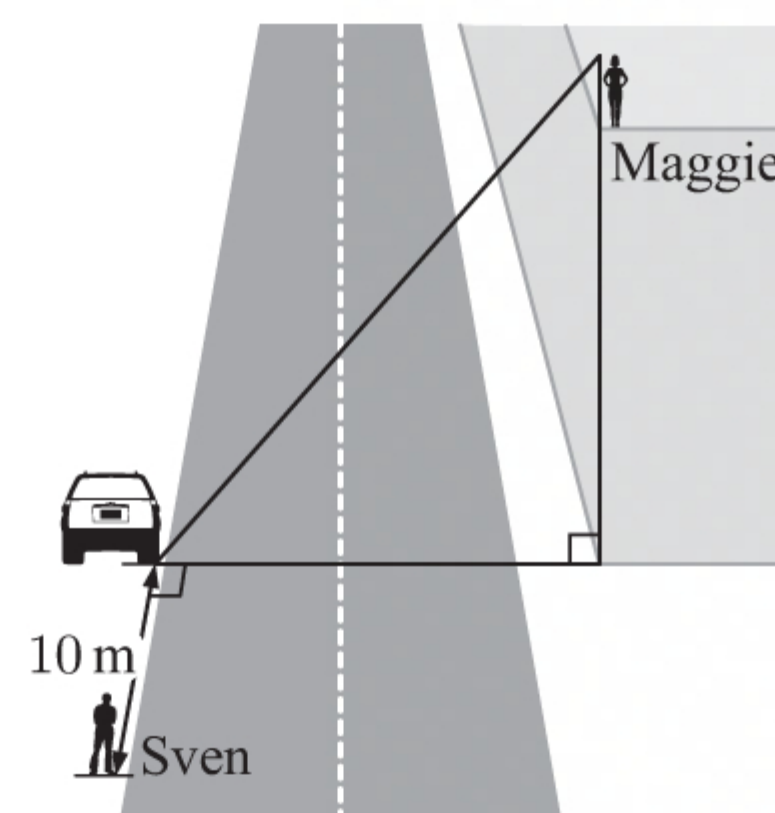
6 Maggie is 155 cm tall and is standing on top of a building 50 m tall. A car is parked on the far kerb of the road, directly opposite Maggie. To see the car, Maggie looks down at an angle 67° below horizontal.

a How far is the car from the base of the building?

b Maggie's friend Sven is walking on the same side of the road that the car is parked. He is currently 10 m from the car.

i Find the distance between Maggie and Sven.

ii At what angle must Sven look up to see Maggie?



7 Consider the functions $f(x) = x - 2$ and $g(x) = 3 - x - 2x^2$. Find:

a $f^{-1}(x)$

b $(g \circ f)(x)$

c $(g \circ f)(-1)$

8 The velocity of a truck t seconds after applying its brakes is $v = \frac{20}{\sqrt{2t+1}}$ m s $^{-1}$, $0 \leq t \leq 10$.

a Find the speed of the truck when the brakes are applied.

b Find the acceleration function.

c At what time does the truck have acceleration -2.5 m s $^{-2}$?

d Find the distance travelled by the truck in the first 10 seconds after applying the brakes.

9 The masses of sea lions on a particular island are normally distributed with mean μ and standard deviation σ . 10% of the sea lions have mass greater than 900 kg, and 15% have mass less than 500 kg.

a Find μ and σ .

b A randomly selected sea lion weighs more than 800 kg. Find the probability that the sea lion weighs less than 850 kg.

10 Suppose $f(x) = \sin^2 x - \cos^2 x$.

a Show that $f'(x) = 2 \sin 2x$.

b Hence find $4[f'(x)]^2 + [f''(x)]^2$.

c Given that $f(\theta)$, $f'(\theta)$, and $f''(\theta)$ form an arithmetic sequence, find the value of $\tan 2\theta$.

11 The lengths, in minutes, of games in a chess tournament are displayed in this cumulative frequency graph.

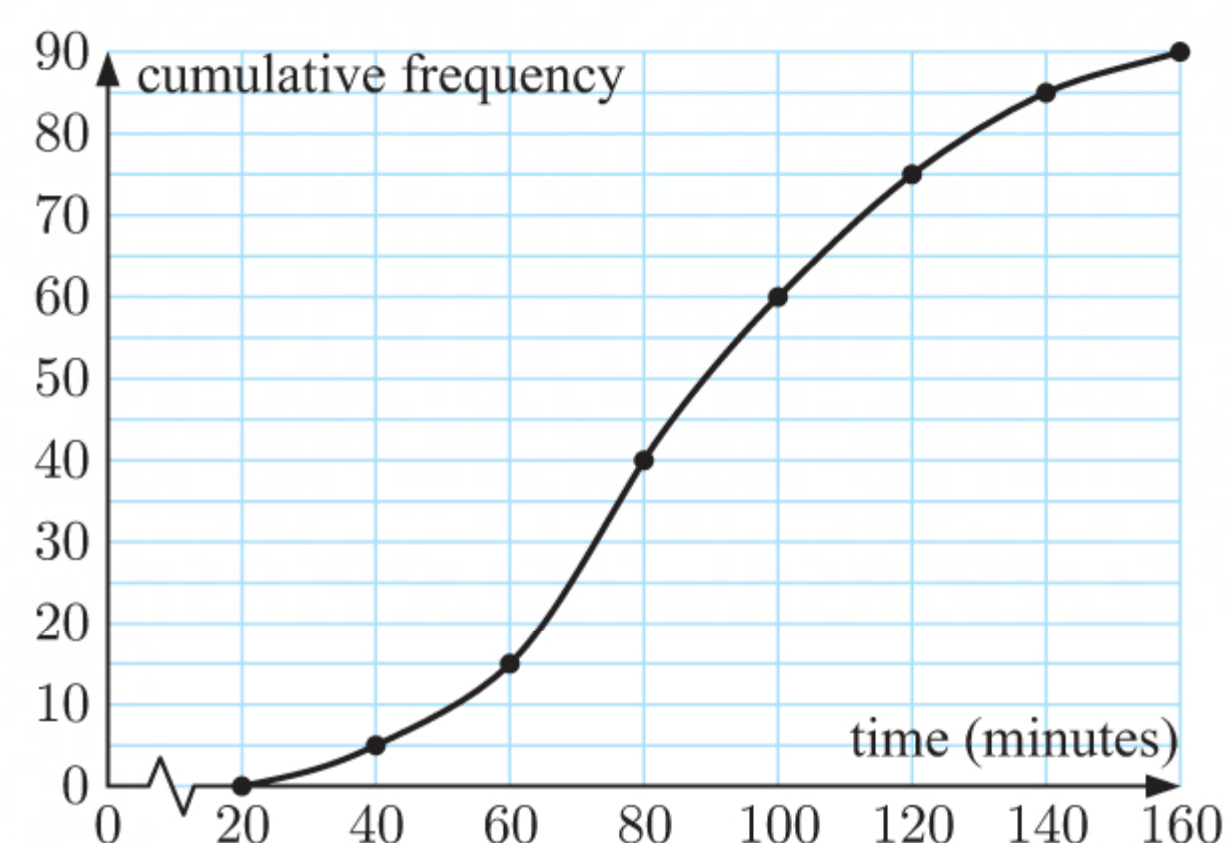
a How many games were played during the tournament?

b Find the median game length.

c Estimate the interquartile range for the data.

d 10% of the games took less than k minutes. Estimate the value of k .

e Draw a frequency histogram to represent the data.



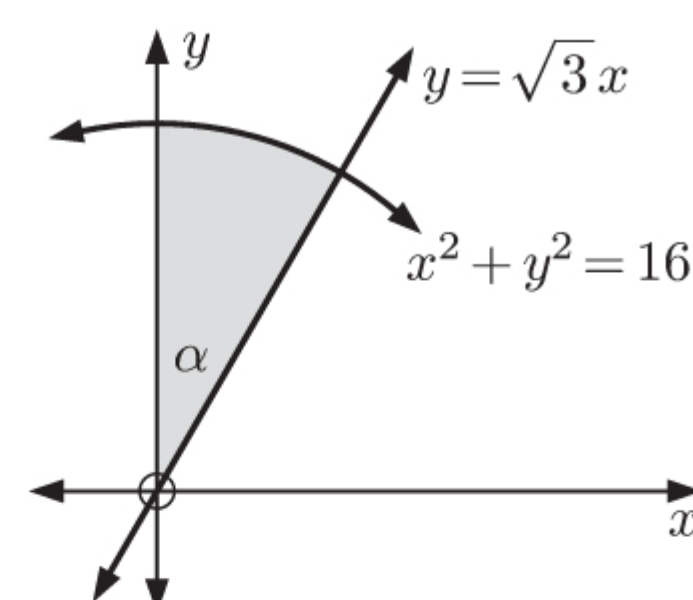
12 The diagram shows a line $y = \sqrt{3}x$ and an arc of the circle $x^2 + y^2 = 16$.

a Show that $\alpha = \frac{\pi}{6}$.

b Hence find the area A of the sector shown.

c By considering A as the area between the two curves $x^2 + y^2 = 16$ and $y = \sqrt{3}x$, show that $A = \int_0^2 \sqrt{16 - x^2} dx - 2\sqrt{3}$.

d Show that $\int_0^2 \sqrt{16 - x^2} dx = \frac{4\pi}{3} + 2\sqrt{3}$.



Trial examination 1

PAPER 1

NO CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 6]

The coefficient of x^2 in the expansion of $(1 - 2x)^n$, where $n \in \mathbb{N}$, is 144.

- a** Write down the number of terms in this expansion, in terms of n . [1]
b Find the value of n . [5]

2 [Maximum mark: 6]

Let $f(x) = e^{2x-4}$.

- a i** Show that [2]

$$f^{-1}(x) = 2 + \ln \sqrt{x}$$

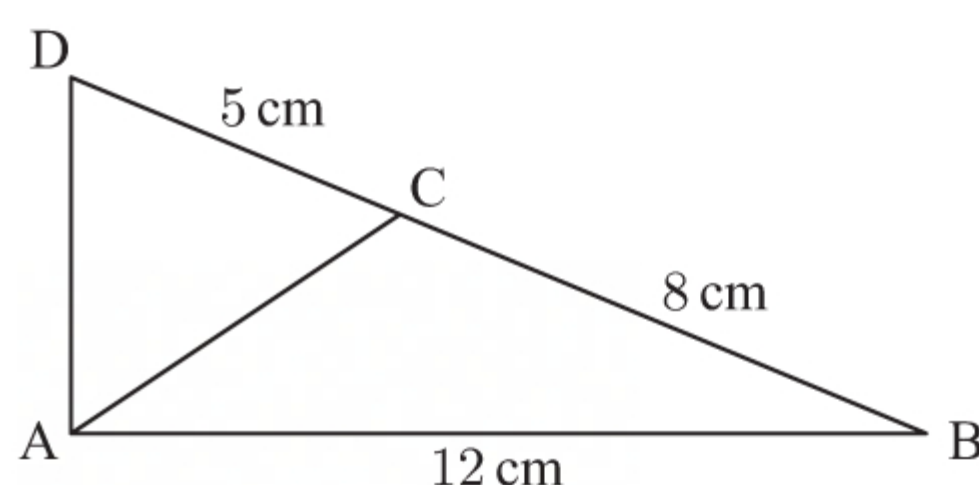
- ii** Write down the domain of $f^{-1}(x)$. [1]

- b** Solve the equation [3]

$$f^{-1}(x) = \frac{1}{2} \ln 2$$

3 [Maximum mark: 5]

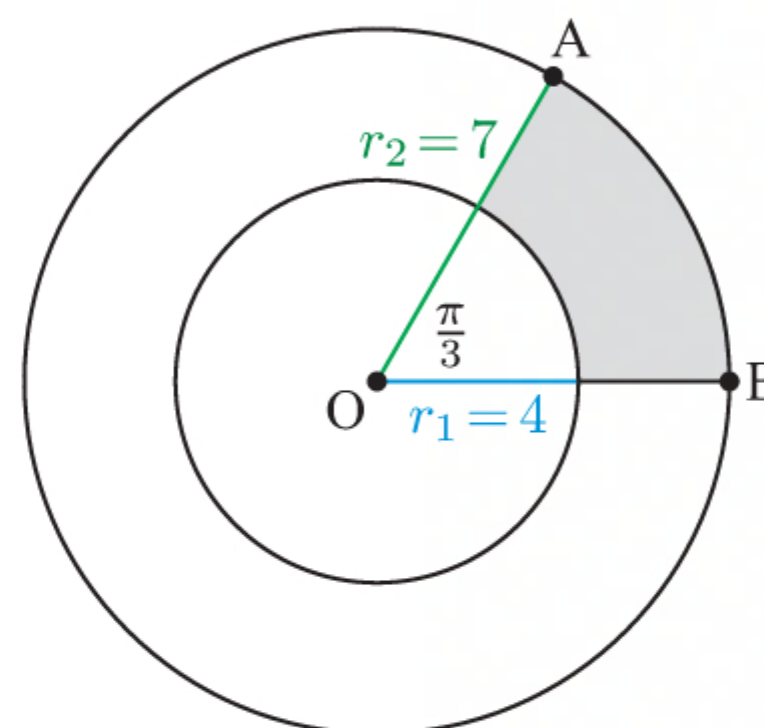
The following diagram shows triangle ABC, with $AB = 12$ cm, $BC = 8$ cm, and $\cos \hat{ABC} = \frac{12}{13}$. Point D is on line (BC) such that $CD = 5$ cm and C is between B and D.



- a** Find AD. [2]
b Write down $\sin \hat{ADC}$. [1]
c Find the **exact** area of triangle ADC. [2]

4 [Maximum mark: 8]

The diagram shows two concentric circles with centre O.



The radius of the smaller circle, r_1 , is 4 units and the radius of the larger circle, r_2 , is 7 units.

Points A and B are on the circumference of the larger circle such that $\hat{AOB} = \frac{\pi}{3}$.

- a** Find the length of the arc AB, in terms of π . [2]
b Find the area of the shaded region. [4]
c Show that the perimeter of the shaded region is $\frac{11\pi}{3} + 6$. [2]

5 [Maximum mark: 8]

The first three terms of a geometric sequence are

$$\ln(x^{27}), \ln(x^9), \ln(x^3)$$

a Find the common ratio. [3]

b Solve [5]

$$\sum_{k=1}^{\infty} 3^{4-k} \ln x = 324$$

6 [Maximum mark: 7]

Solve, for $0 < \theta < 2\pi$, the equation

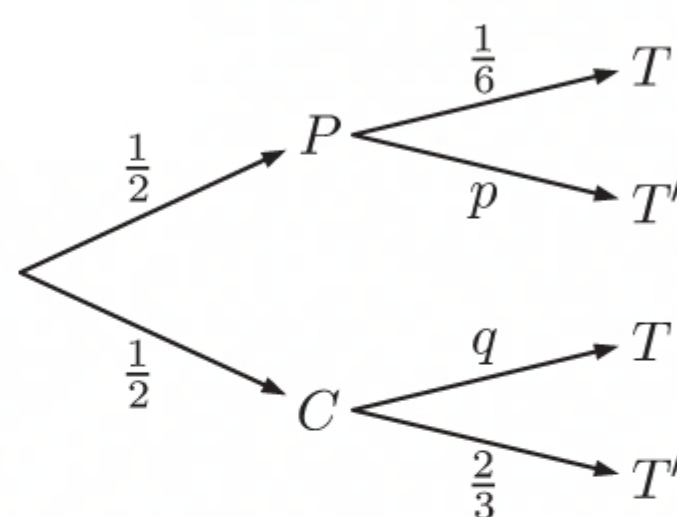
$$\sin \theta = \cos 2\theta$$

SECTION B**7 [Maximum mark: 16]**

Hassan travels to school every day either by public transportation (P) or by car (C) and on any particular day it is equally likely that he chooses to travel by public transportation or by car.

The probability that he arrives on time (T) is $\frac{1}{6}$ if he travels by public transportation. The probability that he does not arrive on time (T') is $\frac{2}{3}$ if he travels by car.

This information is represented by the following tree diagram.



a Write down the value of:

i p [1]

ii q . [1]

b Find the probability that Hassan will travel to school by car and will arrive on time. [3]

c Find the probability that Hassan will not arrive on time. [4]

d Given that Hassan does not arrive to school on time, find the probability that he travelled by public transportation. [3]

e Hassan goes to school three times in a week. Find the probability that he is on time at least once. [4]

8 [Maximum mark: 24]

Consider a quadratic function in the form $f(x) = a(x + p)(x + q)$, $a \neq 0$, $p, q \in \mathbb{Q}$.

a Show that [2]

$$f'(x) = 2ax + ap + aq$$

The line $y = -\frac{2}{5}x - \frac{11}{5}$ is normal to the curve of $f(x)$ at $x = 2$.

The line $y = \frac{2}{3}x - \frac{11}{3}$ is normal to the curve of $f(x)$ at $x = -2$.

b Find the value of a . [6]

The graph of $f(x)$ has axis of symmetry $x = -\frac{1}{2}$.

c Show that $p + q = 1$. [2]

The y -intercept of $f(x)$ is $(0, -6)$.

d Show that $pq = -12$. [2]

e Hence, or otherwise, find the possible values of p and q . [4]

f The line $y = kx - 8$ is tangent to the graph of $f(x)$. Find the values of k . [8]

PAPER 2

CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 6]

A biased eight-sided die is rolled. The following table gives the probability of each score.

Score (x)	1	2	3	4	5	6	7	8
Probability ($P(X = x)$)	0.10	0.13	0.14	k	0.16	0.12	0.11	0.08

- a Find the value of k . [2]
- b Find $E(X)$, the expected value of the score. [2]
- c The die is rolled 75 times. Find the expected number of times to obtain a score of 5 or 6. [2]

2 [Maximum mark: 7]

Let $f(x) = \frac{x+2}{e^x} + 4$ for $-2.5 \leq x \leq 8$.

The graph of $y = f(x)$ has a maximum point at A.

- a Write down the coordinates of A. [2]
- b Find the x -intercept of the graph of $y = f(x)$. [2]
- c Sketch the graph of $y = f(x)$ for $-2.5 \leq x \leq 8$, $-5 \leq y \leq 10$. [3]

3 [Maximum mark: 7]

A local cycling club is organising a race to select primary school students to represent the club in an upcoming competition.

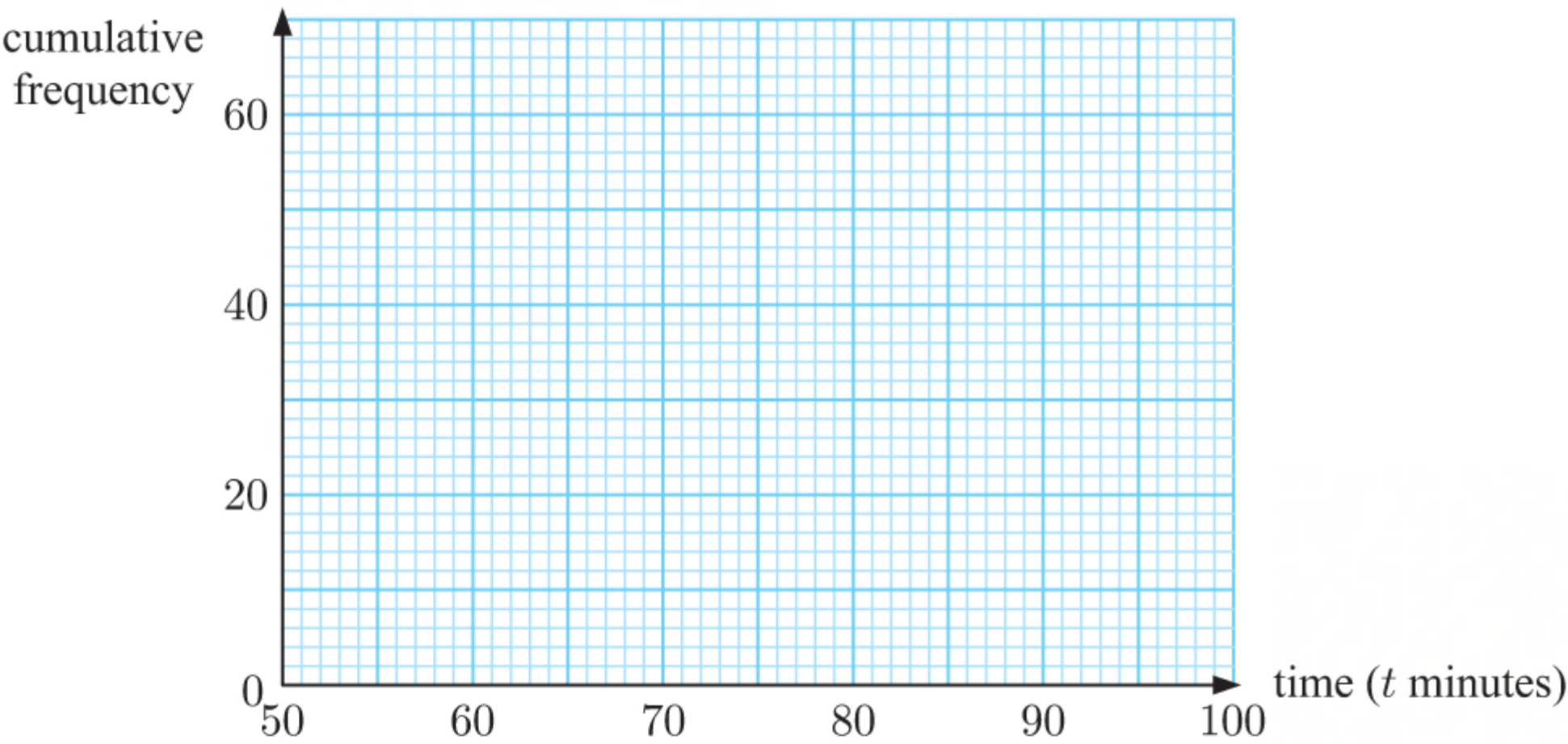
The time taken by a group of primary school students to complete the race are shown in the table below.

Time (t minutes)	$50 \leq t < 60$	$60 \leq t < 70$	$70 \leq t < 80$	$80 \leq t < 90$	$90 \leq t < 100$
Frequency	4	16	p	q	4
Cumulative Frequency	4	20	44	r	60

- a Write down the value of p . [1]
- b Find the value of:

i q [2]

ii r . [2]
- c Sketch the cumulative frequency graph on a grid like the one below to represent the given information. [2]



4 [Maximum mark: 6]

The following table shows the mean length of male and female babies.

Age (x months)	0	2	4	6	8	10	12
Mean length of male babies (y_1 cm)	19.69	23.03	25.20	26.77	27.95	28.74	29.92
Mean length of female babies (y_2 cm)	19.29	22.44	24.41	25.48	27.17	28.15	29.13

The relationship between the age (x) and the length of a male baby (y_1) can be modelled by the regression line with equation $y_1 = a_1x + b_1$.

- a** Find the value of a_1 and b_1 . [3]
- b** Write down the correlation coefficient. [1]

The relationship between the age (x) and the length of a female baby (y_2) can be modelled by the regression line with equation $y_2 = 0.780\,36x + 20.470\,71$.

- c** Use this equation to estimate the mean length of a female baby that is three and a half months old. [2]

5 [Maximum mark: 8]

A ball is dropped from a height of 5 metres. Each time the ball bounces, it reaches 85% of the height it reached on the previous bounce.

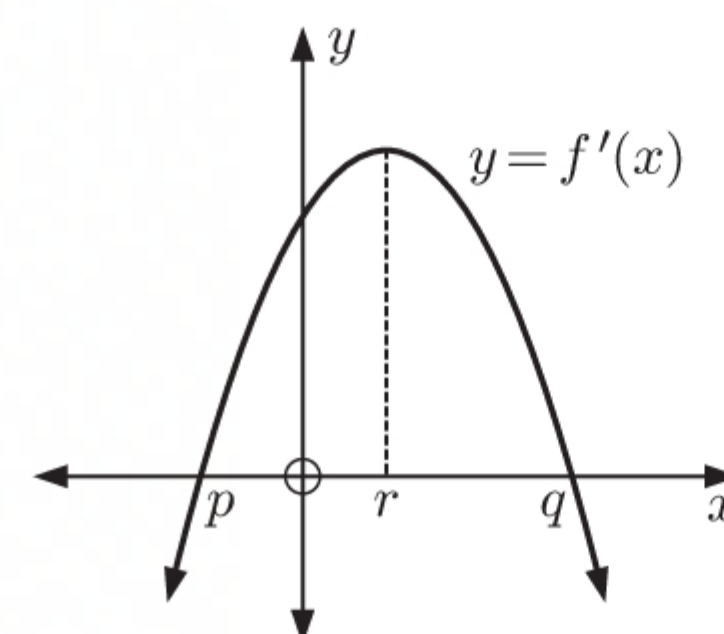
- a** Find the height the ball reaches after its sixth bounce. [2]

The ball bounces k times before it no longer reaches a height of 2 metres.

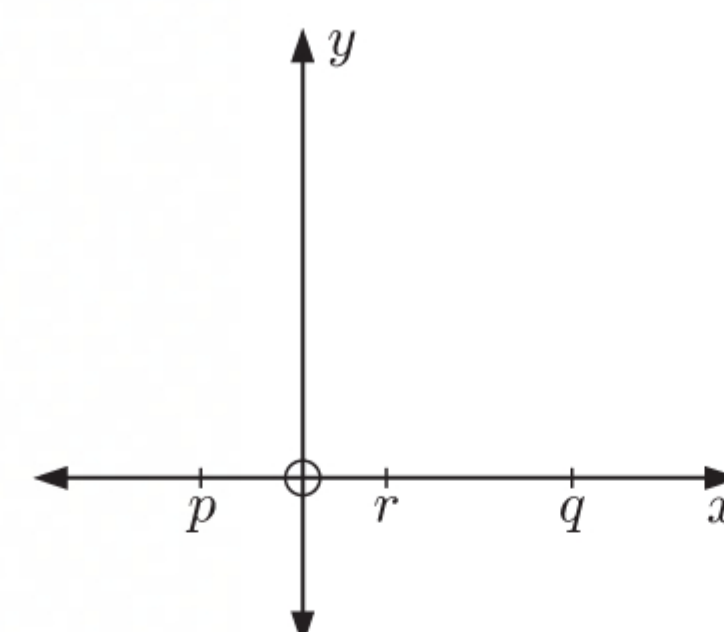
- b** Find the value of k . [3]
- c** Find the total distance travelled by the ball. [3]

6 [Maximum mark: 6]

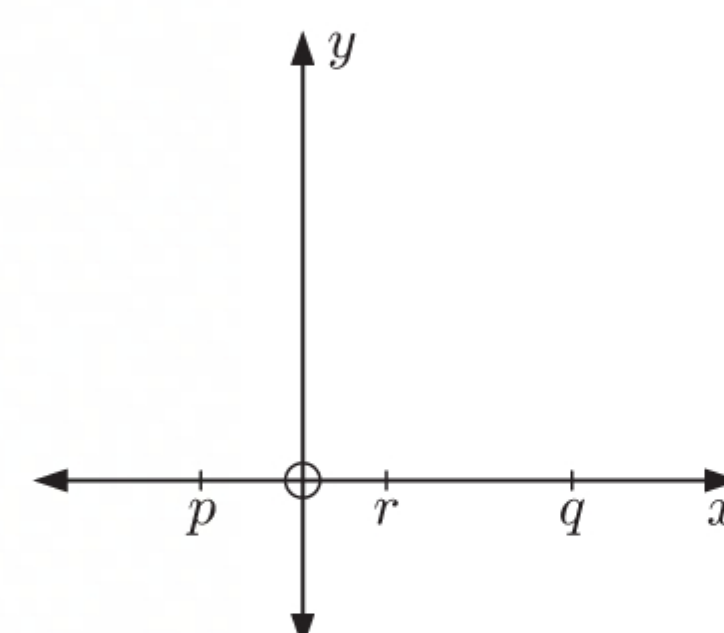
The diagram alongside shows the graph of the gradient function $y = f'(x)$.



- a** On axes like the ones shown, sketch the graph of:
 - i** $y = f(x)$, clearly indicating its stationary points and non-stationary point of inflection [3]



- ii** $y = f''(x)$, clearly indicating its x -intercept. [1]



- b** Hence, or otherwise, write down the values of x for which the graph of $y = f(x)$ is concave-down. [2]

SECTION B

7 [Maximum mark: 10]

The first terms of a geometric sequence are $\frac{\ln x}{\ln 4}, \frac{\ln x}{\ln 16}, \frac{\ln x}{\ln 256}, \dots$

- a Find the common ratio. [2]
- b Show that $u_8 = \frac{\ln x}{2^8 \ln 2}$. [2]
- c Find the **exact** value of S_8 . [3]

The sum of the infinite geometric sequence is given as $S_\infty = \frac{\ln x}{\ln a}$.

- d Find the value of a . [3]

8 [Maximum mark: 8]

Let $f(x) = \frac{x}{4} + 2$ and $g(x) = 2 \sin(2x)$.

- a Find an expression for $h(x) = (f \circ g)(x)$. [2]
- b Write down:
 - i the period of $h(x)$ [1]
 - ii the amplitude of $h(x)$. [1]

The graph of $y = h(x)$ can be obtained from the graph of $y = g(x)$ by two transformations:

- (1) a vertical stretch of scale factor p , followed by
- (2) a vertical translation of b units.

- c Write down the value of:
 - i p [1]
 - ii b . [1]
- d Sketch the graph of $y = h(x)$ for $-10 \leq x \leq 10$, $-1 \leq y \leq 4$. [2]

9 [Maximum mark: 12]

A jar contains a large number of chocolate chip cookies. The weights of the cookies are normally distributed with mean 15 grams and standard deviation 1.2 grams.

One cookie is chosen randomly from the jar.

- a Find the probability that the cookie weighs:
 - i less than 17 grams [2]
 - ii between 12 grams and 17 grams. [1]
- b Twelve per cent of the cookies in the jar weigh less than p grams.
 - i Copy and complete the following normal distribution diagram to represent this information by indicating p and shading the appropriate region. [1]



- ii Find the value of p . [2]
- c The weights of chocolate chip cookies in another jar are normally distributed with mean 13 grams and standard deviation σ . It is known that 3% of the cookies in this second jar weigh less than 9 grams.
 - i Find the value of σ . [4]
 - ii 20 cookies were selected from this jar, each being replaced before making the next selection. Find the probability that exactly 2 of the cookies weigh less than 9 grams. [2]

10 [Maximum mark: 10]

Consider $f(x) = \frac{3x-2}{x+1}$, $x \neq -1$.

a Write down the equation of the:

i horizontal asymptote

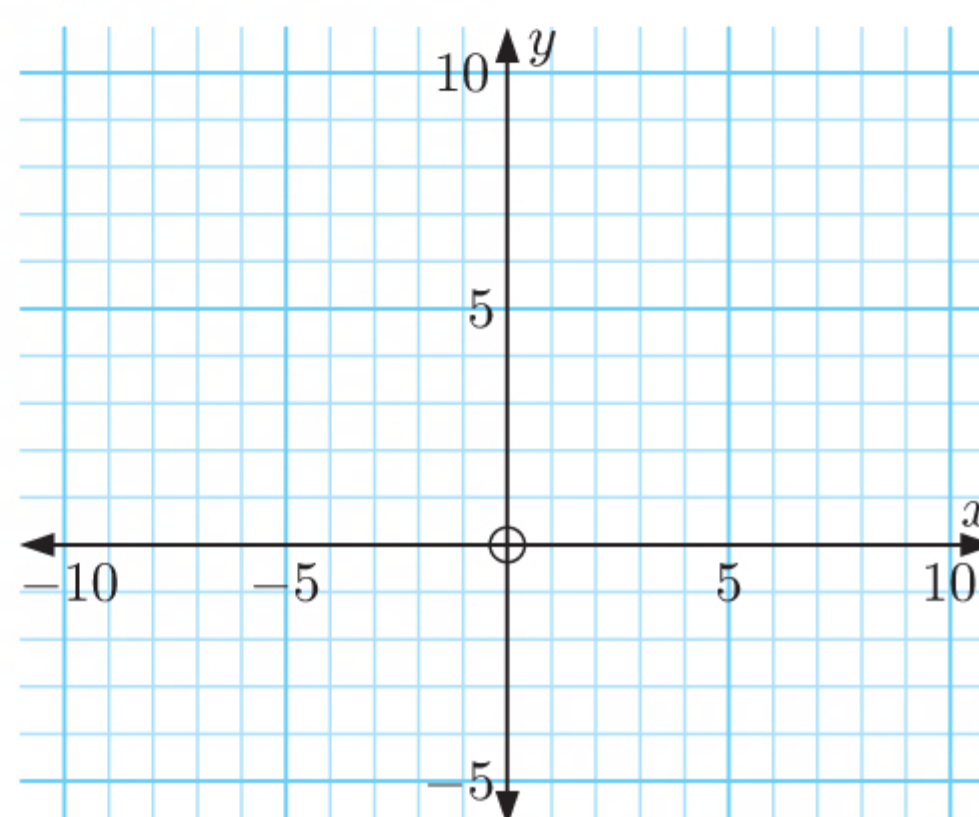
[1]

ii vertical asymptote.

[1]

b On a grid like the one provided, sketch the graph of $y = f(x)$, including any asymptotes.

[4]



c Find the x -intercept of the graph of $y = f(x)$.

[2]

d Hence, find the area of the region enclosed by the graph of $y = f(x)$, the x -axis, and the lines $x = 0$ and $x = 5$.

[2]

Trial examination 2

PAPER 1

NO CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 4]

Consider the first four terms of a geometric sequence 40, 20, 10, 5,

- a i Write down the common ratio r . [1]
- ii Hence explain why the sum to infinity of the corresponding series exists. [1]
- b Find the sum to infinity of the series $40 + 20 + 10 + 5 + \dots$ [2]

2 [Maximum mark: 5]

- a For $n \in \mathbb{Z}^+$, explain why $(2n - 1)$ and $(4n + 1)$ are odd. [1]
- b Expand and simplify $(2n - 1)(4n + 1)$. [1]
- c Hence, or otherwise, prove that the product of two odd numbers of this form is always odd. [3]

3 [Maximum mark: 5]

A data set consisting of 12 values has a mean of 18 and variance of 9.

- a Write down the standard deviation of the data set. [1]
- b If each value in the data set is halved, determine the value of the new:
 - i mean [1]
 - ii variance. [3]

4 [Maximum mark: 7]

Consider the function $f(x) = xe^x$, $x \in \mathbb{R}$.

- a Find $f'(x)$. [3]
- b Hence find $\int f(x) dx$. [4]

5 [Maximum mark: 5]

- a Sketch the graph of the function $f(x) = 3 \sin 2x$, $0 \leq x \leq \frac{3\pi}{2}$. [3]
- b Hence, or otherwise, **write down** the number of solutions to the equation $f(x) + 2 = 0$. [2]

6 [Maximum mark: 7]

- a Find $g(x)$ given that $g'(x) = \frac{2}{3-x}$ and $g(x)$ passes through the point $(2, -1)$. [5]
- b State the domain and range of $g(x)$. [2]

7 [Maximum mark: 8]

Solve the following equation for $x \in \mathbb{R}^+$:

$$3 \log_2 x + 2 \log_4 (2x + 1) = 6 \log_8 x$$

SECTION B

8 [Maximum mark: 12]

Consider the function $f(x) = \frac{x+4}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

- a Write down the equations of the horizontal and vertical asymptotes. [2]
- b Find the coordinates of all axes intercepts of $f(x)$. [2]
- c Hence, sketch $f(x)$ clearly labelling the coordinates of all axes intercepts and the equations of all asymptotes. [3]

- d** Use **your** sketch to solve the following inequality: $\frac{x+4}{2x-1} < 0$. [2]

The function f contains the point $A(2, 2)$. The graph of g is the image of the graph of f given by the following transformations:

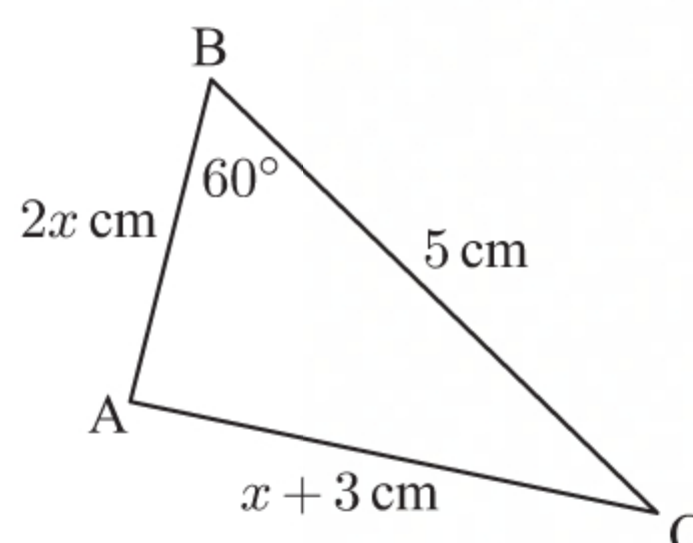
$$g(x) = 2f(x-1) - 1$$

Let A' be the image of A under the transformation.

- e** Find the coordinates of A' . [3]

9 [Maximum mark: 15]

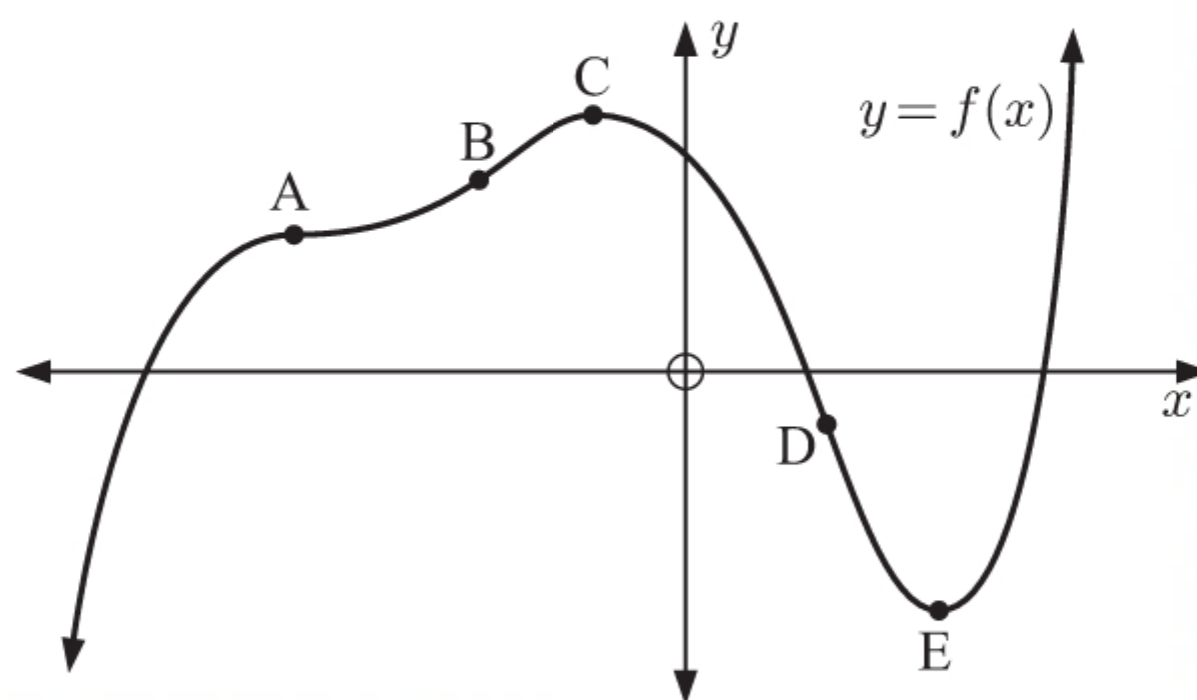
Consider the triangle ABC below which shows $BC = 5$ cm, $BA = 2x$ cm, $AC = x + 3$ cm, and $\widehat{ABC} = 60^\circ$.



- a** Use the cosine rule to find two possible values for x . [6]
- b** For **each** value of x , determine the exact value for the area of triangle ABC . [3]
- c** Suppose x takes the value corresponding to the *larger* area of triangle ABC . Given that $\widehat{ACB} = \theta$, find the exact value of $\sin 2\theta$. [6]

10 [Maximum mark: 12]

Consider the graph of the function $y = f(x)$ below.



Points A , B , C , D , and E have been labelled on the sketch such that A , C , and E are stationary points and A , B , and D are inflection points.

- a** How many solutions are there to the equation $f(x) = 0$? [1]
- b** Write down the points on $y = f(x)$ at which:
- i** $f'(x) = 0$ [2]
- ii** $f''(x) = 0$. [2]

Let the x -coordinate of A be represented as x_A , the x -coordinate of B as x_B , and so on.

- c** State all intervals where:
- i** $f''(x) \geq 0$ [2]
- ii** $f''(x) \leq 0$. [2]
- d** Sketch the graph of the curve $y = f'(x)$. [3]

PAPER 2

CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 4]

The radius of Mercury, the closest planet to the sun in our solar system, is 2440 km. Assuming that Mercury is a sphere, find its volume in cubic metres giving your answer in standard form ($a \times 10^b$ where $1 \leq a < 10$ and $b \in \mathbb{Z}$) correct to three decimal places.

2 [Maximum mark: 6]

In an arithmetic series, $S_6 = 402$ and $S_{12} = 1236$. Find u_1 and d .

3 [Maximum mark: 7]

Charlie raises chickens in a number of different pens and collects data about the weekly number of eggs he collects from each pen.

The data for six of his chicken pens is shown below.

Number of chickens (N)	24	26	28	34	36	40
Weekly number of eggs produced (W)	44	51	57	66	70	82

The relationship between the variables is modelled by the regression line with equation $W = mN + b$.

- a

i

Write down the value of m and of b .

[2]
- ii

Write down the value of the correlation coefficient, r .

[1]
- b

Write down two words that describe the value of r .

[2]
- c

Use your regression line to predict the number of eggs produced in one week for Charlie’s pen that contains 30 chickens.

[2]

4 [Maximum mark: 6]

Let $f(x) = \frac{x^3}{8} - 2x + \ln(10 + e^{x-1})$, $-4 \leq x \leq 4$.

- a

For the graph of $y = f(x)$, find the coordinates of:
- i

the y -intercept

[1]
- ii

the x -intercepts

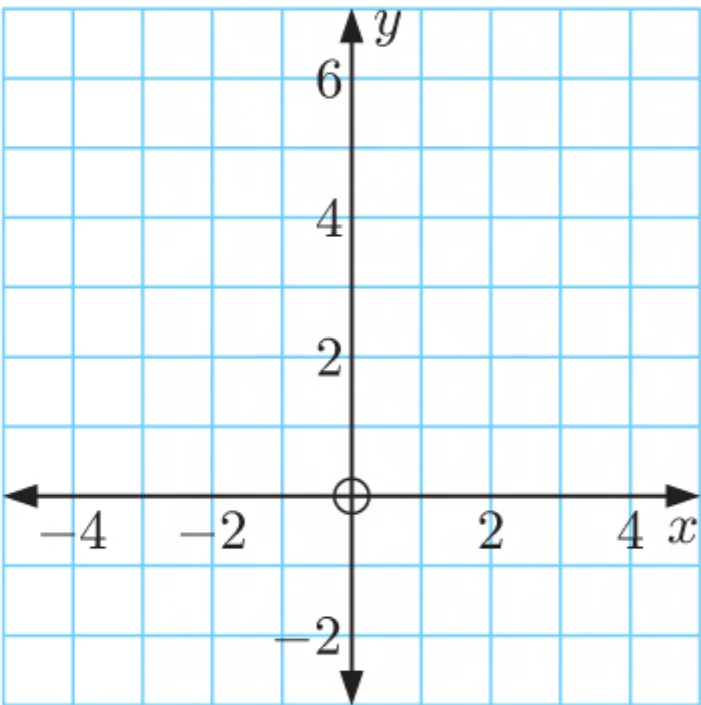
[1]
- iii

all turning points.

[1]
- b

On a grid like the one below, sketch the graph of $y = f(x)$, for $-4 \leq x \leq 4$.

[3]



5 [Maximum mark: 4]

X is a normal random variable with mean 28. Given that $P(X < 36) = 0.833$, find the standard deviation of X .

6 [Maximum mark: 6]

Find the area of the region enclosed by the graphs $f(x) = e^{\cos x}$ and $g(x) = \sin\left(\frac{e^x}{5}\right)$, $0 \leq x \leq \pi$.

7 [Maximum mark: 6]

Consider the function $h(x) = x^2 - \frac{k}{2}x + k - 3$, where $k > 0$, $k \in \mathbb{R}$.

Find the possible values of k such that $y = h(x)$ does not intersect with the x -axis.

SECTION B

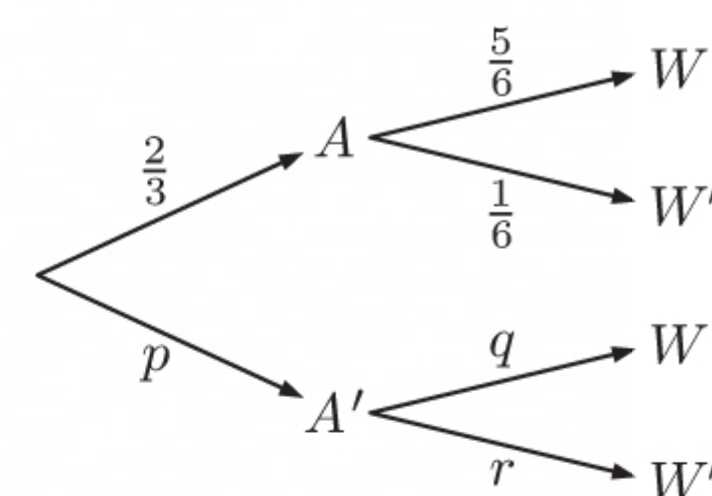
8 [Maximum mark: 14]

Maria uses an alarm clock to wake up for work. The probability that the alarm rings is $\frac{2}{3}$. If the alarm rings, there is a probability of $\frac{5}{6}$ that Maria arrives at work on time. If the alarm does not ring, the probability that Maria arrives at work on time is $\frac{2}{5}$.

Let A represent the alarm clock ringing.

Let W represent Maria arriving at work on time.

Some of this information is represented by the tree diagram.



- Find the value of p , q , and r . [3]
- Find the probability that the alarm rings and Maria is late to work. [2]
- Find the probability that Maria will arrive at work on time. [4]
- Given that Maria arrives at work on time, find the probability that the alarm did not ring. [3]
- In a random sample of 5 days on which Maria goes to work, find the probability that she arrives at work on time on exactly three of those days. [2]

9 [Maximum mark: 11]

A fair game is a game in which the expected profit of any player is zero. A fair coin has two sides; heads (H) and tails (T).

Consider a game in which a player pays \$5 to toss three fair coins and then receives a payment of \$ X , based on the number of tails that occur. The payment is as follows:

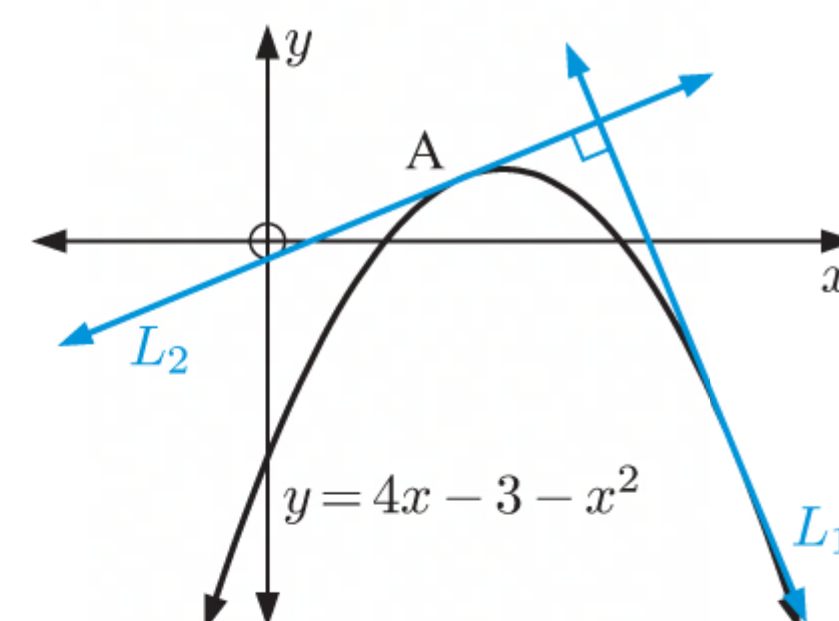
- $X = 0$, if there are no tails;
- $X = 4$, if there is one tail;
- $X = m$, if there are two tails;
- $X = 10$, if there are three tails.

The player's profit is \$ Y , where $Y = X - 5$.

- Show that $E(Y) = \frac{3}{8}(m - 6)$. [7]
- Given that this is a fair game, determine the value of m . [2]
- Determine the profit earned by a player if he tosses 5 tails when playing the game twice. [2]

10 [Maximum mark: 16]

The diagram alongside shows a curve C with equation $y = 4x - 3 - x^2$, its tangent line L_1 at the point where $x = 4$, and its tangent line L_2 at A.



- Find $\frac{dy}{dx}$ for the curve C . [2]
- Show that the gradient of the tangent line L_1 is -4 , and hence find an equation for L_1 in the form $y = ax + b$, where $a, b \in \mathbb{Z}$. [5]
- Show that:
 - the x -coordinate of A is $\frac{15}{8}$ [3]
 - L_2 has equation $y = \frac{1}{4}x + \frac{33}{64}$. [2]
- The region bounded by C , the line L_2 , and the vertical line $x = k$, $k > \frac{15}{8}$, is $\frac{43}{512}$ units².
 - Show that k satisfies $64k^3 - 360k^2 + 675k - 438 = 0$. [3]
 - Hence find k , correct to 3 significant figures. [1]

Trial examination 3

PAPER 1

NO CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 7]

- a Show that for $n \in \mathbb{Z}$,

$$(n - 3)(n)(n + 3) = n^3 - 9n \quad [2]$$

- b Hence, or otherwise, prove that the product of three consecutive multiples of 3 is even. [5]

2 [Maximum mark: 6]

Consider two events, A and B , such that $P(A) = \frac{11}{20}$, $P(A \cap B) = \frac{1}{5}$, and $P(A | B) = \frac{4}{9}$.

- a Find $P(B)$. [2]
 b Find $P(A \cup B)$. [2]
 c Show that the events A and B are **not** independent. [2]

3 [Maximum mark: 5]

Solve the equation for x .

$$\log_4(x - 2) + \frac{1}{2} \log_2(x) = \log_4(15) + \frac{3}{2}$$

4 [Maximum mark: 7]

- a Prove the identity

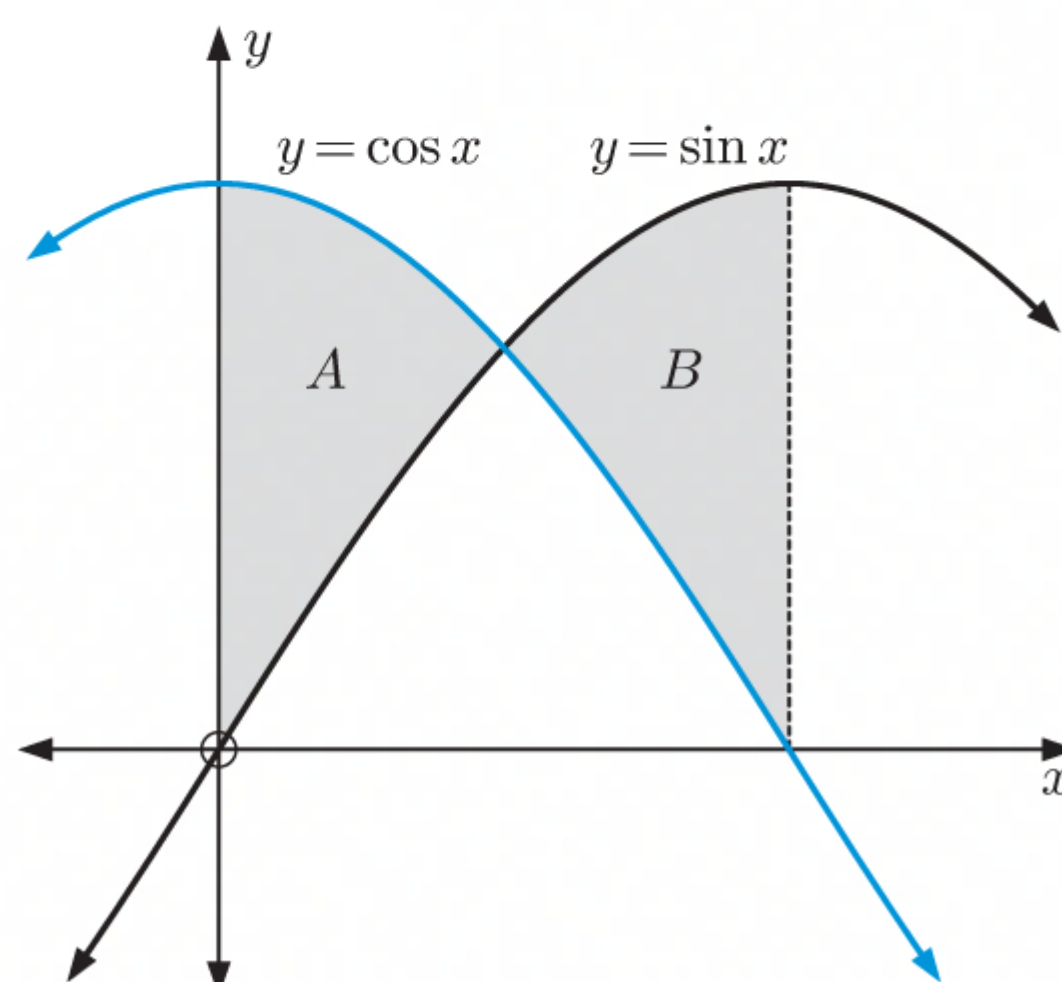
$$\frac{\sin x + \cos x}{\sin x - \cos x} = -\frac{\cos 2x}{1 - \sin 2x} \quad [4]$$

- b Hence, find the **exact** value of [3]

$$\frac{\sin \frac{\pi}{12} + \cos \frac{\pi}{12}}{\sin \frac{\pi}{12} - \cos \frac{\pi}{12}}$$

5 [Maximum mark: 8]

Regions A and B are bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$, as shown below.



- a Write definite integrals to represent the area $A + B$. [4]
 b Calculate the **exact** area $A + B$. [4]

6 [Maximum mark: 7]

A particle moves along a straight line such that its velocity, $v \text{ m s}^{-1}$, at time t seconds is given by

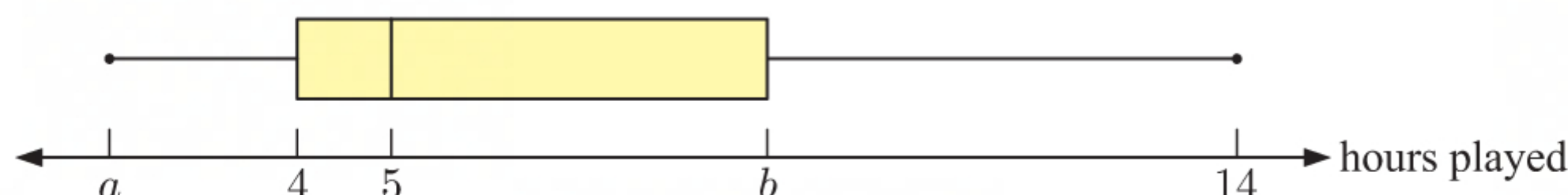
$$v(t) = t^2 - t - 6$$

- a Find the displacement of the particle during the time period $2 \leq t \leq 6$. [3]
 b Find the total distance travelled during this time period. [4]

SECTION B

7 [Maximum mark: 11]

A group of 15 boys recorded the number of hours they spent playing computer games during their 5-day school week. Their results are summarised in the box-and-whisker plot shown below.



- a The range of data is 12. Find the value of a . [2]
- b The group of boys played a total 90 hours of computer games. Find the mean number of hours they played during that 5-day school week. [2]
- c The interquartile range of the data is 5. Find the value of b . [2]

A group of 10 girls also recorded the number of hours they spent playing computer games in the same 5-day school week. Their results are summarised below.

$\bar{x} = 5$	$\sigma = 3$
---------------	--------------

- d Find the number of hours the 10 girls spent playing computer games in the same 5-day school week. [2]
- e Find the mean number of hours **all 25 boys and girls** spent playing computer games during the same 5-day school week. [3]

8 [Maximum mark: 9]

The second and the fourth terms of a geometric sequence are $8 \sin \theta \cos^3 \theta$ and $\sin 2\theta$, respectively.

- a Find the two possible common ratios, in simplest form. [4]
- b Find the two possible first terms. [3]
- c Find an expression for the 8th term. [2]

9 [Maximum mark: 20]

Consider $f(x) = 3x^2 + 18x + c$. The equation $f(x) = 0$ has two equal roots.

- a i Find an expression for the discriminant in terms of c . [2]
- ii Hence, show that $c = 27$. [2]

The graph of $f(x)$ has its vertex on the x -axis.

- b Find the coordinates of the vertex. [4]
- c Write down the solution to the equation $f(x) = 0$. [1]

The function $f(x)$ can be written in the form $f(x) = a(x - h)^2 + k$.

- d Write down the value of:
 - i a [1]
 - ii h [1]
 - iii k . [1]
- e The graph of function $g(x)$ is obtained from the graph of $f(x)$ by a reflection of $f(x)$ in the x -axis, followed by a vertical stretch with scale factor $\frac{1}{3}$.
Find $g(x)$. [4]
- f The gradient of the tangent drawn to $g(x)$ at $x = a$ is 8. Find the value of a . [4]

PAPER 2

CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 6]

Let $f(x) = \frac{6-2x}{4x-5}$ and $g(x) = \frac{5}{3x+3}$

- a Write down the equations of the two asymptotes of $f(x)$. [2]
- b Write down the equation of the vertical asymptote of $g^{-1}(x)$. [1]
- c Find $h(x)$ given that $(h \circ g)(x) = 3x$. [3]

2 [Maximum mark: 6]

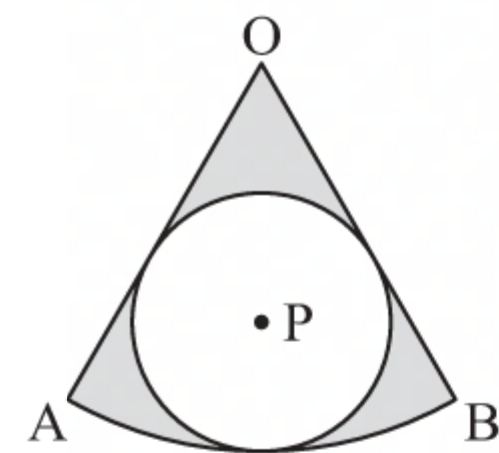
Jose buys a motorcycle during the annual expo in his town. After the 5% discount, he is asked to pay \$17 575.

As he cannot afford the motorcycle at the moment, he takes a loan of \$17 575 for 6 months at a nominal annual interest rate of 15%, **compounded monthly**.

- a Calculate the original price of the motorcycle. [2]
- b Calculate the difference between the *original* price of the motorcycle and the *total amount* Jose will pay. [4]

3 [Maximum mark: 8]

The diagram shows sector OAB of a circle with centre O for which $OA = 6$ and $\widehat{AOB} = \frac{\pi}{3}$ radians. A circle with centre P, inside the sector, is drawn such that it touches OA, OB, and the arc AB. The area outside the circle and inside the sector is shaded.

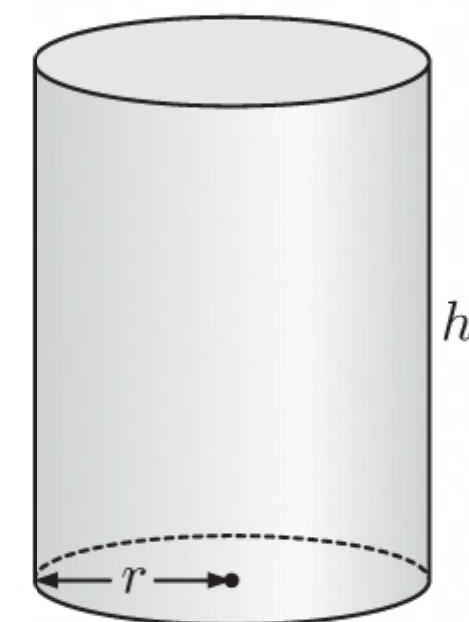


- a Find the **exact** area of sector OAB. [2]
- b Find the radius of the circle with centre P. [3]
- c Find the area of the shaded region. [3]

4 [Maximum mark: 8]

Mark makes a fully closed metal container in the shape of a cylinder, as shown in the diagram (not drawn to scale). The container has radius r metres, height h metres, and its volume is 1.3 cubic metres.

Let $A(r)$ represent the surface area of the container, in terms of the radius.



- a Show that $A(r) = 2\pi r^2 + \frac{2.6}{r}$. [3]
- b Find $\frac{dA}{dr}$. [2]
- c Given that the surface area of the fully closed metal container is minimised, find the radius, r , and the height, h , of the container. [3]

5 [Maximum mark: 7]

The duration of direct flights from Shanghai, China to Paris, France in a particular year followed a normal distribution with mean μ and standard deviation σ . It is known that 91% of the flights took under 12 hours 45 minutes and 95.5% of the flights took over 11 hours 45 minutes. Find μ and σ , rounded to the nearest minute.

6 [Maximum mark: 5]

A piece of artwork is created by randomly using 500 small tiles, each of which is either red or blue. The probability that a small red tile is used is 0.15. Let the random variable R represent the number of small red tiles used in a piece of artwork.

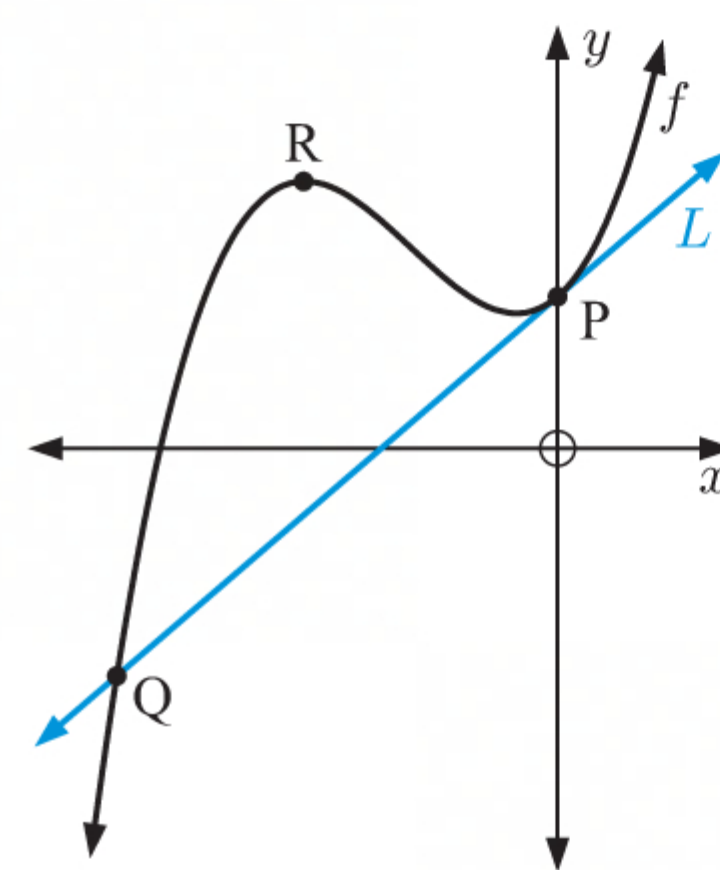
- a State the distribution of the random variable R , including the values of any parameters. [2]
- b Write down the mean of R . [1]
- c Find $P(R > 78)$. [2]

SECTION B

7 [Maximum mark: 20]

Let $f(x) = 2x^3 + 5x^2 + ax + 2$. Part of the graph of $f(x)$ is shown in the diagram.

The graph of $f(x)$ crosses the y -axis at point P. The line L is the tangent to the graph of $f(x)$ at P.



a Find the coordinates of P. [2]

b i Find $f'(x)$ in terms of a . [2]

ii Hence find the equation of L in terms of a . [4]

Line L intersects the graph of $f(x)$ at Q, as shown.

c Find the x -coordinate of Q. [2]

Line L intersects the x -axis at $(-1, 0)$.

d Find:

i the value of a [2]

ii the y -coordinate of Q. [2]

e The graph of $f(x)$ has a local maximum at point R, as shown. Find the coordinates of R. [6]

8 [Maximum mark: 20]

Consider the triangle ABC for which $\widehat{BAC} = 60^\circ$, $AB = (x - 2)^2$, and $AC = 2\sqrt{3}x$, where $x > 0$, $x \neq 2$.

a Show that the area, $A \text{ cm}^2$, of the triangle is given by [2]

$$A = \frac{3}{2}x^3 - 6x^2 + 6x$$

b i Write down $\frac{dA}{dx}$. [3]

ii Show that $\frac{dA}{dx} = 0$ when $x = \frac{2}{3}$. [2]

c i Find $\frac{d^2A}{dx^2}$. [2]

ii Hence justify that $x = \frac{2}{3}$ gives the maximum area for the triangle ABC. [2]

d Find the **exact** values of AB and AC for which the area of the triangle ABC is maximum. [4]

e Find the maximum area of the triangle ABC. [2]

f Find BC when the area of the triangle ABC is maximum. [3]

Trial examination 4

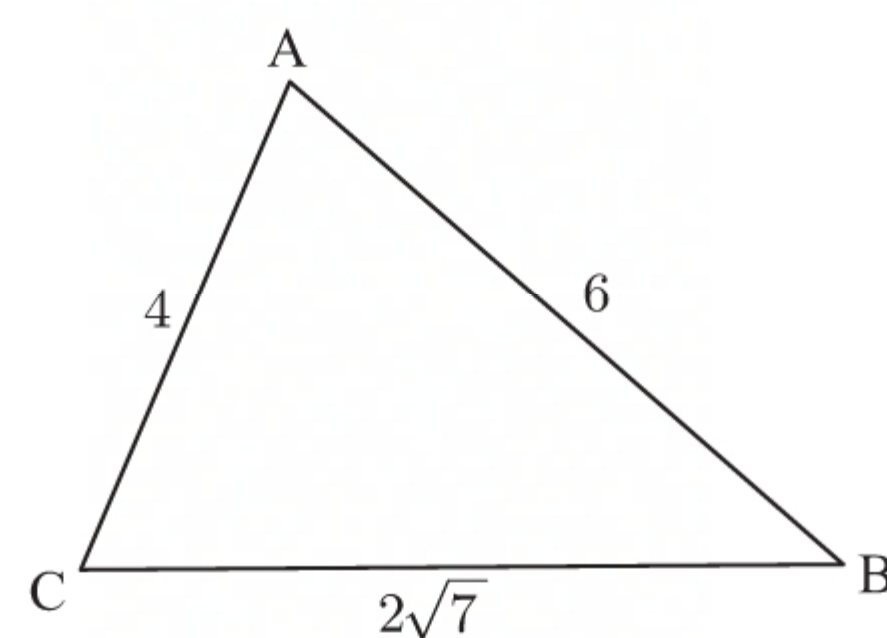
PAPER 1

NO CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 6]

In triangle ABC, $AB = 6$, $AC = 4$, and $BC = 2\sqrt{7}$.



a Find the measure of \widehat{BAC} in degrees.

[4]

b Hence find the area of the triangle.

[2]

2 [Maximum mark: 5]

X and Y are independent events such that $P(X \cap Y) = \frac{1}{5}$ and $P(X' \cap Y) = \frac{1}{2}$. Find $P(X \cup Y)$.

3 [Maximum mark: 4]

Consider the three digit number “ abc ”.

Prove that if $a + b + c$ is divisible by 3, then “ abc ” is also divisible by 3.

4 [Maximum mark: 6]

At the point where $x = 0$, the tangent to $f(x) = e^{\sin kx} + c$ has equation $y = -x + 3$.

Find the value of:

a c

[3]

b k

[3]

5 [Maximum mark: 5]

Consider the functions $f : x \mapsto 3 - x$ and $g : x \mapsto \frac{2x+1}{3}$.

a Show that $g^{-1}(x) = \frac{3x-1}{2}$.

[2]

b Solve for x : $(f \circ g^{-1})(x) = 4$.

[3]

6 [Maximum mark: 5]

Solve for x : $9^x + 18 = 3^{x+2}$

SECTION B

7 [Maximum mark: 15]

A nursery wants to compare the growth of seedlings under different lighting conditions. Two samples A and B are compared after 30 days.

a The heights (in cm) of the sample A seedlings are:

8.5	10.1	14.2	7.8	11.3	9.6	10.0	13.2
9.9	6.8	12.1	8.7	11.5	10.3	8.2	

i Find the five-number summary for this data.

[4]

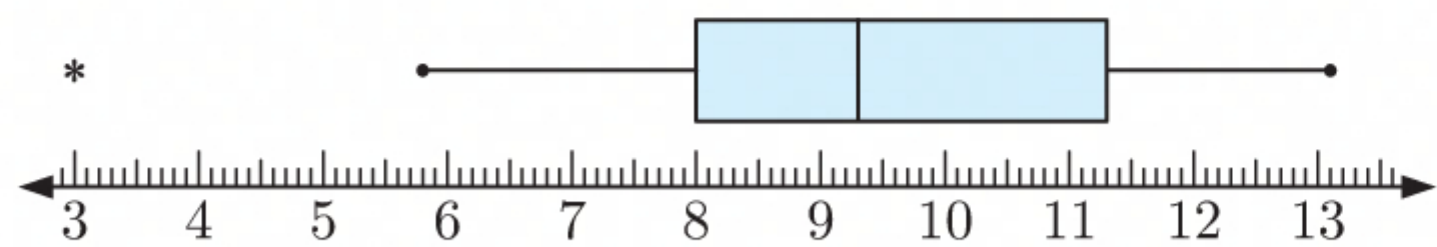
ii Discuss whether there are any outliers in the data.

[3]

iii Construct a box plot to display the data.

[3]

b The box plot for the sample B seedlings is:



- i Find the range and interquartile range for this sample. [2]
- ii Compare the two samples to decide which growing conditions are more favourable. [3]

8 [Maximum mark: 19]

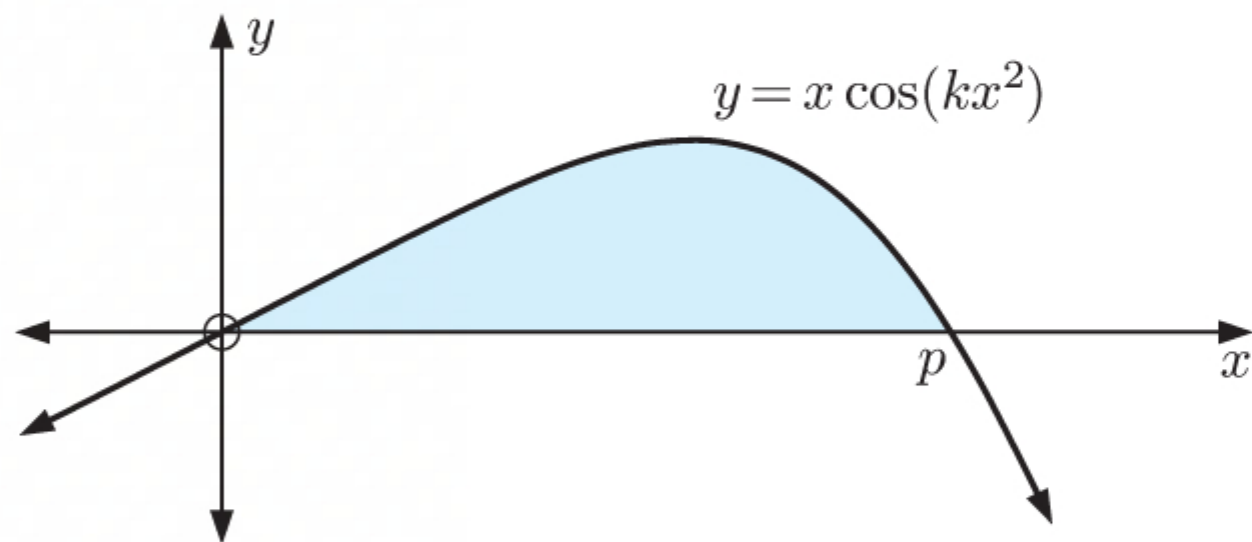
Suppose $f(x) = x^3 - 6x^2 + 9x - 2$.

- a Find $f'(x)$ and draw its sign diagram. [4]
- b Find and describe the turning points of $y = f(x)$. [2]
- c Find the inflection point of the function. [3]
- d i Find the equation of the normal to the curve at the inflection point. [3]
- ii Find the exact x -coordinates of the points where the normal meets the curve again. [7]

9 [Maximum mark: 15]

Consider the function $f(x) = x \cos(kx^2)$, $0 < k < 2\pi$. The tangent to $y = f(x)$ at $x = 1$ passes through the origin.

- a Find the value of k . [6]
- b The graph of $y = f(x)$ is shown below. Find the value of p . [2]



- c Find $\int f(x) dx$ and hence find the shaded area. [7]

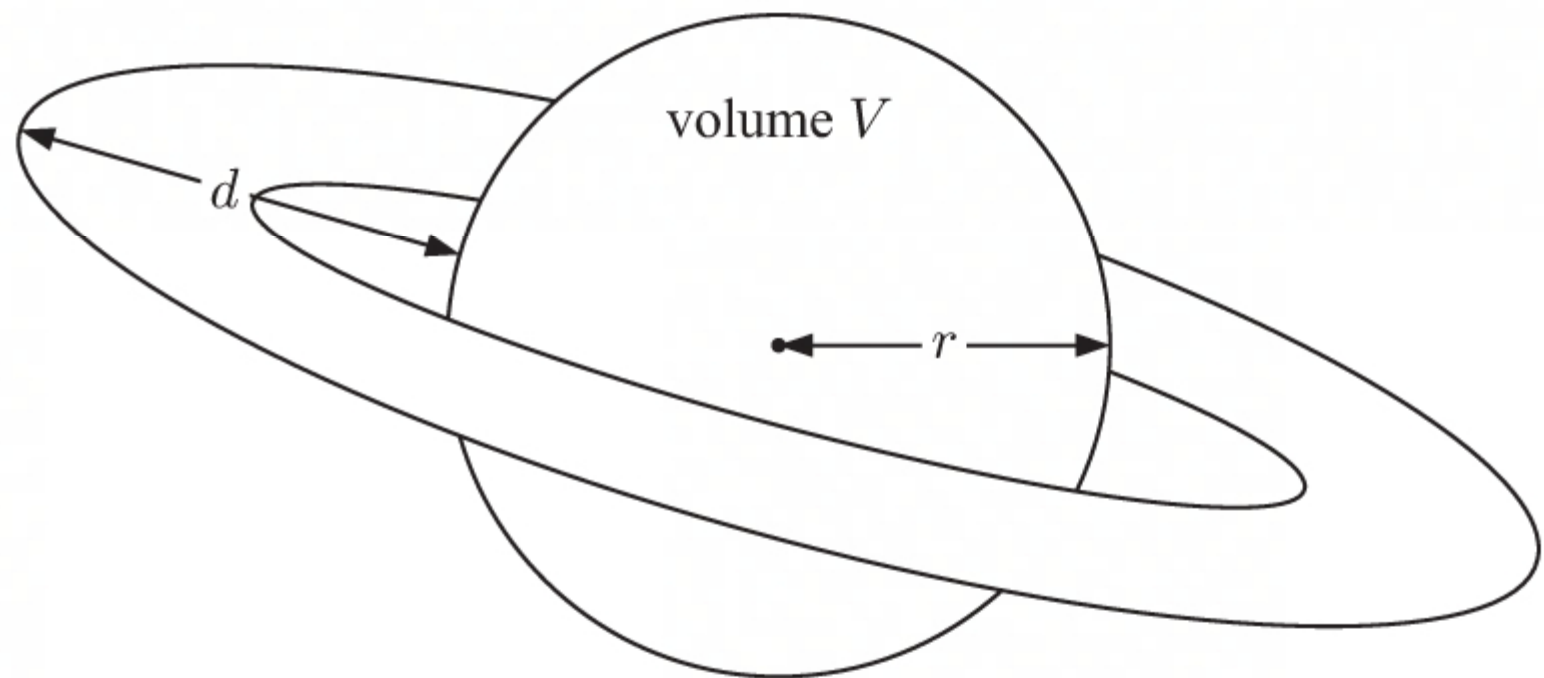
PAPER 2

CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 5]

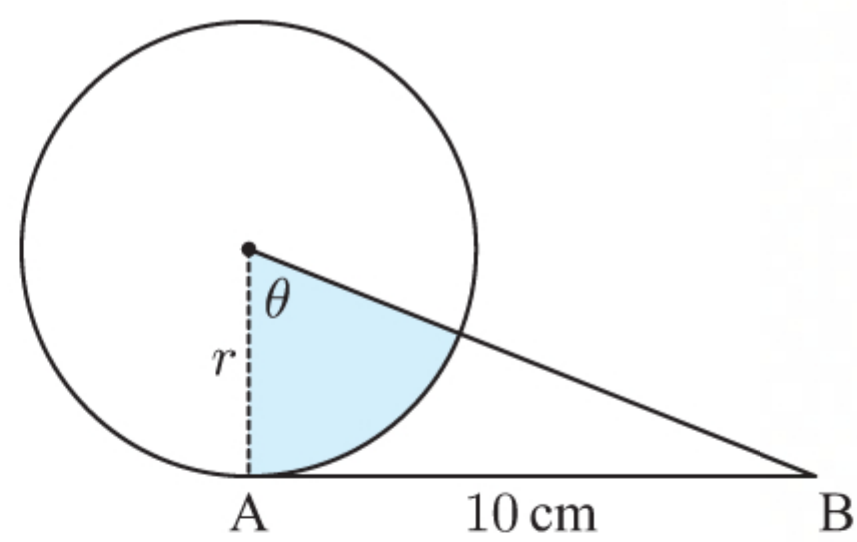
The data below is given by NASA for the planet Saturn.



Equatorial radius	$r \approx 5.8232 \times 10^4 \text{ km}$
Volume	$V \approx 8.2713 \times 10^{14} \text{ km}^3$
Ring system	up to $d \approx 2.82 \times 10^5 \text{ km}$ from planet

- a Use the equatorial radius given to estimate the volume of Saturn. [2]
- b Estimate the circumference of the orbit of an ice-covered rock at the far edge of Saturn's rings. [3]

2 [Maximum mark: 7]



AB is a tangent to the circle. The shaded area is 20 cm^2 . Find the radius of the circle r , and the angle at its centre θ in radians.

3 [Maximum mark: 8]

Portia has €10 000 she wants to invest. She is given two options:

- A: 5% per annum simple interest paid quarterly.
- B: 4.4% per annum interest compounded quarterly.

- a Write a formula for the value of the simple interest investment after n quarters. [2]
- b i Find a formula for the value of the compound interest investment after n quarters. [2]
ii Hence find the value of this investment after 7 quarters. [1]
- c Find how long Portia would need to invest her money, for the compound interest investment to be the better option. [3]

4 [Maximum mark: 5]

- a A pair of dice is rolled. Find the probability that the sum of the dice is 5. [2]
- b A pair of dice is rolled 10 times. Find the probability that their sum will be 5 at least twice. [3]

5 [Maximum mark: 6]

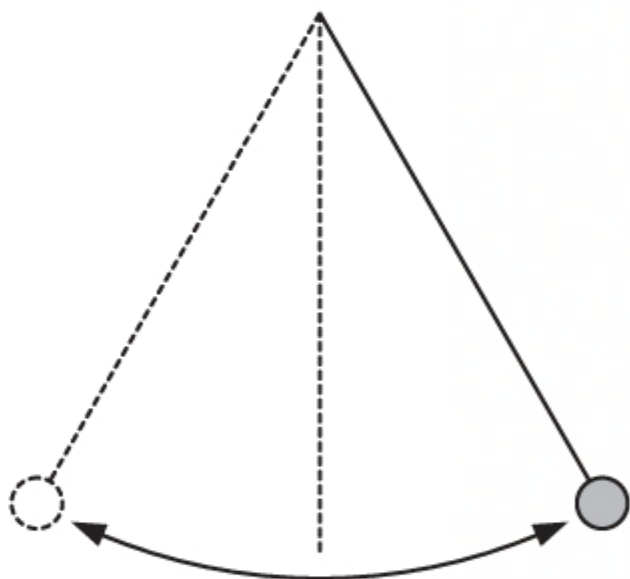
Jody asks 8 office workers in a big city about the distance they travel to work each day, and the time it takes them to get there. The workers record their journeys to work by GPS.

Distance travelled to work (x km)	2.2	6.8	15.4	3.1	5.6	9.0	17.2	1.4	4.1
Travel time (y minutes)	16	27	43	14	26	32	61	12	19

- a Calculate Pearson’s correlation coefficient r for the data. [1]
- b Explain why x is the independent variable. [1]
- c What Jody actually wants to know is the relationship between the straight-line distance the workers live from the office, and the time it takes them to get to work.
 - i If Jody is to use the data she has collected, explain why it is more appropriate to use the x against y regression line. [1]
 - ii Calculate the x against y regression line for the given data. [2]
 - iii Hence estimate the distance a worker lives from the office, if it takes them 50 minutes to get to work. [1]

6 [Maximum mark: 7]

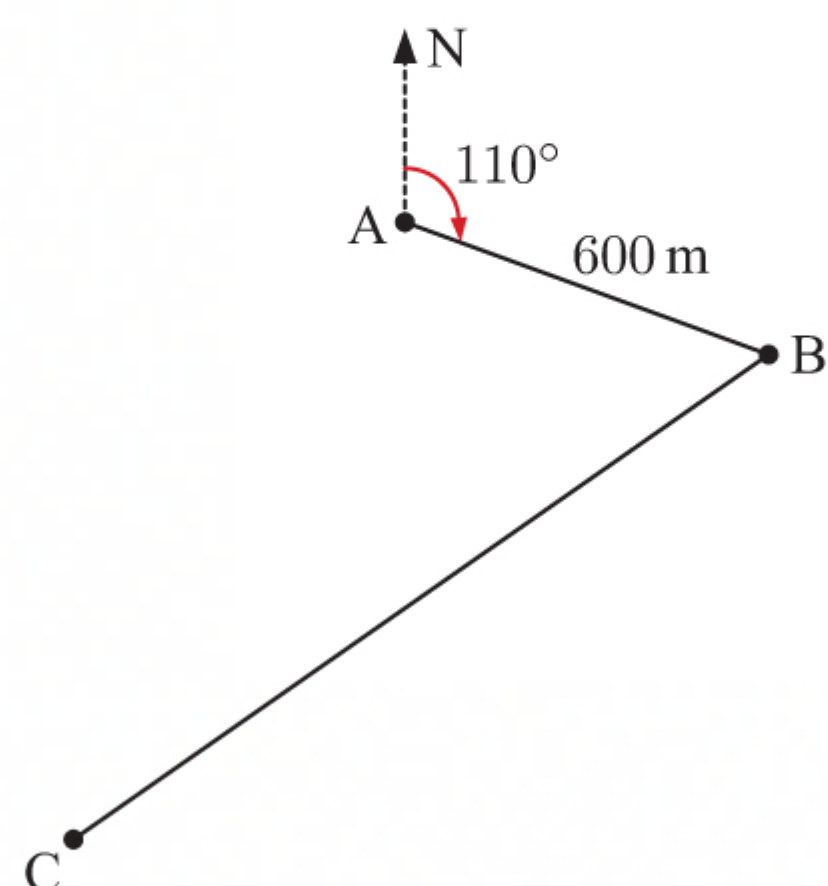
When a pendulum is released, its velocity along the arc of motion is given by $v(t) = -32 \sin 2t \text{ cm s}^{-1}$.



- a Find the initial acceleration of the pendulum. [2]
- b Find the length of the arc along which the pendulum travels. [3]
- c Hence find the average speed of the pendulum. [2]

SECTION B

7 [Maximum mark: 12]



Alan runs at 3 m s^{-1} from A to B. It takes him 200 seconds to reach B. At exactly the same time, Belinda starts cycling at 8 m s^{-1} on the bearing 230° from B to C.

- Find \widehat{ABC} . [2]
- Find the distance Alan is from B t seconds after he leaves A, $0 \leq t \leq 200$. [2]
- Find the distance Belinda is from B t seconds after she leaves B, $0 \leq t \leq 200$. [2]
- Hence write a formula for the distance between Alan and Belinda after t seconds, $0 \leq t \leq 200$. [3]
- Find the minimum distance between Alan and Belinda, and the time when this occurs. [3]

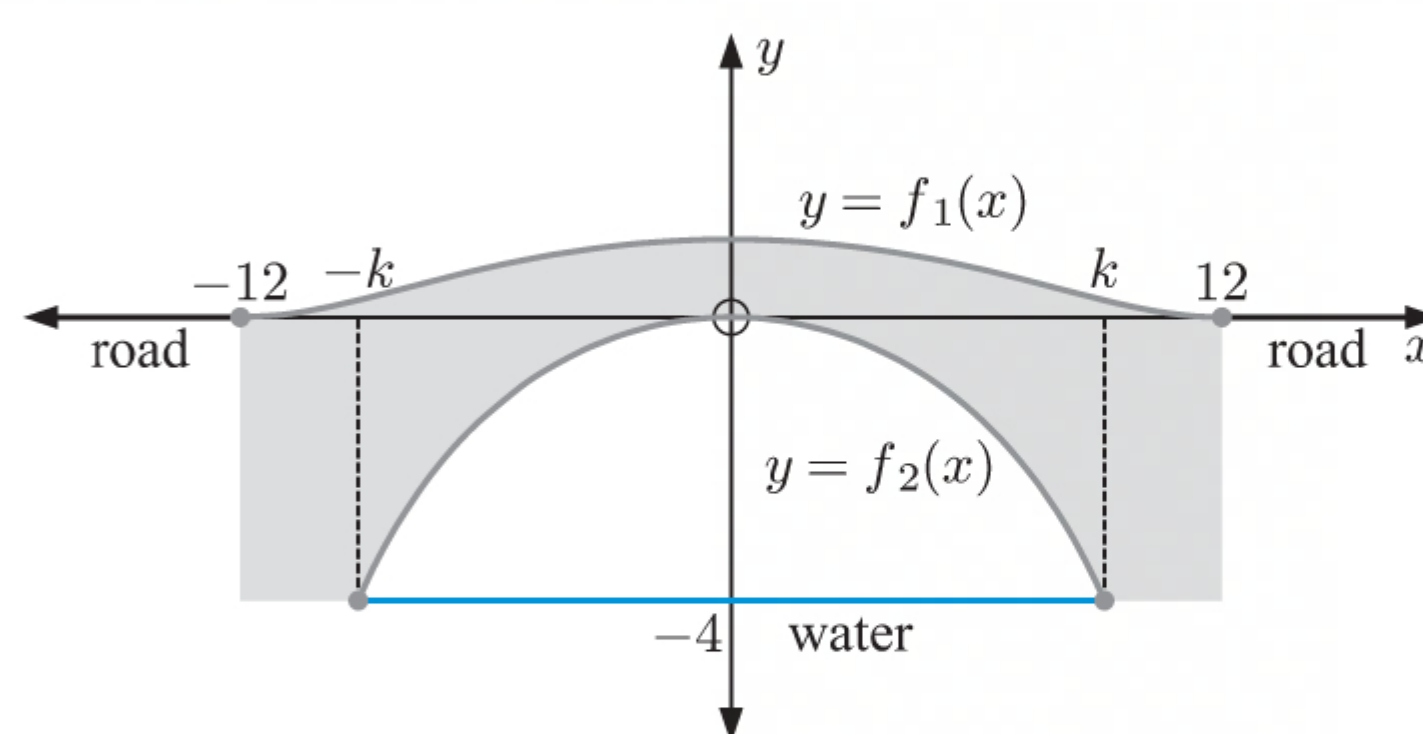
8 [Maximum mark: 11]

A large sample of fallen acorns is collected from a forest. 18% of acorns are longer than 4.6 cm, and 15% of acorns are shorter than 2.8 cm.

- Suppose the population of acorns is normally distributed with mean μ and standard deviation σ .
 - Find *two* linear equations connecting μ and σ . [4]
 - Hence find μ and σ . [2]
- A random sample of 12 fallen acorns is chosen. Let Y represent the number of acorns longer than 4 cm. Find $E(Y)$. [5]

9 [Maximum mark: 19]

An arched bridge over a river is shown in the diagram. x and y are both in metres.



The defining functions are $f_1(x) = \ln\left(\cos \frac{\pi x}{12} + 2\right)$, $-12 \leq x \leq 12$
and $f_2(x) = 2 \ln\left(\cos \frac{\pi x}{12} + 1\right) + a$, $-k \leq x \leq k$.

- Find the value of a . [2]
- Show that $\cos \frac{k\pi}{12} = \frac{2}{e^2} - 1$. [4]
 - Hence find k . [1]
- Find exactly the maximum gradient of the road, and when this occurs. [8]
- Find the shaded cross-sectional area of the bridge. [4]

Trial examination 5

PAPER 1

NO CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 5]

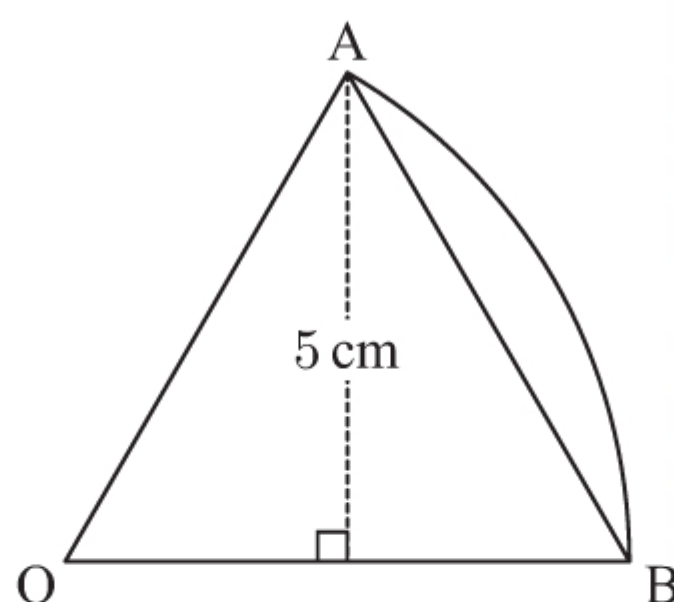
The probability distribution of a discrete random variable, X , is given by the following table, where A and p are constants.

x	A	5	7	9
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{9}$	p

- a** Find the value of p . [2]
b Given that $E(X) = 6$, find the value of A . [3]

2 [Maximum mark: 5]

The diagram shows a sector AOB of a circle with centre O and radius r . The triangle AOB is equilateral and has perpendicular height 5 cm.



- a** Show that the radius of the sector is $\frac{10\sqrt{3}}{3}$ cm. [2]
b Find in terms of π , the perimeter P of the sector. [3]
- #### 3 [Maximum mark: 5]
- Events A and B are such that $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$, and $P(A | B) = \frac{1}{4}$.
- a** Find $P(A \cap B)$. [2]
b Find $P(A \cup B)$. [2]
c State with a reason whether or not events A and B are independent. [1]

4 [Maximum mark: 7]

The function $f(x) = \frac{4x-1}{ax+b}$ has asymptotes $x = \frac{1}{2}$ and $y = 2$.

- a** Find the values a and b . [2]
b Find $f^{-1}(x)$. [3]
c Find the domain and range of $f^{-1}(x)$. [2]

5 [Maximum mark: 7]

- a** Expand and simplify $(2n+1)^3$. [3]
b Hence show that if k is an odd number, $k^3 + 3k^2 - k$ is always 3 more than a multiple of 8. [4]

6 [Maximum mark: 5]

Find $\int \frac{e^x}{(e^x - 1)^2} dx$.

SECTION B

7 [Maximum mark: 18]

Let $f(x) = kx(x+1)^2$ where $k > 0$ is a constant.

- a Show that $f'(x) = 3kx^2 + 4kx + k$. [3]
- b Hence show that $f(x)$ has two turning points at $x = -1$ and $x = -\frac{1}{3}$, and state the coordinates at these points. [4]
- c By finding $f''(x)$, classify the nature of each turning point. [3]
- d Find the values of x for which $f(x)$ is strictly increasing. [3]
- e The area between $y = f(x)$ and the x -axis for $-1 \leq x \leq 0$ is 0.5 units². Find the value of k . [5]

8 [Maximum mark: 14]

Let $\sin \theta = \frac{\sqrt{3}}{5}$ where θ is obtuse.

- a Find $\cos \theta$. [3]
- b Find $\cos 2\theta$. [2]

Now consider the function $f(x) = 1 + \log_2(\tan x) + \log_2(1 - \sin^2 x)$.

- c Show that $f(x) = \log_2(\sin 2x)$. [4]
- d Hence solve: $1 + \log_2(\tan x) + \log_2(1 - \sin^2 x) = -1$ for $0 < x < 2\pi$. [5]

9 [Maximum mark: 14]

The first three terms of an infinite geometric sequence are: $\log(x^3)$, $\log x$, $\log(x^{\frac{1}{3}})$ where $x > 0$.

The first three terms of an arithmetic sequence are: $\log x$, $\log\left(\frac{x}{3}\right)$, $\log\left(\frac{x}{9}\right)$ where $x > 0$.

- a Find the common ratio of the geometric sequence. [2]
- b Find an expression for the sum to infinity of the geometric sequence, S_G , giving your answer in the form $p \log x$, where $p \in \mathbb{Q}$. [3]
- c Find the common difference of the arithmetic sequence. [3]
- d Find an expression for the sum of the first 10 terms of the arithmetic sequence, S_{10} . [3]
- e Given that S_{10} from part d is equal to $\frac{2}{9}S_G$, find x , giving your answer in the form 3^q where $q \in \mathbb{Q}$. [3]

PAPER 2

CALCULATOR, 90 MINUTES

SECTION A

1 [Maximum mark: 7]

Peter has a 35% chance of winning a game of table tennis. He enters a tournament, in which he plays 15 games. Let X be the number of games that Peter wins in the tournament.

- a Find the probability that Peter wins at least half of his games. [3]
- b If μ and σ denote the mean and standard deviation of X , respectively, find $P(\mu - \sigma < X < \mu + \sigma)$. [4]

2 [Maximum mark: 6]

A particle P moves in a straight line with velocity function $v(t) = (2t + 3)(5t^{\frac{3}{2}} + 8)$ m/s at time t (in seconds), where $t \geq 0$. The particle is initially located at the origin.

- a Find the displacement function $s(t)$. [4]
- b Find the initial acceleration of P. [2]

3 [Maximum mark: 5]

The times taken for a group of runners to complete a half-marathon are found to follow a normal distribution with mean μ and standard deviation σ . Given that 24% of runners complete the race in less than 80 minutes and one in three runners take more than 120 minutes, find the values of μ and σ .

4 [Maximum mark: 7]

A manufacturer is producing paper cups that are able to hold 240 cm^3 of liquid when filled to the top. The cups are to be modelled as an open-topped cylinder with base radius r cm.

- a** Show that the total surface area $A \text{ cm}^2$ of paper required per cup is $A = \frac{480}{r} + \pi r^2$. [3]
- b** Show that the surface area of paper required per cup is minimised when $r = h$. [4]

5 [Maximum mark: 5]

Find the exact solutions to the equation $3(4^x) - 10(2^x) = -3$.

6 [Maximum mark: 6]

Consider the functions $f(x) = x^3 e^{-x}$ and $g(x) = \ln x$.

- a** Use calculus to prove that $f(x)$ has two stationary points. [4]
- b** Solve the equation $f(x) = g(x)$. [2]

SECTION B**7 [Maximum mark: 15]**

The functions $f(x) = x^3 - x$ and $g(x) = e^{2x}$ are defined for all $x \in \mathbb{R}$.

The function $g(x)$ is translated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then vertically stretched with scale factor 3, then further translated by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. The resultant function is denoted $h(x)$.

- a** Find the algebraic form of the function $h(x)$. [3]
- b** Find the exact coordinates of where the graph $y = h(x)$ crosses the x -axis. [4]
- c** Find the equation of the normal to the function $y = f(x)$ at $x = 2$. [4]
- d** Write down the composite function $gf(x)$, and find the total area of the regions enclosed by $y = f(x)$ and $y = gf(x) - 1$. [4]

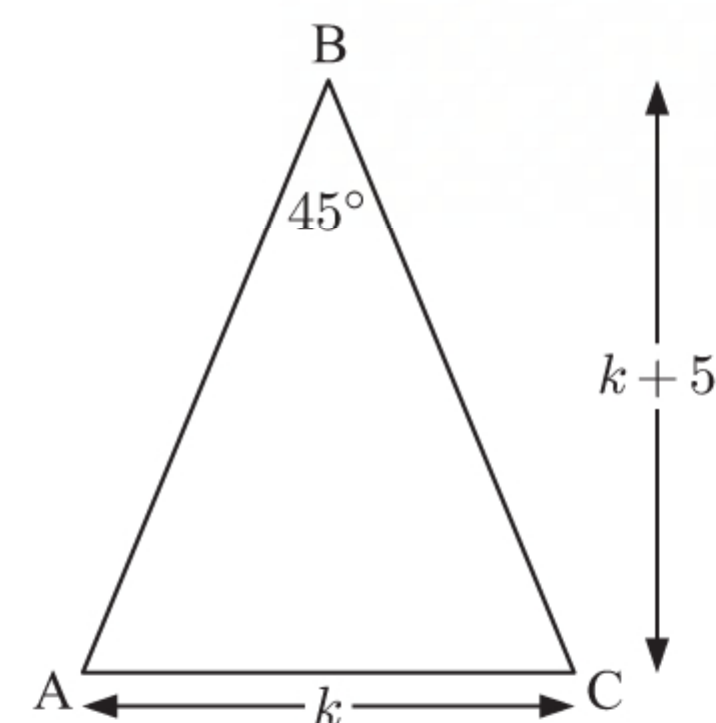
8 [Maximum mark: 14]

Consider the functions $f(x) = (2x - 1)(x + 3)$ and $g(x) = x^2 + (a + 1)x + a$, where $a \in \mathbb{Z}$.

- a** State the coordinates of the x -intercepts and y -intercept of f . [2]
- b** Find the equation of the axis of symmetry of f . [2]
- c** Hence, or otherwise, find the exact coordinates of the turning point of f . [2]
- d** Find the value of a for which g has two equal roots. [3]
- e** Using the value of a from **d**, prove algebraically that the graphs of $y = f(x)$ and $y = g(x)$ intersect at $x = -4$ and $x = 1$. Hence, find the values of x for which $f(x) < g(x)$. [5]

9 [Maximum mark: 15]

- a** Use the fact that $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$ to show that $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$. [2]
- b** Given that $\tan 135^\circ = -1$, show that $\tan 67.5^\circ = 1 + \sqrt{2}$. [4]
- c** An isosceles triangle ABC is such that $AB = BC$ and $\widehat{ABC} = 45^\circ$. The triangle has base length k and height $k + 5$.



- i** Show that $\tan \widehat{BCA} = \frac{2k + 10}{k}$. [2]
- ii** Hence find k , giving your answer in the form $m + n\sqrt{2}$, $m, n \in \mathbb{Z}$. [4]
- iii** Find the exact area of triangle ABC. [3]

Worked solutions

TOPIC 1 SKILL BUILDER QUESTIONS

$$\begin{aligned} 1 \quad \mathbf{a} \quad (-3m^3)^4 &= (-3)^4 \times (m^3)^4 \\ &= 81m^{12} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \left(\frac{xy^2}{2}\right)^5 &= \frac{(xy^2)^5}{2^5} \\ &= \frac{x^5y^{10}}{32} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 7s^2t \times (4st^3)^3 &= 7s^2t \times 4^3s^3t^9 \\ &= 7 \times 64s^5t^{10} \\ &= 448s^5t^{10} \end{aligned}$$

$$\begin{aligned} 2 \quad \mathbf{a} \quad 4^0 + 4^{-1} &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \left(2\frac{3}{4}\right)^{-2} &= \left(\frac{11}{4}\right)^{-2} \\ &= \left(\frac{4}{11}\right)^2 \\ &= \frac{4^2}{11^2} \\ &= \frac{16}{121} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2^2 + 2^1 + 2^{-1} &= 4 + 2 + \frac{1}{2} \\ &= 6 + \frac{1}{2} \\ &= \frac{13}{2} \end{aligned}$$

$$\begin{aligned} 3 \quad \mathbf{a} \quad (x^2 + x^{-2})^2 &= (x^2)^2 + 2x^2 \times x^{-2} + (x^{-2})^2 \\ &= x^4 + 2 + x^{-4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x^4 - x^2)(x^3 + 3) &= x^4 \times x^3 + 3x^4 + (-x^2) \times x^3 + (-x^2) \times 3 \\ &= x^7 + 3x^4 - x^5 - 3x^2 \end{aligned}$$

$$\begin{aligned} 4 \quad \mathbf{a} \quad a^2b^{-3} &= a^2 \times \frac{1}{b^3} \\ &= \frac{a^2}{b^3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{2m^{-2}n^3}{m^5n^{-5}} &= 2 \times m^{-2-5} \times n^{3-(-5)} \\ &= 2 \times m^{-7} \times n^8 \\ &= 2 \times \frac{1}{m^7} \times n^8 \\ &= \frac{2n^8}{m^7} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{12a^{-3}}{b^{-5}} &= 12 \times a^{-3} \times \frac{1}{b^{-5}} \\ &= 12 \times \frac{1}{a^3} \times b^5 \\ &= \frac{12b^5}{a^3} \end{aligned}$$

5 Using technology:

$$\mathbf{a} \quad (3.57 \times 10^6) \times (2.38 \times 10^3) = 8.4966 \times 10^9$$

$$\mathbf{b} \quad \frac{4.61 \times 10^{-7}}{3.45 \times 10^8} \approx 1.34 \times 10^{-15}$$

$$\mathbf{c} \quad (0.000\,08)^4 = 4.096 \times 10^{-17}$$

$$\begin{aligned} 6 \quad \mathbf{a} \quad \text{Amount of fluoride} &= \text{concentration} \times \text{volume} \\ &= (3 \times 10^{-4}) \times (5.6 \times 10^8) \\ &= 1.68 \times 10^5 \text{ g} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Volume} &= \frac{\text{amount of fluoride}}{\text{concentration}} \\ &= \frac{4.13 \times 10^7}{3 \times 10^{-4}} \\ &\approx 1.38 \times 10^{11} \text{ litres} \end{aligned}$$

7 **a** If 3, k , 11 are consecutive terms of an arithmetic sequence, then

$$k - 3 = 11 - k \quad \{\text{equating differences}\}$$

$$\therefore 2k = 14$$

$$\therefore k = 7$$

b If -2 , $k + 4$, $k^2 + 11$ are consecutive terms of an arithmetic sequence, then

$$k + 4 - (-2) = k^2 + 11 - (k + 4) \quad \{\text{equating differences}\}$$

$$\therefore k + 6 = k^2 + 11 - k - 4$$

$$\therefore k^2 - 2k + 1 = 0$$

$$\therefore (k - 1)^2 = 0$$

$$\therefore k = 1$$

- c** If $k - 5$, $2k$, $2k^2$ are consecutive terms of an arithmetic sequence, then

$$2k - (k - 5) = 2k^2 - 2k \quad \{\text{equating differences}\}$$

$$\therefore k + 5 = 2k^2 - 2k$$

$$\therefore 2k^2 - 3k - 5 = 0$$

$$\therefore 2k^2 + 2k - 5k - 5 = 0$$

$$\therefore 2k(k + 1) - 5(k + 1) = 0$$

$$\therefore (k + 1)(2k - 5) = 0$$

$$\therefore k = -1 \text{ or } k = \frac{5}{2}$$

8 a Average mass = $\frac{\text{total mass} - \text{mass of cage}}{\text{number of hamsters}}$

$$= \frac{1400 - 800}{5}$$

$$= \frac{600}{5}$$

$$= 120 \text{ g}$$

b $u_n = 800 + 120n$

9 a $u_5 = u_1 r^4 = 324 \quad \dots (1)$

and $u_{10} = u_1 r^9 = 78\,732 \quad \dots (2)$

Now $\frac{u_1 r^9}{u_1 r^4} = \frac{78\,732}{324} \quad \{(2) \div (1)\}$

$$\therefore r^5 = 243$$

$$\therefore r = \sqrt[5]{243}$$

$$\therefore r = 3$$

Using (1), $u_1(3)^4 = 324$

$$\therefore 81u_1 = 324$$

$$\therefore u_1 = 4$$

Thus $u_n = 4 \times 3^{n-1}$

b $u_8 = u_1 r^7 = -10 \quad \dots (1)$

and $u_{12} = u_1 r^{11} = -160 \quad \dots (2)$

Now $\frac{u_1 r^{11}}{u_1 r^7} = \frac{-160}{-10} \quad \{(2) \div (1)\}$

$$\therefore r^4 = 16$$

$$\therefore r = \pm \sqrt[4]{16}$$

$$\therefore r = \pm 2$$

If $r = 2$, then using (1), $u_1(2)^7 = -10$

$$\therefore 128u_1 = -10$$

$$\therefore u_1 = \frac{-10}{128} = -\frac{5}{64}$$

Thus $u_n = -\frac{5}{64} \times 2^{n-1}$

If $r = -2$, then using (1), $u_1(-2)^7 = -10$

$$\therefore -128u_1 = -10$$

$$\therefore u_1 = \frac{10}{128} = \frac{5}{64}$$

Thus $u_n = \frac{5}{64} \times (-2)^{n-1}$

10 $2, 2\sqrt{3}, 6, 6\sqrt{3}$

a $\frac{2\sqrt{3}}{2} = \sqrt{3}, \quad \frac{6}{2\sqrt{3}} = \frac{\cancel{2} \times 3}{\cancel{2}\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}, \quad \frac{6\sqrt{3}}{6} = \sqrt{3}$

Consecutive terms have a common ratio of $\sqrt{3}$.

\therefore the sequence is geometric with $u_1 = 2$ and $r = \sqrt{3}$.

b $u_n = u_1 r^{n-1}$

$$= 2(\sqrt{3})^{n-1}$$

$$= 2 \times 3^{\frac{n-1}{2}}$$

c $u_{10} = 2 \times 3^{\frac{10-1}{2}}$

$$= 2 \times 3^{\frac{9}{2}}$$

$$= 2 \times 3^4 \times 3^{\frac{1}{2}}$$

$$= 2 \times 81 \times \sqrt{3}$$

$$= 162\sqrt{3}$$

d We need to find n such that $u_n = 2 \times 3^{\frac{n-1}{2}} > 1000$.

Using a graphics calculator with $Y_1 = 2 \times 3 \wedge ((X - 1) \div 2)$, we view a table of values:

Math (Rad) (Norm) [d/c] (Real)	
Y1=2*3^(((X-1)/2))	
X	Y1
11	486
12	841.77
13	1458
14	2525.3

The first term to exceed 1000 is $u_{13} = 1458$.

- 11** There is a fixed percentage increase each year, so the population forms a geometric sequence with $u_0 = 217$ and $r = 1.42$.

\therefore the population after n years is $u_n = 217 \times (1.42)^n$.

$$\begin{aligned} \text{a i } u_5 &= 217 \times (1.42)^5 \\ &\approx 1252.86 \end{aligned}$$

The expected population size after 5 years is approximately 1250 birds.

$$\begin{aligned} \text{ii } u_{10} &= 217 \times (1.42)^{10} \\ &\approx 7233.41 \end{aligned}$$

The expected population size after 10 years is approximately 7230 birds.

$$\begin{aligned} \text{b } 217 \times (1.42)^n &= 30\,000 \\ \therefore (1.42)^n &= \frac{30\,000}{217} \\ \therefore n \log 1.42 &= \log\left(\frac{30\,000}{217}\right) \\ \therefore n &= \frac{\log\left(\frac{30\,000}{217}\right)}{\log 1.42} \\ \therefore n &\approx 14.1 \end{aligned}$$

It will take approximately 14.1 years for the population to reach 30 000.

- 12 a** If the interest rate per annum is 7.2%, then the interest rate per month $i = \frac{7.2\%}{12} = 0.6\% = 0.006$.

$$\begin{aligned} r &= 1 + i \\ &= 1 + 0.006 \\ &= 1.006 \end{aligned}$$

- b** The interest is calculated monthly, so $n = 3 \times 12 = 36$ time periods.

$$\begin{aligned} u_{36} &= u_0 \times r^{36} \\ &= 500 \times (1.006)^{36} \\ &\approx 620.15 \end{aligned}$$

The value of the account after 3 years is €620.15.

$$\begin{aligned} \text{c } \text{real value} \times (1.02)^3 &= \text{€}620.15 \\ \therefore \text{real value} &= \frac{\text{€}620.15}{(1.02)^3} \\ &= \text{€}584.38 \end{aligned}$$

$$\begin{aligned} \text{13 a } u_3 &= u_0 \times (1 - d)^3 \\ &= 2000 \times (0.7)^3 \quad \{30\% = 0.3\} \\ &= 686 \end{aligned}$$

$$\begin{aligned} \text{b } \text{Depreciation} &= \text{£}2000 - \text{£}686 \\ &= \text{£}1314 \end{aligned}$$

So, after 3 years the value of the television is £686.

- 14 a** The series is arithmetic with $u_1 = 11$, $d = 4$, and $n = 20$.

$$\begin{aligned} \text{Now } S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore S_{20} &= \frac{20}{2}(2 \times 11 + 19 \times 4) \\ &= 10(22 + 76) \\ &= 980 \end{aligned}$$

$$\text{b } 7 + 12.5 + 18 + 23.5 + \dots + 106$$

The series is arithmetic with $u_1 = 7$, $d = 5.5$, and $u_n = 106$.

First we need to find n .

$$\begin{aligned} \text{Now } u_n &= 106 \\ \therefore u_1 + (n-1)d &= 106 \\ \therefore 7 + 5.5(n-1) &= 106 \\ \therefore 5.5(n-1) &= 99 \\ \therefore n-1 &= 18 \\ \therefore n &= 19 \end{aligned}$$

$$\begin{aligned} \text{Using } S_n &= \frac{n}{2}(u_1 + u_n) \\ \therefore S_{19} &= \frac{19}{2}(7 + 106) \\ &= \frac{19}{2} \times 113 \\ &= 1073.5 \end{aligned}$$

- c** $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms can be expressed as two separate arithmetic series:

$$1 + 3 + 5 + 7 + \dots \text{ where } u_1 = 1, d = 2, n = 50$$

$$\text{and } -2 - 4 - 6 - 8 - \dots \text{ where } u_1 = -2, d = -2, n = 50$$

$$\begin{aligned} \text{Using } S_n &= \frac{n}{2}(2u_1 + (n-1)d), \text{ the sum of the first series} = \frac{50}{2}(2(1) + 49(2)) \\ &= 25(2 + 98) \\ &= 2500 \end{aligned}$$

$$\begin{aligned} \text{and the sum of the second series} &= \frac{50}{2}(2(-2) + 49(-2)) \\ &= 25(-4 - 98) \\ &= -2550 \end{aligned}$$

$$\begin{aligned} \therefore \text{ the sum of both series} &= 2500 + (-2550) \\ &= -50 \end{aligned}$$

So, $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms is -50 .

- d** The integers from 1 to 200 which are not divisible by 3 are 1, 2, 4, 5, 7, 8, ..., 200.

The sum of these integers can be expressed as two separate arithmetic series A and B :

$$S_A = 1 + 4 + 7 + \dots + 196 + 199 \text{ where } u_1 = 1, d = 3, u_n = 199$$

$$\text{and } S_B = 2 + 5 + 8 + \dots + 197 + 200 \text{ where } u_1 = 2, d = 3, u_n = 200$$

$$\text{Now for } S_A, u_n = u_1 + (n-1)d \text{ and for } S_B, u_n = u_1 + (n-1)d$$

$$\therefore 199 = 1 + 3(n-1) \qquad \qquad \qquad \therefore 200 = 2 + 3(n-1)$$

$$\therefore 198 = 3(n-1) \qquad \qquad \qquad \therefore 198 = 3(n-1)$$

$$\therefore 66 = n-1 \qquad \qquad \qquad \therefore 66 = n-1$$

$$\therefore n = 67 \qquad \qquad \qquad \therefore n = 67$$

$$\text{Using } S_n = \frac{n}{2}(u_1 + u_n), \quad S_A = \frac{67}{2}(1 + 199) = 6700 \quad \text{and} \quad S_B = \frac{67}{2}(2 + 200) = 6767$$

$$\begin{aligned} \text{The total sum} &= S_A + S_B \\ &= 6700 + 6767 \\ &= 13467 \end{aligned}$$

$$\begin{aligned} \mathbf{15} \quad \mathbf{a} \quad u_7 &= 1 \qquad \therefore u_1 + 6d = 1 \qquad \dots (1) \quad \{\text{using } u_n = u_1 + (n-1)d\} \\ u_{15} &= -23 \quad \therefore u_1 + 14d = -23 \quad \dots (2) \end{aligned}$$

We now solve (1) and (2) simultaneously:

$$-u_1 - 6d = -1 \quad \{\text{multiplying both sides of (1) by } -1 \}$$

$$\underline{u_1 + 14d = -23}$$

$$8d = -24 \quad \{\text{adding the equations}\}$$

$$\therefore d = -3$$

$$\begin{aligned} \text{So, in (1):} \quad u_1 + 6(-3) &= 1 \\ \therefore u_1 - 18 &= 1 \\ \therefore u_1 &= 19 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad u_n &= u_1 + (n-1)d \\ \therefore u_{27} &= 19 + 26(-3) \quad \{\text{using } \mathbf{a}\} \\ &= -59 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad S_n &= \frac{n}{2}(u_1 + u_n) \\ \therefore S_{27} &= \frac{27}{2}(19 + (-59)) \quad \{\text{from } \mathbf{b}\} \\ &= \frac{27}{2} \times (-40) \\ &= -540 \end{aligned}$$

$$\begin{aligned} \mathbf{16} \quad \mathbf{a} \quad S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore -210 &= \frac{n}{2}(2 \times 18 - 3(n-1)) \\ \therefore \frac{n}{2}(36 - 3n + 3) &= -210 \\ \therefore \frac{n}{2}(39 - 3n) &= -210 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{From } \mathbf{a}, \quad \frac{n}{2}(39 - 3n) &= -210 \\ \therefore n(39 - 3n) &= -420 \\ \therefore 39n - 3n^2 &= -420 \\ \therefore 3n^2 - 39n - 420 &= 0 \\ \therefore n^2 - 13n - 140 &= 0 \\ \therefore (n-20)(n+7) &= 0 \\ \therefore n &= 20 \quad \{n > 0\} \end{aligned}$$

- 17 a** The sequence is arithmetic with $u_1 = 7$ and $d = 3$. **b**

$$\text{Now } S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\therefore S_n = \frac{n}{2}(2(7) + 3(n-1))$$

$$\therefore S_n = \frac{n}{2}(14 + 3n - 3)$$

$$\therefore S_n = \frac{n}{2}(11 + 3n)$$

$$S_n = 140$$

$$\therefore \frac{n}{2}(11 + 3n) = 140 \quad \{\text{using a}\}$$

$$\therefore n(11 + 3n) = 280$$

$$\therefore 11n + 3n^2 = 280$$

$$\therefore 3n^2 + 11n - 280 = 0$$

$$\therefore 3n^2 - 24n + 35n - 280 = 0$$

$$\therefore 3n(n-8) + 35(n-8) = 0$$

$$\therefore (n-8)(3n+35) = 0$$

$$\therefore n = 8 \quad \{n > 0\}$$

18 a $r = \frac{0.25}{0.125} = 2$

b Using $u_n = u_1 \times r^{n-1}$,

$$u_{20} = 0.125 \times 2^{19}$$

$$= 65\,536$$

c Using $S_n = \frac{u_1(r^n - 1)}{r - 1}$,

$$S_{10} = \frac{0.125(2^{10} - 1)}{2 - 1}$$

$$= 127.875$$

- 19 a** The interest is calculated annually, so $n = 7$ time periods.

$$u_7 = u_0 \times (1 + i)^7$$

$$= 2000 \times (1.0825)^7 \quad \{8.25\% = 0.0825\}$$

$$\approx 3483.58$$

The total value of Kapil's investment on January 1st 2019 is 3484 rupees.

- b** There are $n = 7 \times 12 = 84$ time periods.

Each time period the investment increases by $i = \frac{8\%}{12} \approx 0.667\%$

$$\therefore \text{the value after 7 years is } u_{84} = u_0 \times (1 + i)^{84}$$

$$\approx 2000 \times (1.006\,67)^{84} \quad \{0.667\% = 0.006\,67\}$$

$$\approx 3494.84$$

The total value of Kapil's investment on January 1st 2019 for this account is 3495 rupees, which is 11 rupees more than the account in **a**.

\therefore investing in the account paying 8% per annum interest compounded monthly is the better option.

- 20 a i** $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k = 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + \dots$ is an infinite geometric series with $u_1 = 2\left(\frac{2}{3}\right) = \frac{4}{3}$ and $r = \frac{2}{3}$.

ii $S = \frac{u_1}{1 - r}$

$$\therefore S = \frac{\frac{4}{3}}{1 - \frac{2}{3}}$$

$$= \frac{\frac{4}{3}}{\frac{1}{3}}$$

$$= 4$$

- b i** $\sum_{k=1}^n (k - 4) = -3 - 2 - 1 + 0 + 1 + \dots + (n - 4)$ is an arithmetic series with $u_1 = -3$ and $d = 1$.

ii $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$\therefore S_n = \frac{n}{2}(2(-3) + (n-1))$$

$$= \frac{n}{2}(-6 + n - 1)$$

$$= \frac{n}{2}(n - 7)$$

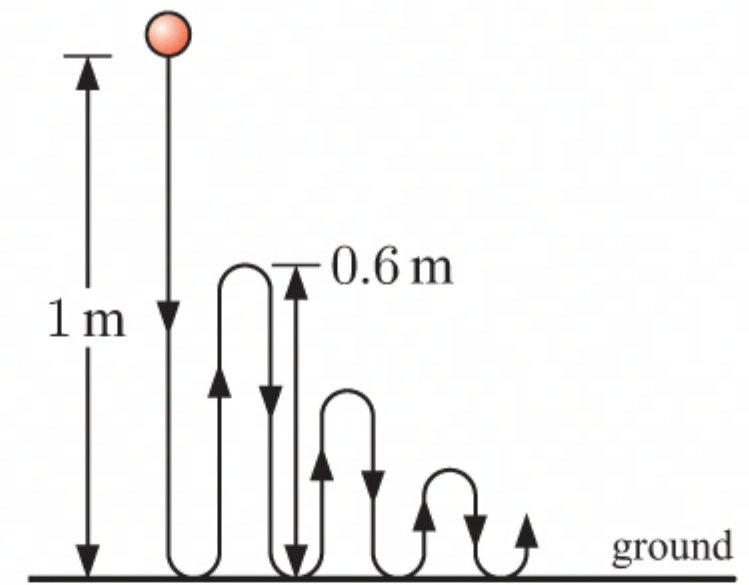
$$\begin{aligned}
 \mathbf{c} \quad S_n &= S \\
 \therefore \frac{n}{2}(n-7) &= 4 \\
 \therefore n(n-7) &= 8 \\
 \therefore n^2 - 7n - 8 &= 0 \\
 \therefore (n-8)(n+1) &= 0 \\
 \therefore n &= 8 \quad \{n > 0\}
 \end{aligned}$$

21 a $S = 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$ is an infinite geometric series with $u_1 = 1$ and $r = 0.6$.

$$\begin{aligned}
 \therefore S &= \frac{u_1}{1-r} \\
 &= \frac{1}{1-0.6} \\
 &= \frac{1}{0.4} \\
 &= 2.5
 \end{aligned}$$

b Each time the ball bounces upward, it must travel the same distance on its way downward.

$$\begin{aligned}
 \therefore \text{total distance} &= 1 + 2(0.6) + 2(0.6)^2 + 2(0.6)^3 + \dots \\
 &= 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\
 &\quad + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\
 &= S + (S - 1) \\
 &= 2S - 1 \\
 &= 2(2.5) - 1 \quad \{\text{using a}\} \\
 &= 5 - 1 \\
 &= 4 \text{ m}
 \end{aligned}$$



22 a The series will converge if $|\text{common ratio}| < 1$

$$\begin{aligned}
 \therefore |x-2| &< 1 \\
 \therefore -1 < x-2 < 1 \\
 \therefore 1 < x < 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{k=1}^{\infty} 12(x-2)^{k-1} &= \frac{12}{1-(x-2)} \quad \{u_1 = 12, r = x-2\} \\
 &= \frac{12}{3-x}
 \end{aligned}$$

and $x = \sqrt{5}$ satisfies $1 < x < 3$

$$\therefore \text{when } x = \sqrt{5}, \text{ the sum of the series} = \frac{12}{3-\sqrt{5}} \approx 15.7.$$

$$\begin{aligned}
 \mathbf{23} \quad \sum_{k=1}^{\infty} \left(\frac{4x}{3}\right)^{k-1} &= \frac{5}{2} \\
 \therefore \frac{1}{1-\frac{4x}{3}} &= \frac{5}{2} \quad \left\{u_1 = 1, r = \frac{4x}{3}\right\} \\
 \therefore \frac{3}{3-4x} &= \frac{5}{2} \\
 \therefore 6 &= 5(3-4x) \\
 \therefore 6 &= 15-20x \\
 \therefore 20x &= 9 \\
 \therefore x &= \frac{9}{20}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{24 a} \quad 10 \times 9 \times 8 \times 7 &= \frac{10 \times 9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\
 &= \frac{10!}{6!}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2 \times 1} &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\
 &= \frac{9!}{6!3!}
 \end{aligned}$$

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & 1 & & 2 & & 1 \\
 & 1 & & 3 & & 3 & & 1 \\
 1 & & 4 & & 6 & & 4 & & 1 \quad \leftarrow \text{the 4th row for } (a+b)^4
 \end{array}$$

$$\therefore (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{aligned}
 \text{b i } (2+x)^4 &= 2^4 + 4(2)^3(x) + 6(2)^2(x)^2 + 4(2)(x)^3 + x^4 \\
 &= 16 + 32x + 24x^2 + 8x^3 + x^4
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } (3x-y)^4 &= (3x)^4 + 4(3x)^3(-y) + 6(3x)^2(-y)^2 + 4(3x)(-y)^3 + (-y)^4 \\
 &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \left(x - \frac{2}{x}\right)^4 &= x^4 + 4(x)^3\left(-\frac{2}{x}\right) + 6(x)^2\left(-\frac{2}{x}\right)^2 + 4(x)\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4 \\
 &= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}
 \end{aligned}$$

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & 1 & & 2 & & 1 \\
 & 1 & & 3 & & 3 & & 1 \\
 1 & & 4 & & 6 & & 4 & & 1 \\
 1 & & 5 & & 10 & & 10 & & 5 & & 1
 \end{array}$$

$$\begin{aligned}
 \text{b i } \left(x + \frac{1}{x}\right)^5 &= x^5 + 5(x)^4\left(\frac{1}{x}\right) + 10(x)^3\left(\frac{1}{x}\right)^2 + 10(x)^2\left(\frac{1}{x}\right)^3 + 5(x)\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5 \\
 &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } (1 - \sqrt{2})^5 &= 1^5 + 5(1)^4(-\sqrt{2}) + 10(1)^3(-\sqrt{2})^2 + 10(1)^2(-\sqrt{2})^3 + 5(1)(-\sqrt{2})^4 + (-\sqrt{2})^5 \\
 &= 1 - 5\sqrt{2} + 20 - 20\sqrt{2} + 20 - 4\sqrt{2} \\
 &= 41 - 29\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{27 } (a+bx)^n &= 1 - 12x + 54x^2 - \dots, \quad a > 0, \quad n \in \mathbb{Z}^+ \\
 \therefore \sum_{r=0}^n \binom{n}{r} a^{n-r} (bx)^r &= 1 - 12x + 54x^2 - \dots \quad \{\text{binomial theorem}\}
 \end{aligned}$$

$$\therefore \binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \dots = 1 - 12x + 54x^2 - \dots$$

$$\therefore a^n + na^{n-1}bx + \frac{n(n-1)}{2}a^{n-2}b^2x^2 + \dots = 1 - 12x + 54x^2 - \dots$$

Equating coefficients:

$$a^n = 1 \quad \dots (1)$$

$$na^{n-1}b = -12 \quad \dots (2)$$

$$\frac{n(n-1)}{2}a^{n-2}b^2 = 54 \quad \dots (3)$$

From (1), $a = 1$ since $a > 0$.

Substituting into (2) we have $nb = -12$

$$\therefore b = \frac{-12}{n} \quad \dots (4)$$

$$\text{Substituting (4) into (3) gives } \frac{n(n-1)}{2} \left(\frac{144}{n^2}\right) = 54$$

$$\therefore \frac{144(n-1)}{n} = 108$$

$$\therefore 144(n-1) = 108n$$

$$\therefore 144n - 144 = 108n$$

$$\therefore 36n = 144$$

$$\therefore n = \frac{144}{36} = 4$$

$$\therefore b = \frac{-12}{4} = -3 \quad \{\text{using (4)}\}$$

28 a The general term of the expansion is $T_{r+1} = \binom{6}{r} a^{6-r} b^r$, $r = 0, 1, 2, \dots, 6$.

b For $a^4 b^2$, $r = 2$

$$\begin{aligned}\therefore T_3 &= \binom{6}{2} a^4 b^2 \\ &= \binom{6}{4} a^4 b^2 \quad \left\{ \binom{6}{2} = \binom{6}{4} \right\} \\ &= 15a^4 b^2\end{aligned}$$

\therefore the coefficient of $a^4 b^2$ is 15.

29 $(x+2)(1-x)^{10}$

$$\begin{aligned}&= (x+2) \left[1^{10} + \binom{10}{1}(1)^9(-x) + \binom{10}{2}(1)^8(-x)^2 + \binom{10}{3}(1)^7(-x)^3 + \binom{10}{4}(1)^6(-x)^4 + \binom{10}{5}(1)^5(-x)^5 + \dots \right] \\ &= (x+2) \left[1 - \binom{10}{1}x + \binom{10}{2}x^2 - \binom{10}{3}x^3 + \binom{10}{4}x^4 - \binom{10}{5}x^5 + \dots \right]\end{aligned}$$

So, the terms containing x^5 are $\binom{10}{4}x^5$ from (1)
and $-2\binom{10}{5}x^5$ from (2)

\therefore the coefficient of x^5 is $\binom{10}{4} - 2\binom{10}{5} = -294$

30 a

r	7C_r
0	1
1	7
2	21
3	35
4	35
5	21
6	7
7	1

b ${}^7C_r = 35$
 $\therefore r = 3$ or 4 {from **a**}

c The coefficient of x^3 is ${}^7C_4 2^3 k^4 = 35 \times 8k^4 = 280k^4$ (1)

The coefficient of x is ${}^7C_6 2k^6 = 7 \times 2k^6 = 16k^6$ (2)

(1) is 10 times larger than (2), so $280k^4 = 10 \times 16k^6$

$$\therefore 28k^4 = 16k^6$$

$$\therefore k^2 = \frac{7}{4} \quad \{k \neq 0\}$$

$$\therefore k = \frac{\sqrt{7}}{2} \quad \{k > 0\}$$

31 a $(x-2)^3 = x^3 + 3(x)^2(-2) + 3(x)(-2)^2 + (-2)^3$
 $= x^3 - 6x^2 + 12x - 8$

b $(3x^2 - 7)(x - 2)^3$

$$= (3x^2 - 7)(x^3 - 6x^2 + 12x - 8) \quad \{\text{using a}\}$$

So, the terms containing x^3 are $36x^3$ from (1)
and $-7x^3$ from (2)

\therefore the coefficient of x^3 is $36 - 7 = 29$.

32 a Using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $\binom{6}{2} = \frac{6!}{2!4!}$

$$\begin{aligned}&= \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{2 \times 1 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\ &= \frac{6 \times 5}{2 \times 1} \\ &= \frac{30}{2} \\ &= 15\end{aligned}$$

b $\binom{6}{4} = \frac{6!}{4!2!} = \binom{6}{2} = 15$

$$\begin{aligned} \text{c } (x-2)^6 &= x^6 + \binom{6}{1}(x)^5(-2)^1 + \binom{6}{2}(x)^4(-2)^2 + \binom{6}{3}(x)^3(-2)^3 + \binom{6}{4}(x)^2(-2)^4 + \binom{6}{5}(x)^1(-2)^5 + (-2)^6 \\ &= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \end{aligned}$$

$$\begin{aligned} \text{33 a } 4^{\frac{5}{2}} &= (2^2)^{\frac{5}{2}} \\ &= 2^5 \\ &= 32 \end{aligned}$$

$$\begin{aligned} \text{b } 49^{-\frac{3}{2}} &= (7^2)^{-\frac{3}{2}} \\ &= 7^{-3} \\ &= \frac{1}{7^3} \\ &= \frac{1}{343} \end{aligned}$$

$$\begin{aligned} \text{c } 27^{\frac{5}{3}} &= (3^3)^{\frac{5}{3}} \\ &= 3^5 \\ &= 243 \end{aligned}$$

$$\begin{aligned} \text{34 a } x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}}) \\ &= x^{\frac{1}{2}} \times x^{-\frac{1}{2}} + x^{\frac{1}{2}} \times 2x - x^{\frac{1}{2}} \times x^{\frac{1}{2}} \\ &= x^0 + 2x^{\frac{3}{2}} - x^1 \\ &= 1 + 2x^{\frac{3}{2}} - x \end{aligned}$$

$$\begin{aligned} \text{b } 5^x(5^{-x} + 5^{3x}) \\ &= 5^x \times 5^{-x} + 5^x \times 5^{3x} \\ &= 5^0 + 5^{4x} \\ &= 1 + 5^{4x} \end{aligned}$$

$$\begin{aligned} \text{c } 2^{-2x}(2^{2x+3} - 2^{-4x} + 3) \\ &= 2^{-2x} \times 2^{2x+3} - 2^{-2x} \times 2^{-4x} + 3 \times 2^{-2x} \\ &= 2^3 - 2^{-6x} + 3 \times 2^{-2x} \\ &= 8 - 2^{-6x} + 3 \times 2^{-2x} \end{aligned}$$

$$\begin{aligned} \text{35 a } 36 - 16^x \\ &= 6^2 - (4^x)^2 \\ &= (6 - 4^x)(6 + 4^x) \end{aligned}$$

$$\begin{aligned} \text{b } 9^x + 4(3^x) - 12 \\ &= (3^x)^2 + 4(3^x) - 12 \\ &= (3^x + 6)(3^x - 2) \end{aligned}$$

$$\begin{aligned} \text{c } 25^x - 5^{x+1} - 24 \\ &= (5^x)^2 - 5(5^x) - 24 \\ &= (5^x - 8)(5^x + 3) \end{aligned}$$

$$\begin{aligned} \text{36 a } 5 \times 2^x &= 160 \\ \therefore 2^x &= 32 \\ \therefore 2^x &= 2^5 \\ \therefore x &= 5 \quad \{\text{equating indices}\} \end{aligned}$$

$$\begin{aligned} \text{b } 8^{2x-3} &= 16^{2-x} \\ \therefore (2^3)^{2x-3} &= (2^4)^{2-x} \\ \therefore 2^{6x-9} &= 2^{8-4x} \\ \therefore 6x-9 &= 8-4x \quad \{\text{equating indices}\} \\ \therefore 10x &= 17 \\ \therefore x &= \frac{17}{10} \end{aligned}$$

$$\begin{aligned} \text{c } \left(\frac{1}{3}\right)^{2x-5} &= 27 \\ \therefore (3^{-1})^{2x-5} &= 3^3 \\ \therefore 3^{5-2x} &= 3^3 \\ \therefore 5-2x &= 3 \quad \{\text{equating indices}\} \\ \therefore -2x &= -2 \\ \therefore x &= 1 \end{aligned}$$

$$\begin{aligned} \text{d } 25^x + 2(5^x) &= 35 \\ \therefore (5^x)^2 + 2(5^x) - 35 &= 0 \\ \therefore (5^x - 5)(5^x + 7) &= 0 \\ \therefore 5^x &= 5 \quad \text{or} \quad 5^x = -7 \\ \therefore 5^x &= 5^1 \quad \{5^x > 0 \text{ for all } x\} \\ \therefore x &= 1 \quad \{\text{equating indices}\} \end{aligned}$$

$$\begin{aligned} \text{37 a } \log(10^9 \times 1000^b) \\ &= \log(10^9 \times (10^3)^b) \\ &= \log(10^9 \times 10^{3b}) \\ &= \log(10^{9+3b}) \\ &= 9 + 3b \end{aligned}$$

$$\begin{aligned} \text{b } \log\left(\frac{10^n}{100}\right) \\ &= \log\left(\frac{10^n}{10^2}\right) \\ &= \log(10^{n-2}) \\ &= n - 2 \end{aligned}$$

$$\begin{aligned} \text{c } \log(2^t \times 5^t) \\ &= \log((2 \times 5)^t) \\ &= \log(10)^t \\ &= t \end{aligned}$$

$$\begin{aligned} \text{38 a } \log_4 8 &= \log_4(2 \times 4) \\ &= \log_4(\sqrt{4} \times 4) \\ &= \log_4(4^{\frac{1}{2}} \times 4^1) \\ &= \log_4(4^{\frac{3}{2}}) \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{b } \log_9\left(\frac{1}{27}\right) &= \log_9\left(\frac{1}{9 \times 3}\right) \\ &= \log_9\left(\frac{1}{9 \times \sqrt{9}}\right) \\ &= \log_9\left(\frac{1}{9^{\frac{3}{2}}}\right) \\ &= \log_9(9^{-\frac{3}{2}}) \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_9\left(\frac{1}{3\sqrt{3}}\right) &= \log_9\left(\frac{1}{3^{\frac{3}{2}}}\right) \\
 &= \log_9\left(\frac{1}{(\sqrt{9})^{\frac{3}{2}}}\right) \\
 &= \log_9\left(\frac{1}{9^{\frac{3}{4}}}\right) \\
 &= \log_9(9^{-\frac{3}{4}}) \\
 &= -\frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \log_{49} 7 &= \log_{49} \sqrt{49} \\
 &= \log_{49}(49^{\frac{1}{2}}) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 39 \text{ a } \log_3 x &= 2 \\
 \therefore x &= 3^2 = 9
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_x 27 &= 3 \\
 \therefore 27 &= x^3 \\
 \therefore x &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_5(2x - 1) &= 1 \\
 \therefore 2x - 1 &= 5^1 \\
 \therefore 2x - 1 &= 5 \\
 \therefore 2x &= 6 \\
 \therefore x &= 3
 \end{aligned}$$

$$\begin{aligned}
 40 \quad \log_a(x + 2) &= \log_a x + 2 \\
 \therefore \log_a(x + 2) - \log_a x &= 2 \\
 \therefore \log_a\left(\frac{x+2}{x}\right) &= 2 \\
 \therefore \frac{x+2}{x} &= a^2 \\
 \therefore x + 2 &= a^2 x \\
 \therefore (1 - a^2)x &= -2 \\
 \therefore x &= \frac{-2}{1 - a^2} = \frac{2}{a^2 - 1} \quad \{a > 1\}
 \end{aligned}$$

$$41 \text{ a } \log_m(m^4) = 4$$

$$\begin{aligned}
 \text{b } \log_n(n^2\sqrt{n}) &= \log_n(n^2 \times n^{\frac{1}{2}}) \\
 &= \log_n(n^{\frac{5}{2}}) \\
 &= \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_a\left(\frac{1}{a^3}\right) &= \log_a(a^{-3}) \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 42 \text{ a } \quad &\frac{1}{4} \ln 81 + \ln 12 - \ln 4 \\
 &= \frac{1}{4} \ln(3^4) + \ln(3 \times 4) - \ln 4 \\
 &= \frac{1}{4}(4 \ln 3) + \ln 3 + \cancel{\ln 4} - \cancel{\ln 4} \\
 &= \ln 3 + \ln 3 \\
 &= 2 \ln 3 \\
 &= \ln(3^2) \\
 &= \ln 9
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \quad &3 \log_9 2 - \log_9 24 \\
 &= 3 \log_9 2 - \log_9(8 \times 3) \\
 &= 3 \log_9 2 - \log_9(2^3 \times 9^{\frac{1}{2}}) \\
 &= 3 \log_9 2 - (\log_9(2^3) + \log_9(9^{\frac{1}{2}})) \\
 &= 3 \log_9 2 - (3 \log_9 2 + \frac{1}{2}) \\
 &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } 5 + \log_2 3 - \frac{1}{2} \log_2 49 &= 5 + \log_2 3 - \log_2(49^{\frac{1}{2}}) \\
 &= \log_2(2^5) + \log_2 3 - \log_2 7 \\
 &= \log_2(32 \times 3) - \log_2 7 \\
 &= \log_2\left(\frac{96}{7}\right)
 \end{aligned}$$

$$43 \quad x = \log_a 5$$

$$\begin{aligned}
 \text{a } \log_a(5a) &= \log_a 5 + \log_a a \\
 &= x + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_a\left(\frac{125}{a^2}\right) &= \log_a 125 - \log_a(a^2) \\
 &= \log_a(5^3) - 2 \\
 &= 3 \log_a 5 - 2 \\
 &= 3x - 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_{25a} 5 &= \frac{\log_a 5}{\log_a(25a)} \\
 &\quad \{\text{change of base rule}\} \\
 &= \frac{x}{\log_a 25 + \log_a a} \\
 &= \frac{x}{\log_a(5^2) + 1} \\
 &= \frac{x}{2 \log_a 5 + 1} \\
 &= \frac{x}{2x + 1}
 \end{aligned}$$

$$44 \quad A = \log_{10} P, \quad B = \log_{10} Q, \quad C = \log_{10} R$$

$$\begin{aligned} \mathbf{a} \quad \log_{10}(PQ) &= \log_{10} P + \log_{10} Q \\ &= A + B \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_{10}\left(\frac{PQ^3}{R^2}\right) &= \log_{10} P + \log_{10}(Q^3) - \log_{10}(R^2) \\ &= A + 3\log_{10} Q - 2\log_{10} R \\ &= A + 3B - 2C \end{aligned}$$

$$45 \quad \mathbf{a} \quad \frac{\log_2 9}{\log_2 3} = \log_3 9 \quad \{\text{change of base rule}\}$$

$$= \log_3(3^2)$$

$$= 2$$

$$\begin{aligned} \mathbf{c} \quad \frac{\log_3(0.25)}{\log_3 64} &= \log_{64}(0.25) \quad \{\text{change of base rule}\} \\ &= \log_{64}\left(\frac{1}{4}\right) \\ &= \log_{64}(4^{-1}) \\ &= \log_{64}(64^{-\frac{1}{3}}) \\ &= -\frac{1}{3} \end{aligned}$$

$$46 \quad \mathbf{a} \quad \ln 20 - \ln 10$$

$$= \ln(2 \times 10) - \ln 10$$

$$= \ln 2 + \cancel{\ln 10} - \cancel{\ln 10}$$

$$= \ln 2$$

$$\begin{aligned} \mathbf{b} \quad & -\ln 13 - 3 \\ &= -(\ln 13 + 3) \\ &= -(\ln 13 + \ln(e^3)) \\ &= -\ln(13e^3) \\ &= \ln\left(\frac{1}{13e^3}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{1}{3} \ln 64 + 2 \ln 2 \\ &= \ln(64^{\frac{1}{3}}) + \ln(2^2) \\ &= \ln 4 + \ln 4 \\ &= 2 \ln 4 \\ &= \ln(4^2) \\ &= \ln 16 \end{aligned}$$

$$47 \quad \mathbf{a} \quad M = ab^3$$

$$\therefore \log_b M = \log_b(ab^3)$$

$$\therefore \log_b M = \log_b a + \log_b(b^3)$$

$$\therefore \log_b M = \log_b a + 3$$

$$\mathbf{b} \quad D = \frac{a}{b^2}$$

$$\therefore \log_b D = \log_b\left(\frac{a}{b^2}\right)$$

$$\therefore \log_b D = \log_b a - \log_b(b^2)$$

$$\therefore \log_b D = \log_b a - 2$$

$$\mathbf{c} \quad F = \sqrt{\frac{b}{a^3}}$$

$$\therefore \log_b F = \log_b \sqrt{\frac{b}{a^3}}$$

$$\therefore \log_b F = \log_b \left(\left(\frac{b}{a^3} \right)^{\frac{1}{2}} \right)$$

$$\therefore \log_b F = \frac{1}{2} \log_b \left(\frac{b}{a^3} \right)$$

$$\therefore \log_b F = \frac{1}{2} [\log_b b - \log_b(a^3)]$$

$$\therefore \log_b F = \frac{1}{2} [1 - 3 \log_b a]$$

$$\therefore \log_b F = \frac{1}{2} - \frac{3}{2} \log_b a$$

$$48 \quad \mathbf{a} \quad 3 \log_5 x = \log_5 24 + \log_5 \left(\frac{1}{3} \right)$$

$$\therefore \log_5(x^3) = \log_5 \left(\frac{24}{3} \right)$$

$$\therefore \log_5(x^3) = \log_5 8$$

$$\therefore x^3 = 8$$

$$\therefore x = 2$$

$$\mathbf{b} \quad \log_2 x = \log_2 12 - \log_2(7 - x)$$

$$\therefore \log_2 x = \log_2 \left(\frac{12}{7 - x} \right)$$

$$\therefore x = \frac{12}{7 - x}$$

$$\therefore x(7 - x) = 12$$

$$\therefore 7x - x^2 = 12$$

$$\therefore x^2 - 7x + 12 = 0$$

$$\therefore (x - 3)(x - 4) = 0$$

$$\therefore x = 3 \text{ or } 4$$

$$\text{c } \ln(x^2 - 3) - \ln(2x) = 0$$

$$\therefore \ln\left(\frac{x^2 - 3}{2x}\right) = 0$$

$$\therefore \frac{x^2 - 3}{2x} = e^0 = 1$$

$$\therefore x^2 - 3 = 2x$$

$$\therefore x^2 - 2x - 3 = 0$$

$$\therefore (x - 3)(x + 1) = 0$$

$$\therefore x = 3 \text{ or } -1$$

But $x = -1$ does not satisfy the original equation, as $\ln(-2)$ is undefined.

\therefore the only solution is $x = 3$.

$$\text{d } \log_3 x + \log_3(x - 2) = 1$$

$$\therefore \log_3(x(x - 2)) = 1$$

$$\therefore x(x - 2) = 3$$

$$\therefore x^2 - 2x - 3 = 0$$

$$\therefore (x - 3)(x + 1) = 0$$

$$\therefore x = 3 \text{ or } -1$$

But $x = -1$ does not satisfy the original equation, as $\log_3(-1)$ is undefined.

\therefore the only solution is $x = 3$.

$$\text{49 a } \log_{\frac{1}{9}} x = \log_9 5$$

$$\therefore \frac{\log_9 x}{\log_9\left(\frac{1}{9}\right)} = \log_9 5 \quad \{\text{change of base rule}\}$$

$$\therefore \frac{\log_9 x}{\log_9(9^{-1})} = \log_9 5$$

$$\therefore \frac{\log_9 x}{-1} = \log_9 5$$

$$\therefore \log_9 x = -\log_9 5$$

$$\therefore \log_9 x = \log_9(5^{-1})$$

$$\therefore x = 5^{-1} = \frac{1}{5}$$

$$\text{b } \log_2 x - \log_8 x = 3$$

$$\therefore \log_2 x - \frac{\log_2 x}{\log_2 8} = 3 \quad \{\text{change of base rule}\}$$

$$\therefore \log_2 x - \frac{\log_2 x}{3} = 3$$

$$\therefore \frac{2}{3} \log_2 x = 3$$

$$\therefore \log_2 x = \frac{9}{2}$$

$$\therefore x = 2^{\frac{9}{2}}$$

$$\therefore x = 2^4 \times 2^{\frac{1}{2}} = 16\sqrt{2}$$

$$\text{c } \log_{27}(x^4) = \log_9 x - \log_3(\sqrt[5]{9})$$

$$\therefore \frac{\log_3(x^4)}{\log_3 27} = \frac{\log_3 x}{\log_3 9} - \log_3(\sqrt[5]{9}) \quad \{\text{change of base rule}\}$$

$$\therefore \frac{\log_3(x^4)}{\log_3(3^3)} = \frac{\log_3 x}{\log_3(3^2)} - \log_3(3^{\frac{2}{5}})$$

$$\therefore \frac{4 \log_3 x}{3} = \frac{\log_3 x}{2} - \frac{2}{5}$$

$$\therefore \log_3 x \left(\frac{4}{3} - \frac{1}{2}\right) = -\frac{2}{5}$$

$$\therefore \frac{5}{6} \log_3 x = -\frac{2}{5}$$

$$\therefore \log_3 x = -\frac{2}{5} \times \frac{6}{5} = -\frac{12}{25}$$

$$\therefore x = 3^{-\frac{12}{25}} = \frac{1}{3^{\frac{12}{25}}}$$

$$\text{50 } \frac{8}{\log_5 9} = \frac{8}{\left(\frac{\log_3 9}{\log_3 5}\right)} \quad \{\text{change of base rule}\}$$

$$= \frac{8 \log_3 5}{\log_3(3^2)}$$

$$= \frac{8 \log_3 5}{2}$$

$$= 4 \log_3 5$$

$$\text{51 a } 3^x = 15$$

$$\therefore \log(3^x) = \log 15$$

$$\therefore x \log 3 = \log 15$$

$$\therefore x = \frac{\log 15}{\log 3}$$

$$\text{b } 3^{x+1} = 8$$

$$\therefore \log(3^{x+1}) = \log 8$$

$$\therefore (x + 1) \log 3 = \log 8$$

$$\therefore x + 1 = \frac{\log 8}{\log 3}$$

$$\therefore x = \frac{\log 8}{\log 3} - 1$$

$$\begin{aligned}
 \text{c} \quad & e^{2x} - 20 = e^x \\
 \therefore & e^{2x} - e^x - 20 = 0 \\
 \therefore & (e^x)^2 - e^x - 20 = 0 \\
 \therefore & (e^x - 5)(e^x + 4) = 0 \\
 \therefore & e^x = 5 \quad \{e^x > 0\} \\
 \therefore & x = \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 3 \times 4^x - 2^x = 0 \\
 \therefore & 3 \times (2^x)^2 - 2^x = 0 \\
 \therefore & 2^x(3 \times 2^x - 1) = 0 \\
 \therefore & 3 \times 2^x - 1 = 0 \quad \{2^x > 0\} \\
 \therefore & 3 \times 2^x = 1 \\
 \therefore & 2^x = \frac{1}{3} \\
 \therefore & \log(2^x) = \log\left(\frac{1}{3}\right) = -\log 3 \\
 \therefore & x \log 2 = -\log 3 \\
 \therefore & x = \frac{-\log 3}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 52 \quad \text{a} \quad & 9^x - 6(3^x) + 8 = 0 \\
 \therefore & (3^x)^2 - 6(3^x) + 8 = 0 \\
 \therefore & (3^x - 4)(3^x - 2) = 0 \\
 \therefore & 3^x = 4 \text{ or } 2 \\
 \therefore & \log(3^x) = \log 4 \text{ or } \log 2 \\
 \therefore & x \log 3 = \log 4 \text{ or } \log 2 \\
 \therefore & x = \frac{\log 4}{\log 3} \text{ or } \frac{\log 2}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 25^x - 5^{x+1} + 6 = 0 \\
 \therefore & (5^x)^2 - 5(5^x) + 6 = 0 \\
 \therefore & (5^x - 2)(5^x - 3) = 0 \\
 \therefore & 5^x = 2 \text{ or } 3 \\
 \therefore & \log(5^x) = \log 2 \text{ or } \log 3 \\
 \therefore & x \log 5 = \log 2 \text{ or } \log 3 \\
 \therefore & x = \frac{\log 2}{\log 5} \text{ or } \frac{\log 3}{\log 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 2 \times 3^{2x} + 3^{x+1} = 5 \\
 \therefore & 2 \times (3^x)^2 + 3(3^x) - 5 = 0 \\
 \therefore & 2 \times (3^x)^2 - 2(3^x) + 5(3^x) - 5 = 0 \\
 \therefore & 2(3^x)(3^x - 1) + 5(3^x - 1) = 0 \\
 \therefore & (3^x - 1)(2(3^x) + 5) = 0 \\
 \therefore & 3^x = 1 \quad \{3^x > 0\} \\
 \therefore & \log(3^x) = \log 1 \\
 \therefore & x \log 3 = 0 \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 53 \quad \text{a} \quad & K(0) = 3200 \times (0.85)^0 \\
 & = 3200 \times 1 \\
 & = 3200
 \end{aligned}$$

\therefore the initial population was 3200 kangaroos.

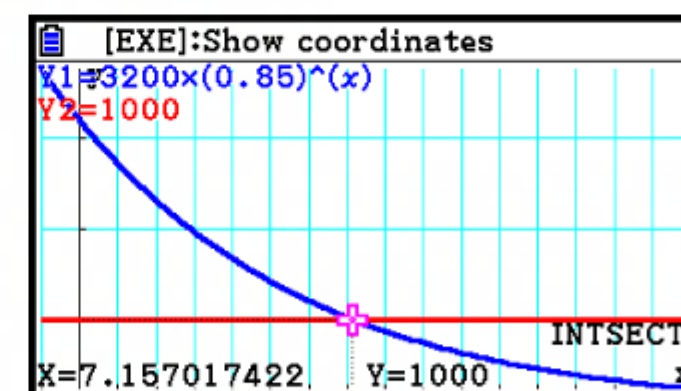
$$\text{c} \quad \text{We need to find when } 3200 \times (0.85)^t = 1000.$$

Using technology, $t \approx 7.16$.

\therefore it will take approximately 7.16 years for the kangaroo population to fall to 1000.

$$\begin{aligned}
 \text{b} \quad & K(5) = 3200 \times (0.85)^5 \\
 & = 1419.857
 \end{aligned}$$

\therefore after 5 years, there were about 1420 kangaroos.



$$\begin{aligned}
 \text{d} \quad & 3200 \times (0.85)^t = 1000 \\
 \therefore & (0.85)^t = \frac{1000}{3200} = 0.3125 \\
 \therefore & t \log(0.85) = \log(0.3125) \\
 \therefore & t = \frac{\log(0.3125)}{\log(0.85)} \\
 & \approx 7.16 \quad \checkmark
 \end{aligned}$$

54 a The statement “If $x > 1$ then $\frac{1}{x} < 1$ ” is true, since the function $f(x) = \frac{1}{x}$ is decreasing for $x > 0$.

b The statement “If $\frac{1}{x} < 1$ then $x > 1$ ” is false, since $-\frac{1}{2} < 1$, but $-2 \not> 1$.

c The statement “ $x > 1$ if and only if $\frac{1}{x} < 1$ ” is false, since $\frac{1}{x} < 1$ does not imply $x > 1$ (from b).

55 Let the middle number be x .

\therefore the sum of the three consecutive odd integers is $(x-2) + x + (x+2) = 3x$ which is divisible by 3.

56

$$2x^3 \geq x$$

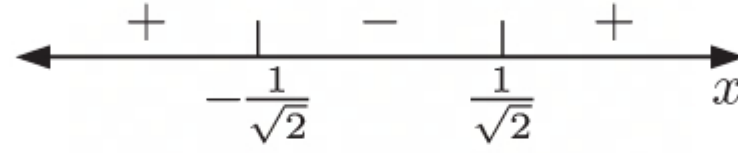
$\therefore 2x^2 \geq 1$ ← incorrect step, if $x = 0$, then we cannot divide both sides by x
and if $x < 0$, then the inequality is reversed.

$$\therefore x^2 \geq \frac{1}{2}$$

$$\therefore x^2 - \frac{1}{2} \geq 0$$

$$\therefore \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0$$

$$\therefore x \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad x \leq -\frac{1}{\sqrt{2}}$$



Correct solution: $2x^3 \geq x$

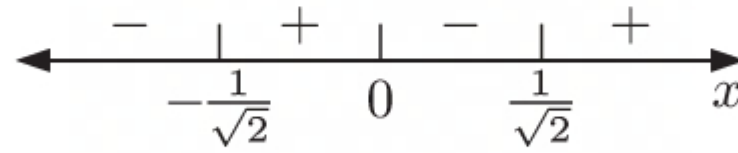
$$\therefore 2x^3 - x \geq 0$$

$$\therefore x(2x^2 - 1) \geq 0$$

$$\therefore x\left(x^2 - \frac{1}{2}\right) \geq 0$$

$$\therefore x\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0$$

$$\therefore -\frac{1}{\sqrt{2}} \leq x \leq 0 \quad \text{or} \quad x \geq \frac{1}{\sqrt{2}}$$



57 a $(a+b)^3 - (a-b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3) - (a^3 - 3a^2b + 3ab^2 - b^3)$
 $= \cancel{a^3} + 3a^2b + \cancel{3ab^2} + b^3 - \cancel{a^3} + 3a^2b - \cancel{3ab^2} + b^3$
 $= 6a^2b + 2b^3$
 $= 2b(3a^2 + b^2) \quad \checkmark$

b If $a = 2$ and $b = 1$, $\text{LHS} = (a+b)^3 - (a-b)^3$ $\text{RHS} = 2b(3a^2 + b^2)$
 $= (2+1)^3 - (2-1)^3$ $= 2(1)(3(2)^2 + (1)^2)$
 $= 3^3 - 1^3$ $= 2(1)(12 + 1)$
 $= 27 - 1$ $= 2(13)$
 $= 26$ $= 26$
 $= \text{LHS} \quad \checkmark$

58 (\Rightarrow) Suppose $xyz = 1$, then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{x^2z + y^2x + z^2y}{xyz}$
 $= x^2z + y^2x + z^2y$

(\Leftarrow) Suppose $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = x^2z + y^2x + z^2y$

$$\therefore \frac{x^2z + y^2x + z^2y}{xyz} = x^2z + y^2x + z^2y$$

$$\therefore xyz = \frac{x^2z + y^2x + z^2y}{x^2z + y^2x + z^2y}$$

$$= 1 \quad \text{since} \quad x, y, z > 0 \Rightarrow x^2z + y^2x + z^2y > 0.$$

TOPIC 2 SKILL BUILDER QUESTIONS

1 a i $4x - 3y + 2 = 0$

$$\therefore 3y = 4x + 2$$

$$\therefore y = \frac{4}{3}x + \frac{2}{3} \quad \text{has gradient } \frac{4}{3}$$

ii The perpendicular bisector has gradient $-\frac{3}{4}$.

b The equation of the perpendicular bisector is $3x + 4y = 3(4) + 4(6)$
 which is $3x + 4y = 36$
 or $3x + 4y - 36 = 0$

2 a The line is parallel to $2x - y = -3$ or $y = 2x + 3$ which has gradient 2.

$$\therefore \text{the line has gradient 2 and passes through } (5, 3).$$

$$\therefore \text{the equation of the line is } y - 3 = 2(x - 5)$$

$$\therefore y - 3 = 2x - 10$$

$$\therefore y = 2x - 7$$

b The line is perpendicular to $y = -4x + 3$, which has gradient -4 .

\therefore the line has gradient $\frac{1}{4}$ and passes through $(-1, 5)$.

\therefore the equation of the line is $y - 5 = \frac{1}{4}(x - (-1))$

$$\therefore y - 5 = \frac{1}{4}(x + 1)$$

$$\therefore y - 5 = \frac{1}{4}x + \frac{1}{4}$$

$$\therefore y = \frac{1}{4}x + \frac{21}{4}$$

3 $L: y = 3 - 2x$

a Substituting $x = 3$ and $y = k$ into the equation gives $k = 3 - 2(3)$

$$\therefore k = 3 - 6$$

$$\therefore k = -3$$

b Line L has gradient -2 .

c From **b**, the line is perpendicular to L , which has gradient -2 .

\therefore the line has gradient $\frac{1}{2}$ and passes through $P(3, -3)$. {from **a**}

\therefore the equation of the line is $y - (-3) = \frac{1}{2}(x - 3)$

$$\therefore y + 3 = \frac{1}{2}x - \frac{3}{2}$$

$$\therefore y = \frac{1}{2}x - \frac{9}{2}$$

4 **a** x adult tickets at \$30 each and y child tickets at \$15 costs \$120 in total.

$$\therefore 30x + 15y = 120$$

b When $y = 4$, $30x + 15(4) = 120$

$$\therefore 30x + 60 = 120$$

$$\therefore 30x = 60$$

$$\therefore x = 2$$

\therefore Tammy bought 2 adult tickets.

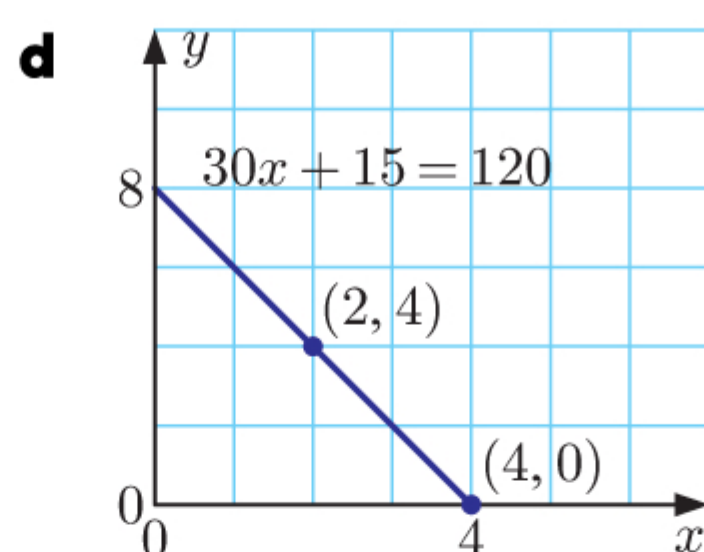
c When $y = 0$, $30x + 15(0) = 120$

$$\therefore 30x = 120$$

$$\therefore x = 4$$

\therefore the x -intercept of the line $30x + 15y = 120$ is 4.

If Tammy did not buy any child tickets, then she bought 4 adult tickets.



5 **a** $2x^2 - 9x = 0$

$$\therefore x(2x - 9) = 0$$

$$\therefore x = 0 \text{ or } 2x - 9 = 0$$

$$\therefore x = 0 \text{ or } \frac{9}{2}$$

c $4x^2 + 11x = 3$

$$\therefore 4x^2 + 11x - 3 = 0$$

$$\therefore (4x - 1)(x + 3) = 0$$

$$\therefore x = \frac{1}{4} \text{ or } -3$$

b $x^2 + 8x - 20 = 0$

$$\therefore (x + 10)(x - 2) = 0$$

$$\therefore x = -10 \text{ or } 2$$

d $(x + 3)(1 - 2x) = -9$

$$\therefore x - 2x^2 + 3 - 6x = -9$$

$$\therefore -2x^2 - 5x + 12 = 0$$

$$\therefore 2x^2 + 5x - 12 = 0$$

$$\therefore (2x - 3)(x + 4) = 0$$

$$\therefore x = \frac{3}{2} \text{ or } -4$$

6 a $x = -2$ is a solution to $x^2 + bx + (b - 2) = 0$
 $\therefore (-2)^2 + b(-2) + b - 2 = 0$
 $\therefore 4 - 2b + b - 2 = 0$
 $\therefore -b = -2$
 $\therefore b = 2$

7 $mx^2 + (m - 2)x + m = 0$ has $a = m$, $b = m - 2$, and $c = m$ $\therefore \Delta = b^2 - 4ac$
 $= (m - 2)^2 - 4(m)(m)$
 $= m^2 - 4m + 4 - 4m^2$
 $= -3m^2 - 4m + 4$

For a repeated root, $\Delta = 0$

$\therefore 0 = -3m^2 - 4m + 4$
 $\therefore 3m^2 + 4m - 4 = 0$
 $\therefore (3m - 2)(m + 2) = 0$
 $\therefore m = \frac{2}{3}$ or -2

8 a $\frac{2}{x} = 5x - 3$
 $\therefore 2 = 5x^2 - 3x$

$\therefore 5x^2 - 3x - 2 = 0$

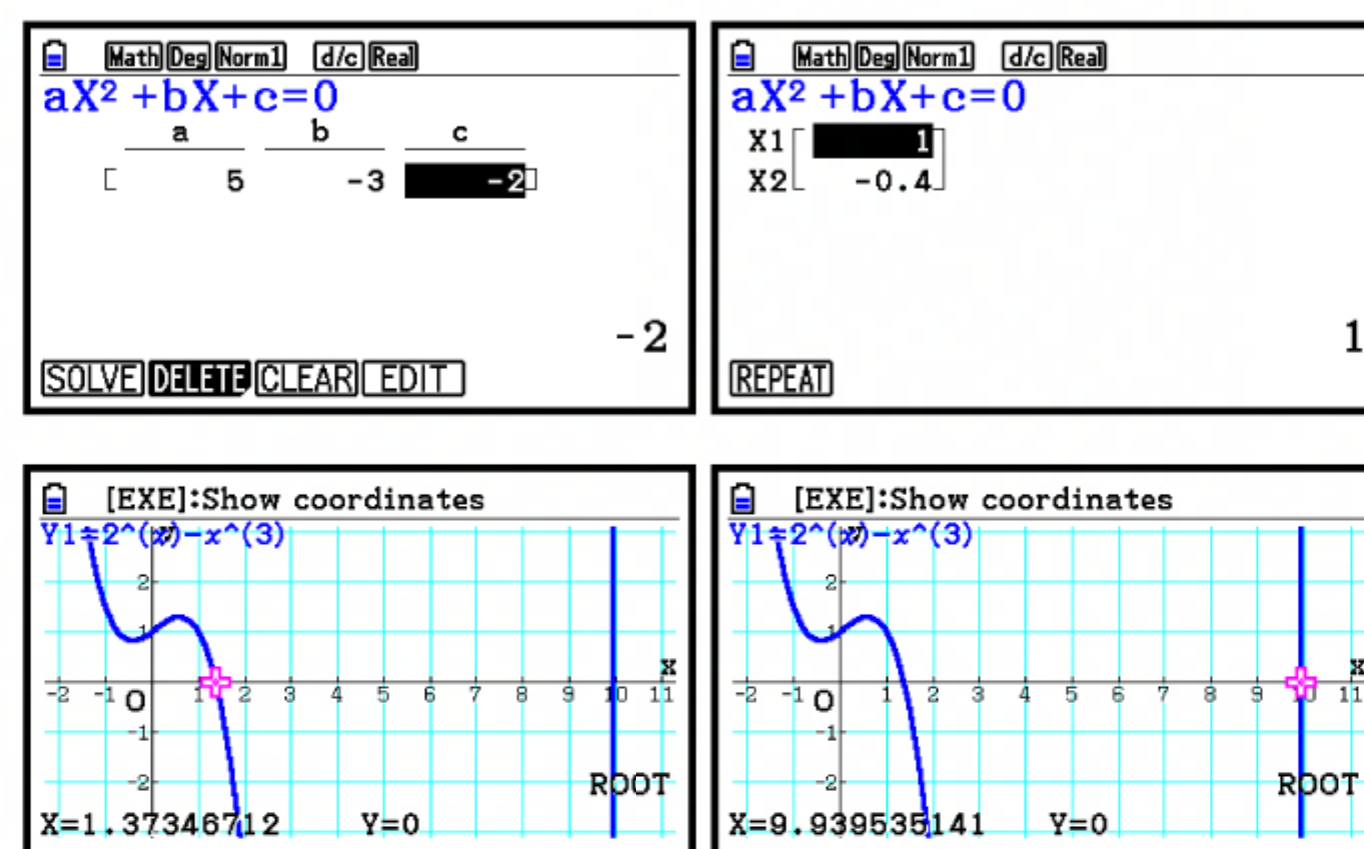
Using technology, $x = -0.4$ or 1

b We graph $y = 2^x - x^3$

The x -intercepts are ≈ 1.37 and 9.94 .

\therefore the solutions are $x \approx 1.37$ or 9.94 .

b $x^2 + 2x = 0$ {using **a**}
 $\therefore x(x + 2) = 0$
 $\therefore x = 0$ or -2
 \therefore the other solution to the equation is $x = 0$.



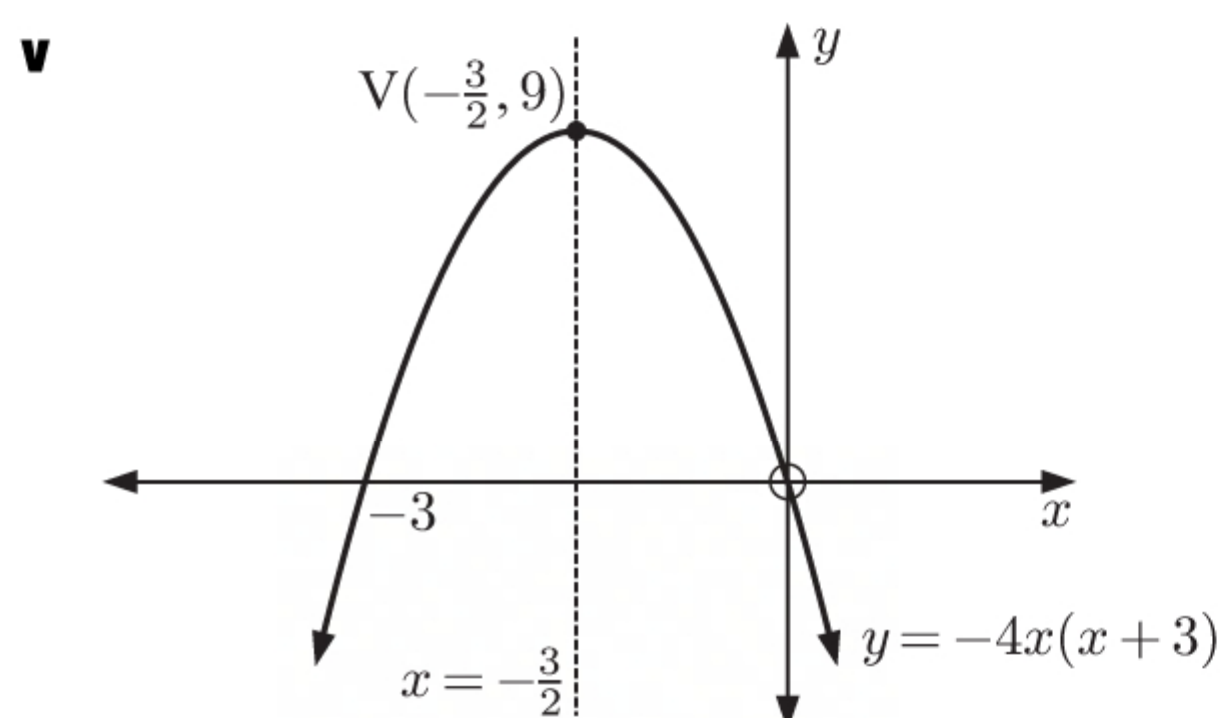
9 a $y = -4x(x + 3)$

i When $y = 0$, $-4x(x + 3) = 0$
 $\therefore x = 0$ or -3

\therefore the x -intercepts are 0 and -3 .

iii When $x = -\frac{3}{2}$, $y = -4(-\frac{3}{2})(-\frac{3}{2} + 3)$
 $= -4(-\frac{3}{2})(\frac{3}{2})$
 $= 9$

\therefore the vertex is $(-\frac{3}{2}, 9)$.



b $y = \frac{1}{2}(x + 6)(x - 4)$

i When $y = 0$, $\frac{1}{2}(x + 6)(x - 4) = 0$
 $\therefore x = -6$ or 4

\therefore the x -intercepts are -6 and 4 .

iii When $x = -1$, $y = \frac{1}{2}(-1 + 6)(-1 - 4)$
 $= \frac{1}{2}(5)(-5)$
 $= -\frac{25}{2}$

\therefore the vertex is $(-1, -\frac{25}{2})$.

ii The axis of symmetry is midway between the x -intercepts.

\therefore the axis of symmetry is $x = \frac{0 + (-3)}{2} = -\frac{3}{2}$.

iv When $x = 0$, $y = 0$

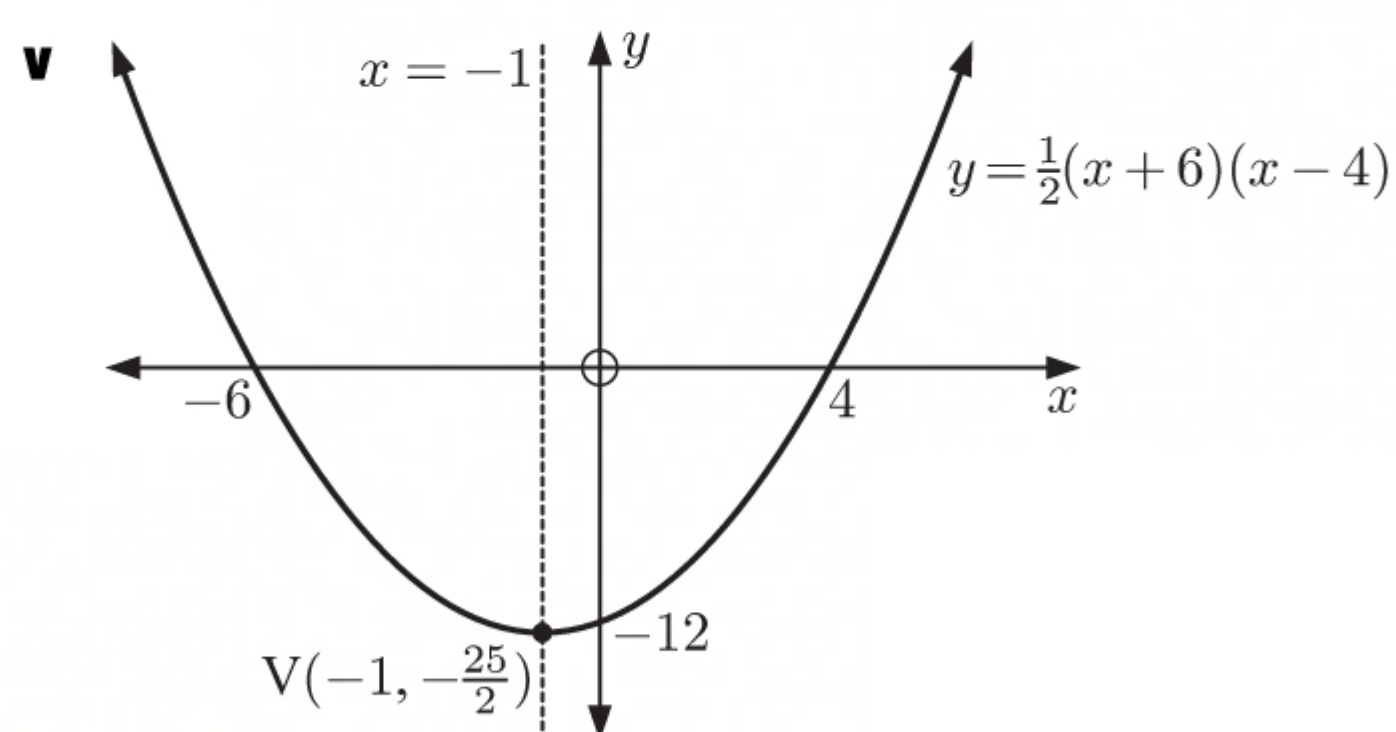
\therefore the y -intercept is 0 .

ii The axis of symmetry is midway between the x -intercepts.

\therefore the axis of symmetry is $x = \frac{-6 + 4}{2} = -1$.

iv When $x = 0$, $y = \frac{1}{2}(6)(-4)$
 $= -12$

\therefore the y -intercept is -12 .

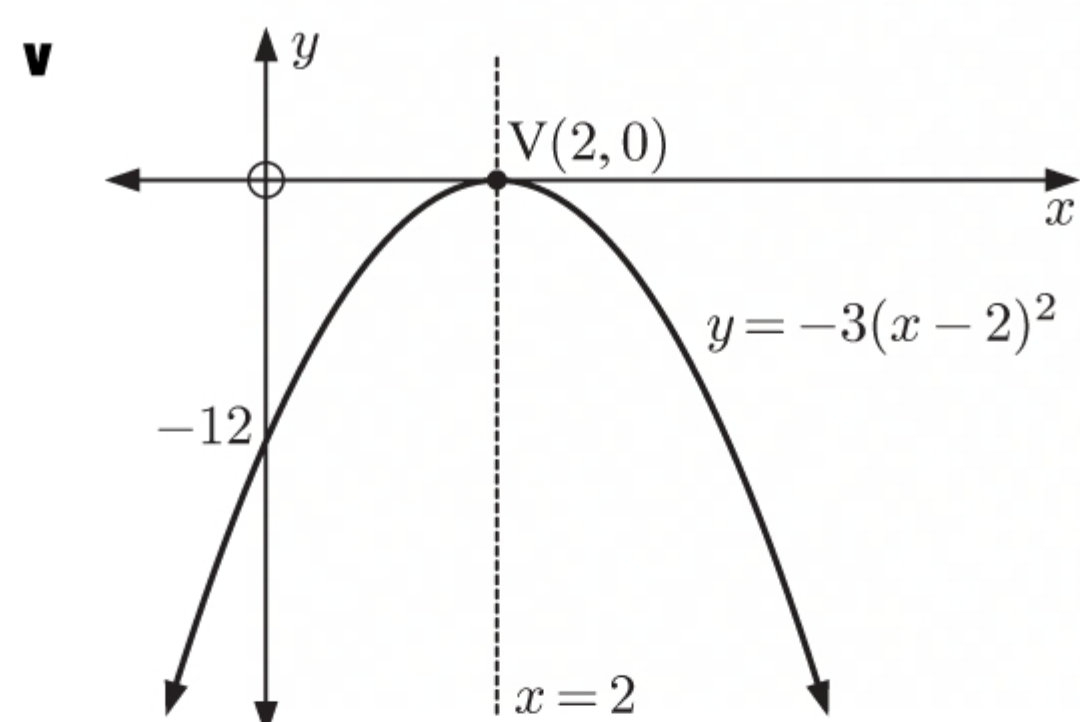


c $y = -3(x - 2)^2$

i When $y = 0$, $-3(x - 2)^2 = 0$
 $\therefore x = 2$

\therefore the x -intercept is 2.

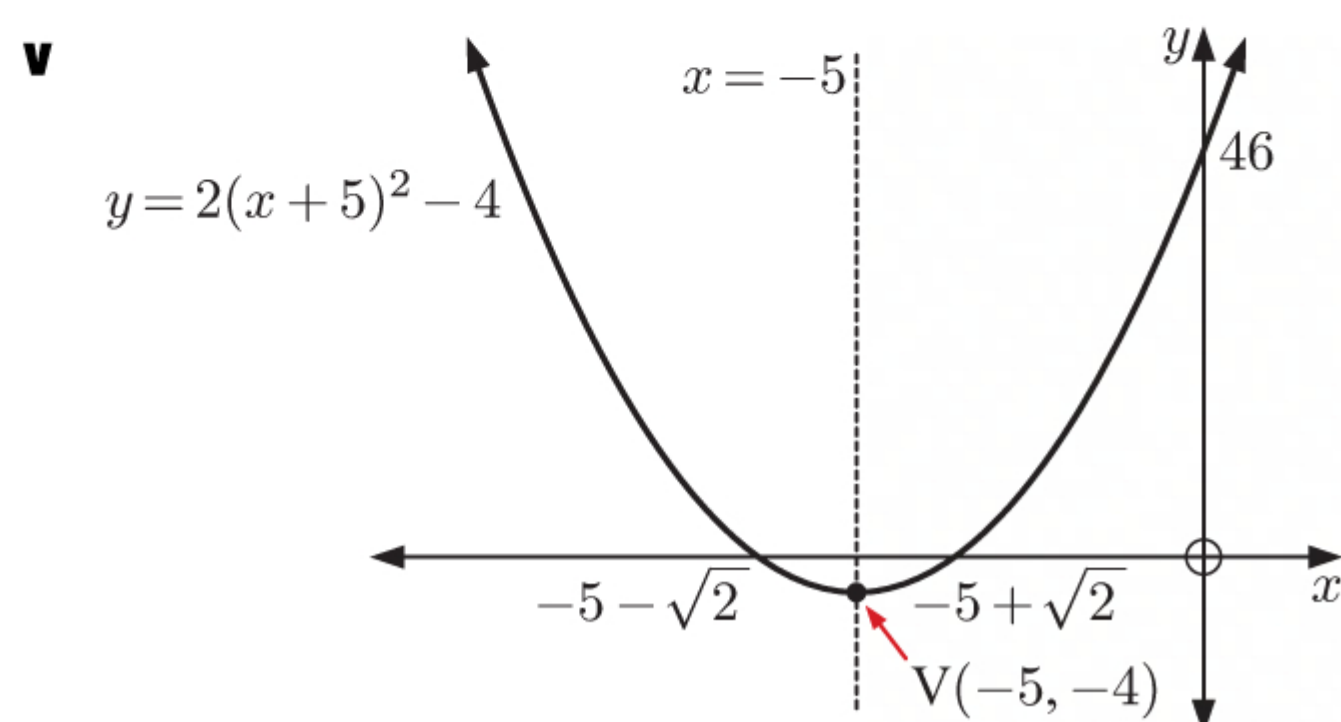
iii The vertex is $(2, 0)$.



d $y = 2(x + 5)^2 - 4$

i When $y = 0$, $2(x + 5)^2 - 4 = 0$
 $\therefore 2(x + 5)^2 = 4$
 $\therefore (x + 5)^2 = 2$
 $\therefore x + 5 = \pm\sqrt{2}$
 $\therefore x = -5 \pm \sqrt{2}$

\therefore the x -intercepts are $-5 - \sqrt{2}$ and $-5 + \sqrt{2}$.



10 a $y = x^2 - 4x + 9$

i The y -intercept is 9.

ii $y = x^2 - 4x + 9$
 $= x^2 - 4x + (-2)^2 + 9 - (-2)^2$
 $\therefore y = (x - 2)^2 + 5$

iii The vertex is $(2, 5)$.

ii The axis of symmetry is $x = 2$.

iv When $x = 0$, $y = -3(-2)^2$
 $= -12$

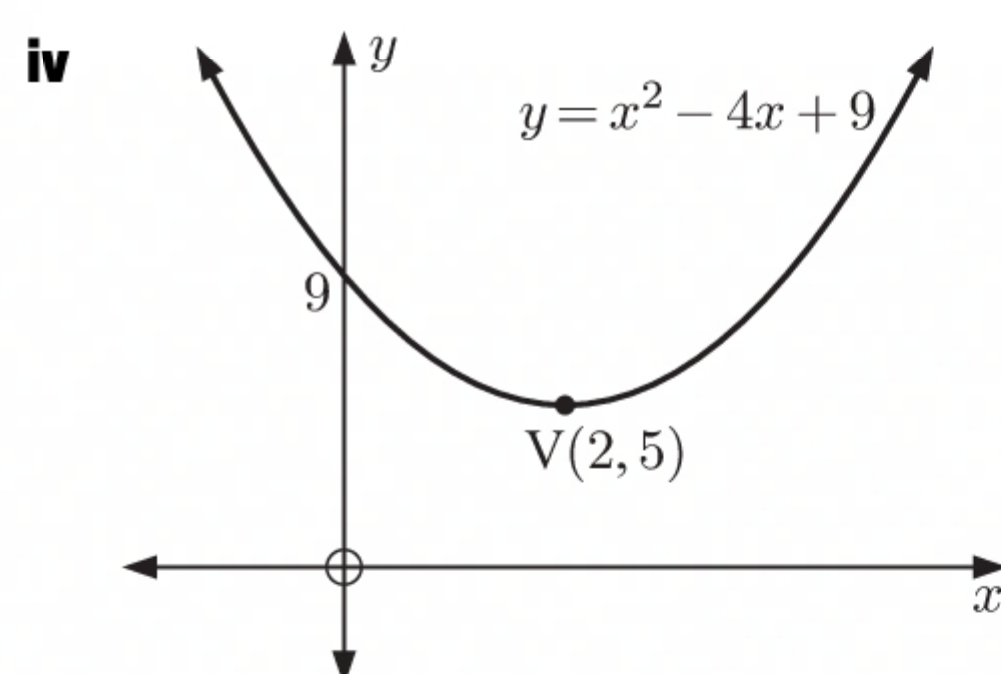
\therefore the y -intercept is -12 .

ii The axis of symmetry is $x = -5$.

iii The vertex is $(-5, -4)$.

iv When $x = 0$, $y = 2(5)^2 - 4$
 $= 50 - 4$
 $= 46$

\therefore the y -intercept is 46.



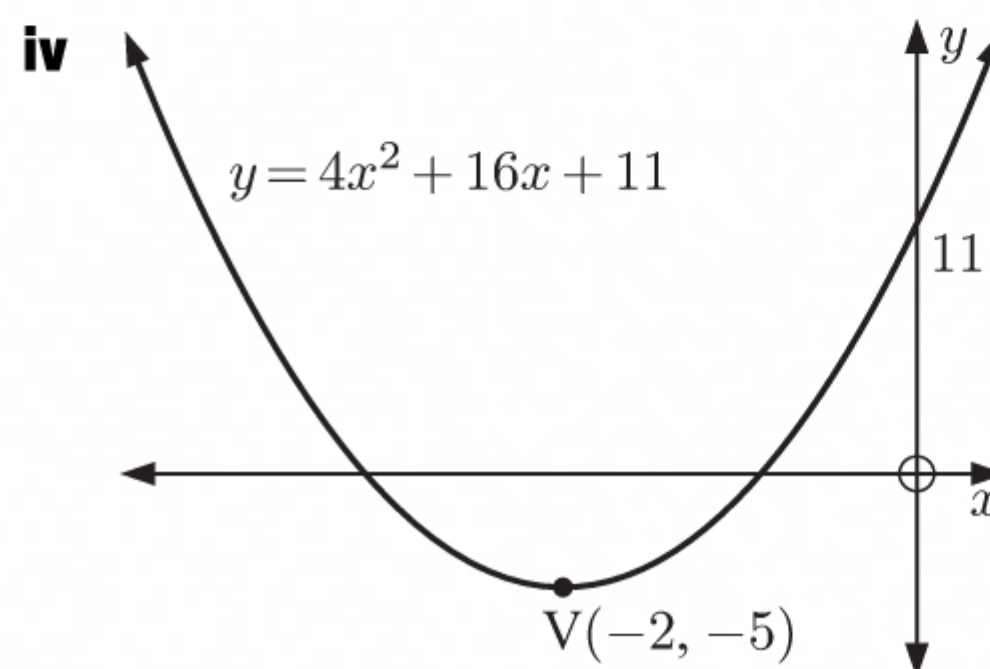
b $y = 4x^2 + 16x + 11$

i The y -intercept is 11.

ii

$$\begin{aligned} y &= 4x^2 + 16x + 11 \\ &= 4\left[x^2 + 4x + \frac{11}{4}\right] \\ &= 4\left[x^2 + 4x + 2^2 + \frac{11}{4} - 2^2\right] \\ &= 4\left[(x+2)^2 - \frac{5}{4}\right] \\ \therefore y &= 4(x+2)^2 - 5 \end{aligned}$$

iii The vertex is $(-2, -5)$.



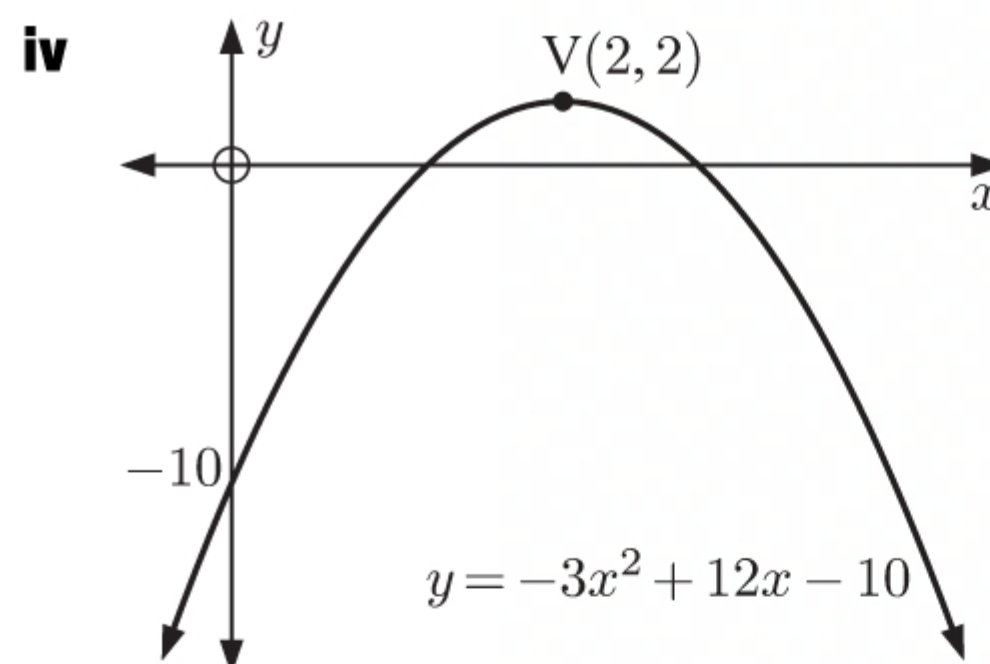
c $y = -3x^2 + 12x - 10$

i The y -intercept is -10 .

ii

$$\begin{aligned} y &= -3x^2 + 12x - 10 \\ &= -3\left[x^2 - 4x + \frac{10}{3}\right] \\ &= -3\left[x^2 - 4x + (-2)^2 + \frac{10}{3} - (-2)^2\right] \\ &= -3\left[(x-2)^2 - \frac{2}{3}\right] \\ \therefore y &= -3(x-2)^2 + 2 \end{aligned}$$

iii The vertex is $(2, 2)$.



11 a i $y = x^2 - 3x - 4$ has $a = 1$, $b = -3$, and $c = -4$

Now $\frac{-b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$

\therefore the axis of symmetry is $x = \frac{3}{2}$.

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 4$

$$\begin{aligned} &= \frac{9}{4} - \frac{9}{2} - 4 \\ &= -\frac{25}{4} \end{aligned}$$

\therefore the vertex is $\left(\frac{3}{2}, -\frac{25}{4}\right)$.

ii $a > 0$, so the shape is

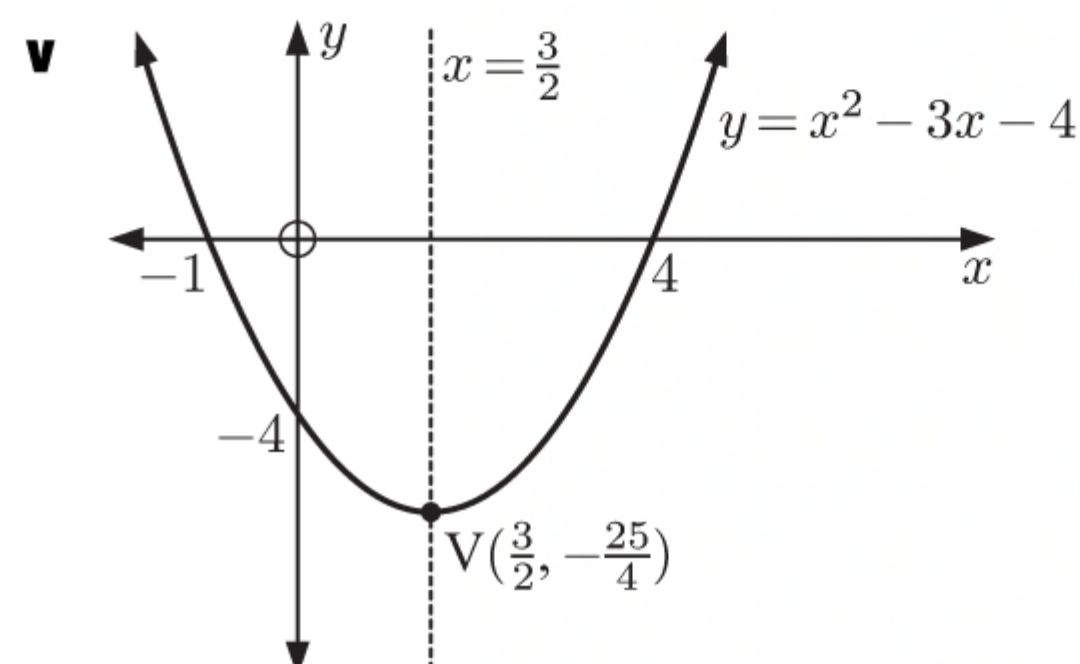
\therefore the vertex $\left(\frac{3}{2}, -\frac{25}{4}\right)$ is a minimum.

iv The y -intercept is -4 .

When $y = 0$,

$$\begin{aligned} x^2 - 3x - 4 &= 0 \\ \therefore (x-4)(x+1) &= 0 \\ \therefore x &= 4 \text{ or } -1 \\ \therefore \text{the } x\text{-intercepts are } 4 \text{ and } -1. \end{aligned}$$

iii The range is $\{y \mid y \geq -\frac{25}{4}\}$.



b i $y = -2x^2 - 5x + 7$ has $a = -2$, $b = -5$, and $c = 7$

Now $\frac{-b}{2a} = \frac{-(-5)}{2(-2)} = -\frac{5}{4}$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

When $x = -\frac{5}{4}$, $y = -2\left(-\frac{5}{4}\right)^2 - 5\left(-\frac{5}{4}\right) + 7$

$$\begin{aligned} &= -\frac{25}{8} + \frac{25}{4} + 7 \\ &= \frac{81}{8} \end{aligned}$$

\therefore the vertex is $\left(-\frac{5}{4}, \frac{81}{8}\right)$.

ii $a < 0$, so the shape is

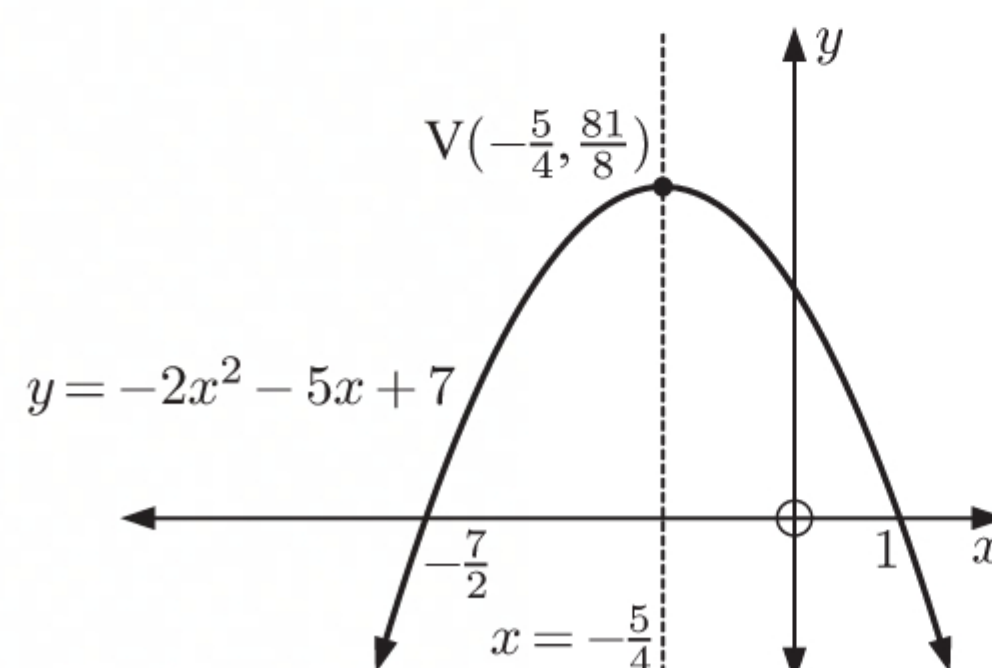
\therefore the vertex $\left(-\frac{5}{4}, \frac{81}{8}\right)$ is a maximum.

iii The range is $\{y \mid y \leq \frac{81}{8}\}$.

iv The y -intercept is 7.

$$\begin{aligned}\text{When } y &= 0, \\ -2x^2 - 5x + 7 &= 0 \\ \therefore -(2x^2 + 5x - 7) &= 0 \\ \therefore -(2x + 7)(x - 1) &= 0 \\ \therefore x &= -\frac{7}{2} \text{ or } 1 \\ \therefore \text{the } x\text{-intercepts are } &-\frac{7}{2} \text{ and } 1.\end{aligned}$$

v



12 $y = (k + 3)x^2 - 2kx + (k - 2)$ has $a = k + 3$, $b = -2k$, $c = k - 2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-2k)^2 - 4(k + 3)(k - 2) \\ &= 4k^2 - 4(k^2 + k - 6) \\ &= 4k^2 - 4k^2 - 4k + 24 \\ &= 24 - 4k\end{aligned}$$

Also, if $k = -3$, then the function is the line $y = 6x - 5$, in which case it cuts the x -axis only *once* at $x = \frac{5}{6}$.

- | | | |
|--|--|---|
| <p>a The graph cuts the x-axis twice if $\Delta > 0$.</p> <p>$\therefore 24 - 4k > 0$</p> <p>$\therefore 4k < 24$</p> <p>$\therefore k < 6, \quad k \neq -3$</p> | <p>b The graph touches the x-axis if $\Delta = 0$.</p> <p>$\therefore 24 - 4k = 0$</p> <p>$\therefore 4k = 24$</p> <p>$\therefore k = 6$</p> | <p>c The graph misses the x-axis if $\Delta < 0$.</p> <p>$\therefore 24 - 4k < 0$</p> <p>$\therefore 4k > 24$</p> <p>$\therefore k > 6$</p> |
|--|--|---|

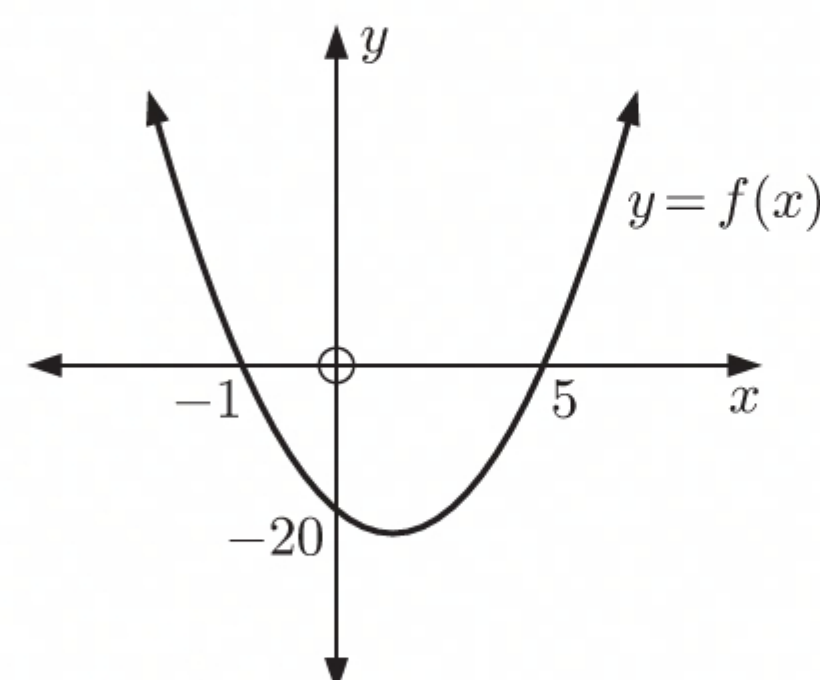
13 a $y = f(x) = a(x - p)(x - q)$ has x -intercepts -1 and 5 .

$\therefore p = 5$ and $q = -1 \quad \{p > q\}$

b From **a**, $f(x) = a(x - 5)(x + 1)$.

From the graph, $f(0) = -20$

$$\begin{aligned}\therefore -20 &= a(-5)(1) \\ \therefore -20 &= -5a \\ \therefore a &= 4\end{aligned}$$



c The axis of symmetry is midway between the x -intercepts.

\therefore the axis of symmetry is $x = \frac{-1 + 5}{2} = 2$.

14 $y = -3x^2 - x + 1$ has $a = -3$, $b = -1$, and $c = 1$

a Since $a < 0$, the graph is concave down.

b $\Delta = b^2 - 4ac$

$$\begin{aligned}&= (-1)^2 - 4(-3)(1) \\ &= 13\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.



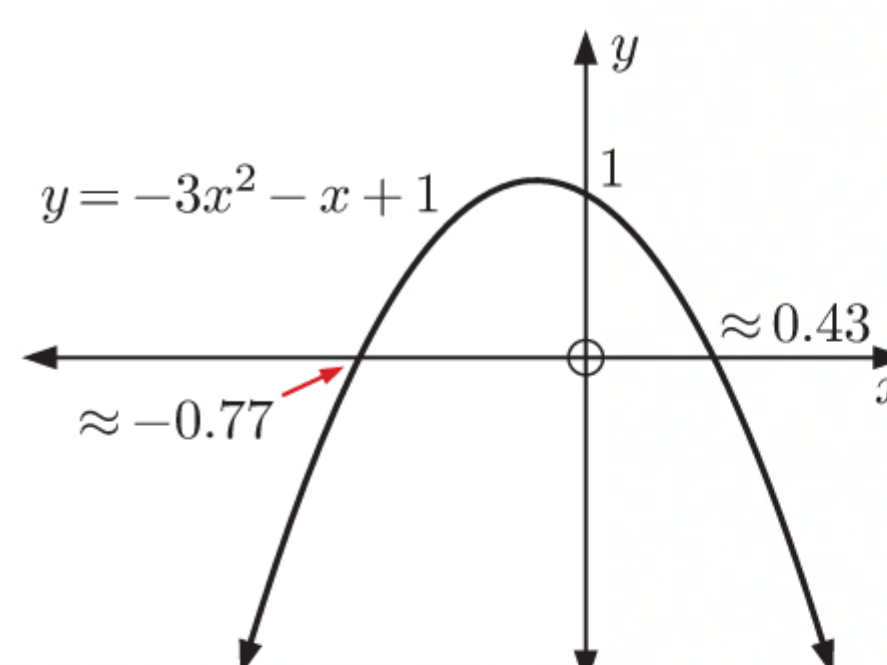
c When $y = 0$, $-3x^2 - x + 1 = 0$

$$\begin{aligned}\therefore x &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ \therefore x &= \frac{-(-1) \pm \sqrt{13}}{2(-3)} \\ \therefore x &= -\frac{1}{6} \pm \frac{\sqrt{13}}{6} \\ \therefore x &\approx 0.43 \text{ or } -0.77\end{aligned}$$

\therefore the x -intercepts are ≈ 0.43 or ≈ -0.77 .

d The y -intercept is 1.

e



15 $y = mx^2 + 4x + 6$ has $a = m$, $b = 4$, and $c = 6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 4^2 - 4(m)(6) \\ &= 16 - 24m\end{aligned}$$

The graph lies entirely above the x -axis if it is positive definite.

$$\begin{aligned}\therefore a &> 0 \quad \text{and} \quad \Delta < 0 \\ \therefore m &> 0 \quad \text{and} \quad 16 - 24m < 0 \\ \therefore m &> 0 \quad \text{and} \quad 24m > 16 \\ \therefore m &> 0 \quad \text{and} \quad m > \frac{2}{3} \\ \therefore m &> \frac{2}{3}\end{aligned}$$

16 $y = kx - 2$ is a tangent to $y = 3x^2 + x + 1$ if they meet at exactly one point (touch).

$$\begin{aligned}y = 3x^2 + x + 1 \text{ meets } y = kx - 2 \text{ where } 3x^2 + x + 1 &= kx - 2 \\ \therefore 3x^2 + (1 - k)x + 3 &= 0\end{aligned}$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\begin{aligned}\therefore (1 - k)^2 - 4(3)(3) &= 0 \\ \therefore 1 - 2k + k^2 - 36 &= 0 \\ \therefore k^2 - 2k - 35 &= 0 \\ \therefore (k - 7)(k + 5) &= 0 \\ \therefore k &= 7 \text{ or } -5\end{aligned}$$

17 a $y = x^2 - 4x - 5$ meets $y = 3x - 11$ where $x^2 - 4x - 5 = 3x - 11$

$$\begin{aligned}\therefore x^2 - 7x + 6 &= 0 \\ \therefore (x - 6)(x - 1) &= 0 \\ \therefore x &= 6 \text{ or } 1\end{aligned}$$

Substituting into $y = 3x - 11$, when $x = 6$, $y = 7$,
and when $x = 1$, $y = -8$.

\therefore the graphs meet at $(6, 7)$ and $(1, -8)$.

b $y = -2x^2 + 5x$ meets $y = 5 - 2x$ where $-2x^2 + 5x = 5 - 2x$

$$\begin{aligned}\therefore -2x^2 + 7x - 5 &= 0 \\ \therefore -(2x^2 - 7x + 5) &= 0 \\ \therefore -(2x - 5)(x - 1) &= 0 \\ \therefore x &= \frac{5}{2} \text{ or } 1\end{aligned}$$

Substituting into $y = 5 - 2x$, when $x = \frac{5}{2}$, $y = 0$,
and when $x = 1$, $y = 3$.

\therefore the graphs meet at $(\frac{5}{2}, 0)$ and $(1, 3)$.

18 $y = 2x + c$ meets $y = 3x^2 + 5x + 7$ where $3x^2 + 5x + 7 = 2x + c$

$$\therefore 3x^2 + 3x + (7 - c) = 0$$

The graphs will never meet if this equation has no real roots $\therefore \Delta < 0$

$$\begin{aligned}\therefore 3^2 - 4(3)(7 - c) &< 0 \\ \therefore 9 - 84 + 12c &< 0 \\ \therefore 12c &< 75 \\ \therefore c &< \frac{25}{4}\end{aligned}$$

- 19 a** The graph touches the x -axis, and has axis of symmetry $x = -3$.

\therefore the graph touches the x -axis at -3 .

\therefore the quadratic has the form $f(x) = a(x + 3)^2$, $a \neq 0$.

The y -intercept is -3 .

$$\text{So, } f(0) = -3$$

$$\therefore -3 = a(3)^2$$

$$\therefore 9a = -3$$

$$\therefore a = -\frac{1}{3}$$

$$\begin{aligned} \text{The quadratic is } f(x) &= -\frac{1}{3}(x + 3)^2 \\ &= -\frac{1}{3}(x^2 + 6x + 9) \\ \therefore f(x) &= -\frac{1}{3}x^2 - 2x - 3 \end{aligned}$$

- b** $y = kx - \frac{9}{4}$ is a tangent to $y = -\frac{1}{3}x^2 - 2x - 3$ if they meet at exactly one point (touch).

$$y = -\frac{1}{3}x^2 - 2x - 3 \text{ meets } y = kx - \frac{9}{4} \text{ where } -\frac{1}{3}x^2 - 2x - 3 = kx - \frac{9}{4}$$

$$\therefore -\frac{1}{3}x^2 - (k + 2)x - \frac{3}{4} = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (-(k + 2))^2 - 4\left(-\frac{1}{3}\right)\left(-\frac{3}{4}\right) = 0$$

$$\therefore k^2 + 4k + 4 - 1 = 0$$

$$\therefore k^2 + 4k + 3 = 0$$

$$\therefore (k + 3)(k + 1) = 0$$

$$\therefore k = -3 \text{ or } -1$$

When $k = -3$, the tangent is $y = -3x - \frac{9}{4}$.

The tangent meets the curve where $-\frac{1}{3}x^2 - (-1)x - \frac{3}{4} = 0$

$$\therefore -\frac{1}{3}x^2 + x - \frac{3}{4} = 0$$

$$\therefore 4x^2 - 12x + 9 = 0$$

$$\therefore (2x - 3)^2 = 0$$

$$\therefore x = \frac{3}{2}$$

Substituting into $y = -3x - \frac{9}{4}$, when $x = \frac{3}{2}$, $y = -\frac{27}{4}$.

\therefore the tangent $y = -3x - \frac{9}{4}$ meets the curve at $(\frac{3}{2}, -\frac{27}{4})$.

When $k = -1$, the tangent is $y = -x - \frac{9}{4}$.

The tangent meets the curve where $-\frac{1}{3}x^2 - x - \frac{3}{4} = 0$

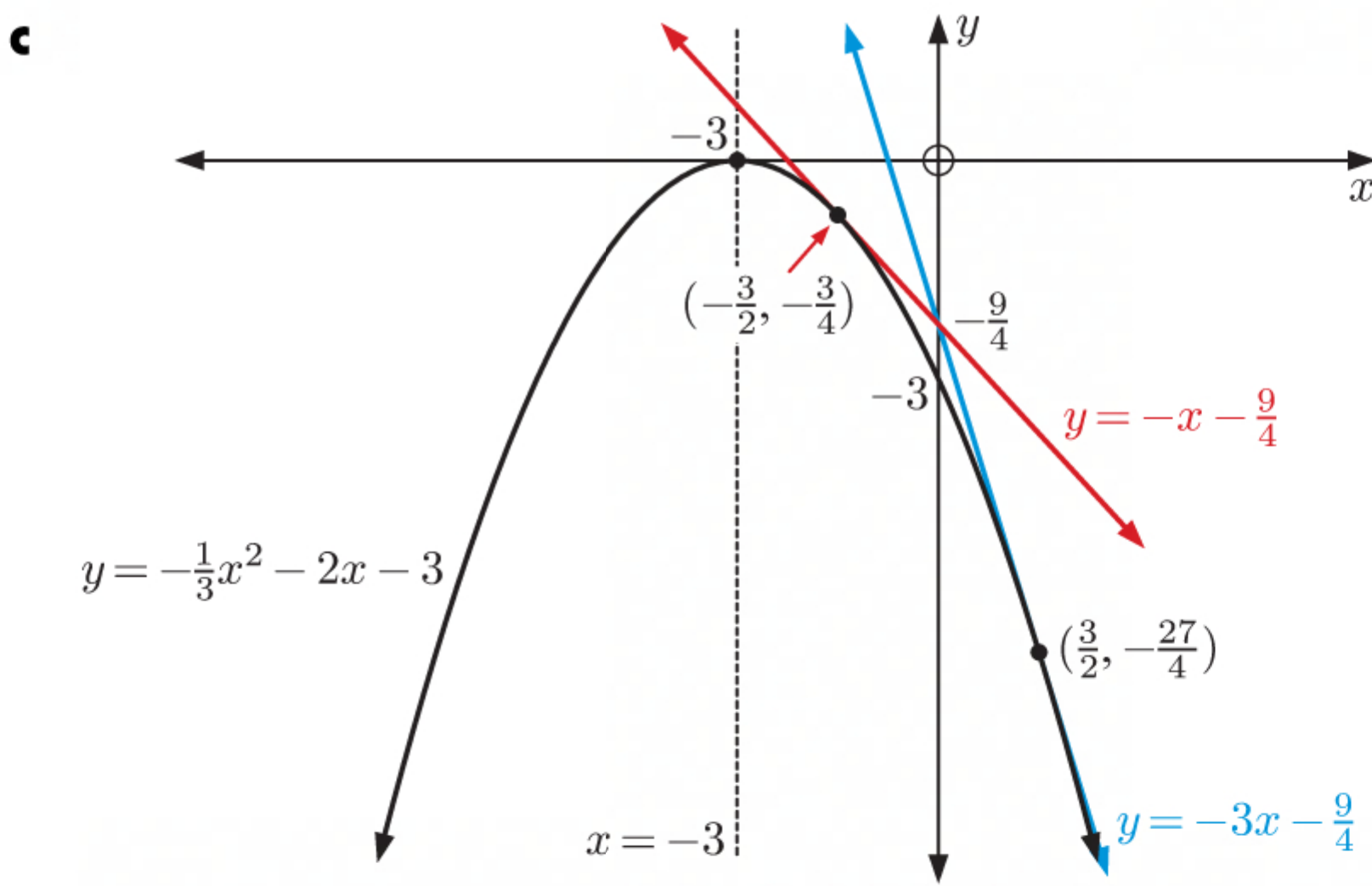
$$\therefore 4x^2 + 12x + 9 = 0$$

$$\therefore (2x + 3)^2 = 0$$

$$\therefore x = -\frac{3}{2}$$

Substituting into $y = -x - \frac{9}{4}$, when $x = -\frac{3}{2}$, $y = -\frac{3}{4}$.

\therefore the tangent $y = -x - \frac{9}{4}$ meets the curve at $(-\frac{3}{2}, -\frac{3}{4})$.



20 Let the quadratic function be $y = ax^2 + bx + c$.

The y -intercept is 4. $\therefore c = 4$

Now, $y = ax^2 + bx + 4$ meets $y = x - 5$ where $ax^2 + bx + 4 = x - 5$
 $\therefore ax^2 + (b - 1)x + 9 = 0$

The graphs touch when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (b - 1)^2 - 4(a)(9) = 0$$

$$\therefore (b - 1)^2 - 36a = 0$$

$$\therefore 36a = (b - 1)^2$$

$$\therefore a = \frac{1}{36}(b - 1)^2 \quad \dots (*)$$

Also, $y = ax^2 + bx + 4$ meets $y = -2x$ where $ax^2 + bx + 4 = -2x$

$$\therefore ax^2 + (b + 2)x + 4 = 0$$

The graphs touch when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (b + 2)^2 - 4(a)(4) = 0$$

$$\therefore (b + 2)^2 - 16a = 0$$

$$\therefore (b + 2)^2 = 16a$$

$$\therefore (b + 2)^2 = \frac{16}{36}(b - 1)^2 \quad \{\text{using } (*)\}$$

$$\therefore b^2 + 4b + 4 = \frac{4}{9}(b^2 - 2b + 1)$$

$$\therefore 9(b^2 + 4b + 4) = 4(b^2 - 2b + 1)$$

$$\therefore 9b^2 + 36b + 36 = 4b^2 - 8b + 4$$

$$\therefore 5b^2 + 44b + 32 = 0$$

$$\therefore (5b + 4)(b + 8) = 0$$

$$\therefore b = -\frac{4}{5} \text{ or } -8$$

Substituting into (*), when $b = -\frac{4}{5}$, $a = \frac{1}{36}\left(-\frac{4}{5} - 1\right)^2 = \frac{9}{100}$,

and when $b = -8$, $a = \frac{1}{36}(-8 - 1)^2 = \frac{9}{4}$.

\therefore the quadratic function is $y = \frac{9}{100}x^2 - \frac{4}{5}x + 4$ or $y = \frac{9}{4}x^2 - 8x + 4$.

21 a Total profit $P = x\left[\left(42 - \frac{x}{15}\right) - \left(26 + \frac{10}{x}\right)\right]$
 $= x\left[-\frac{x}{15} + 16 - \frac{10}{x}\right]$
 $= -\frac{1}{15}x^2 + 16x - 10$ euros

b $P = -\frac{1}{15}x^2 + 16x - 10$ has $a = -\frac{1}{15}$, $b = 16$, and $c = -10$

Since $a < 0$, the shape is

The maximum profit occurs when $x = \frac{-b}{2a} = \frac{-16}{2\left(-\frac{1}{15}\right)} = 120$

So, 120 radios should be made per day to maximise profit.

c When $x = 120$, $P = -\frac{1}{15}(120)^2 + 16(120) - 10$
 $= 950$

So, the maximum profit is €950.

22 a Let the height of the equilateral triangle ends be h cm.

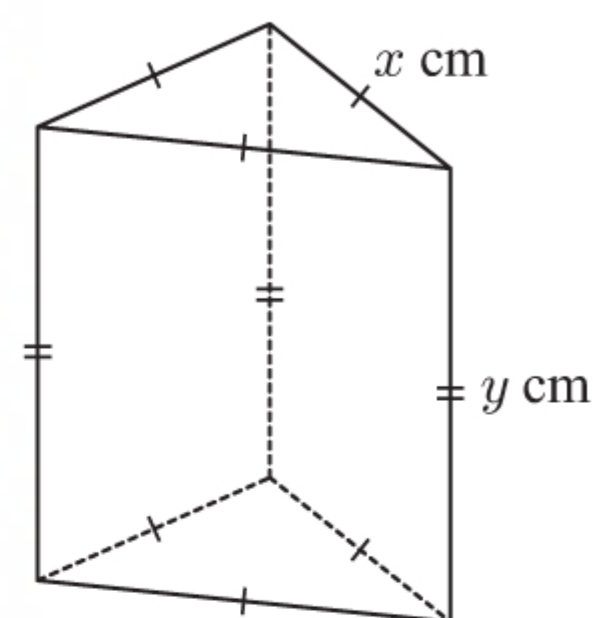
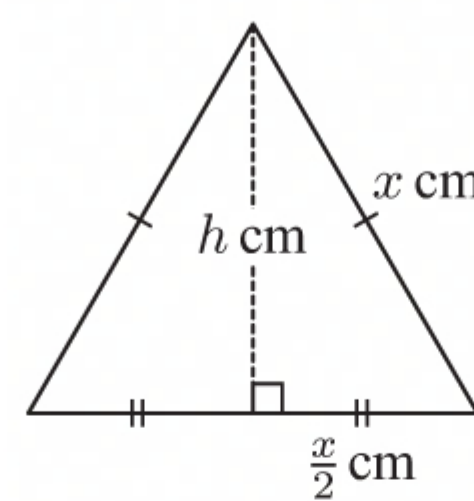
$$\therefore \left(\frac{x}{2}\right)^2 + h^2 = x^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \frac{x^2}{4} + h^2 = x^2$$

$$\therefore h^2 = \frac{3x^2}{4}$$

$$\therefore h = \frac{\sqrt{3}}{2}x \quad \{h > 0\}$$

$$\begin{aligned} \text{So, area of end} &= \frac{1}{2} \times x \times h \\ &= \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x \\ &= \frac{\sqrt{3}}{4}x^2 \text{ cm}^2 \end{aligned}$$



b The sum of all side lengths of the prism must be 1.8 m or 180 cm.

$$\therefore 6x + 3y = 180$$


$$\therefore 3y = 180 - 6x$$

$$\therefore y = 60 - 2x$$

$$\begin{aligned} \text{So, area of rectangular face} &= x \times y \\ &= x(60 - 2x) \\ &= 60x - 2x^2 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \therefore \text{the total surface area } A &= 2 \times \text{area of end} + 3 \times \text{area of rectangular face} \\ &= 2\left(\frac{\sqrt{3}}{4}x^2\right) + 3(60x - 2x^2) \quad \{\text{using a}\} \\ &= \frac{\sqrt{3}}{2}x^2 + 180x - 6x^2 \\ &= \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x \text{ cm}^2 \end{aligned}$$

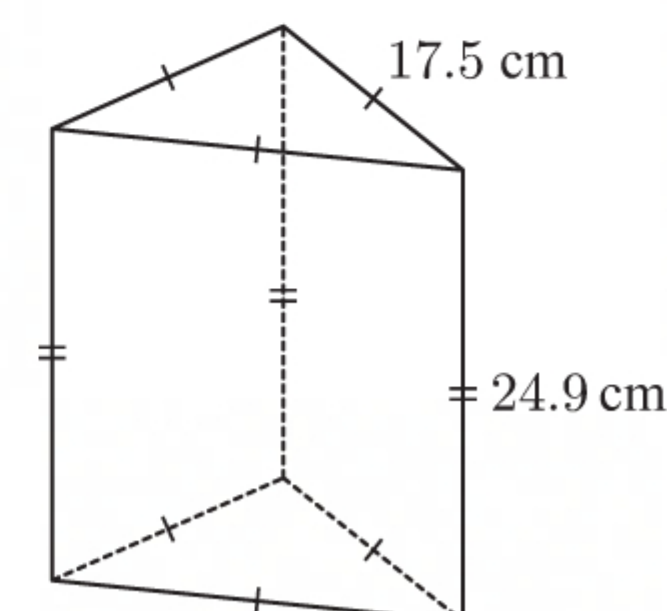
c $A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x$ has $a = \frac{\sqrt{3}}{2} - 6$, $b = 180$, and $c = 0$

Since $a < 0$, the shape is 

The maximum surface area occurs when $x = \frac{-b}{2a} = \frac{-180}{2\left(\frac{\sqrt{3}}{2} - 6\right)} \approx 17.5$

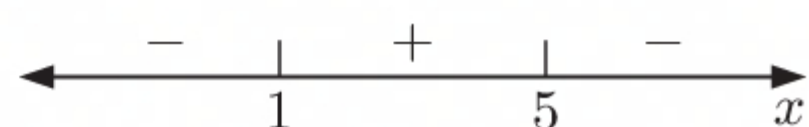
and $y = 60 - 2x \approx 24.9$

So, the dimensions Andreas should choose for the aquarium are shown alongside:



23 a $(x - 1)(5 - x) \leq 0$

Sign diagram of LHS is



$$\therefore x \leq 1 \text{ or } x \geq 5$$

b $x^2 + 8x - 20 < 0$

$$\therefore (x + 10)(x - 2) < 0$$

Sign diagram of LHS is



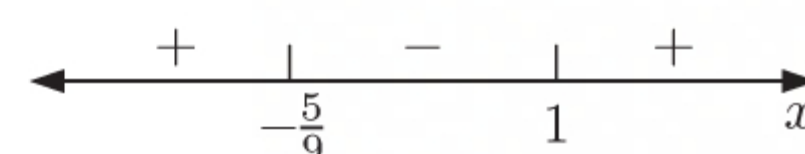
$$\therefore -10 < x < 2$$

c $-9x^2 + 4x + 5 \geq 0$

$$\therefore 9x^2 - 4x - 5 \leq 0$$

$$\therefore (9x + 5)(x - 1) \leq 0$$

Sign diagram of LHS is



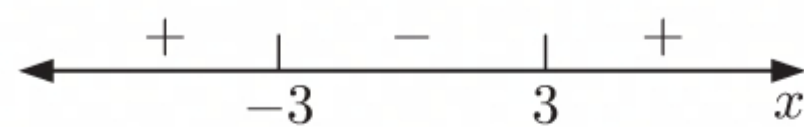
$$\therefore -\frac{5}{9} \leq x \leq 1$$

24 a $x^2 > 9$

$$\therefore x^2 - 9 > 0$$

$$\therefore (x+3)(x-3) > 0$$

Sign diagram of LHS is



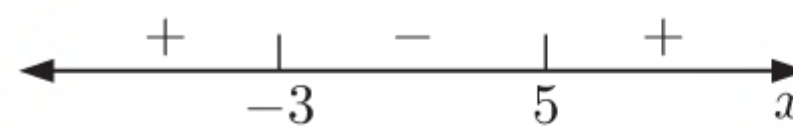
$$\therefore x < -3 \text{ or } x > 3$$

b $x^2 - 15 \leq 2x$

$$\therefore x^2 - 2x - 15 \leq 0$$

$$\therefore (x-5)(x+3) \leq 0$$

Sign diagram of LHS is



$$\therefore -3 \leq x \leq 5$$

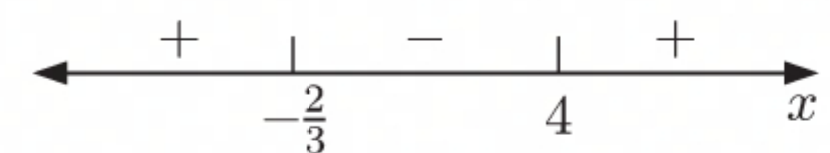
c $3x^2 < 2(5x+4)$

$$\therefore 3x^2 < 10x + 8$$

$$\therefore 3x^2 - 10x - 8 < 0$$

$$\therefore (3x+2)(x-4) < 0$$

Sign diagram of LHS is



$$\therefore -\frac{2}{3} < x < 4$$

25 $y = kx^2 - (k-6)x + (k-6)$ has $a = k$, $b = -(k-6)$, and $c = k-6$

$$\therefore \Delta = b^2 - 4ac$$

$$= [-(k-6)]^2 - 4(k)(k-6)$$

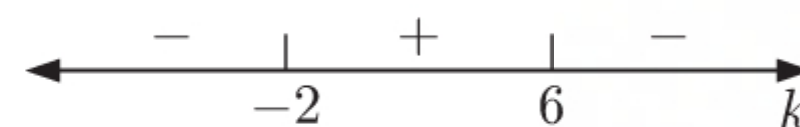
$$= k^2 - 12k + 36 - 4k^2 + 24k$$

$$= -3k^2 + 12k + 36$$

$$= -3(k^2 - 4k - 12)$$

$$= -3(k-6)(k+2)$$

So, Δ has sign diagram



a The graph cuts the x -axis twice if $\Delta > 0$.

$$\therefore -2 < k < 6, k \neq 0$$

b The graph touches the x -axis if $\Delta = 0$.

$$\therefore k = -2 \text{ or } k = 6$$

c The graph misses the x -axis if $\Delta < 0$.

$$\therefore k < -2 \text{ or } k > 6$$

26 $W(t) = 1000 - 0.5t$ litres

a $W(0) = 1000$

The initial amount of water in the tank was 1000 L.

b $W(t) = 700$, so $700 = 1000 - 0.5t$

$$\therefore 0.5t = 300$$

$$\therefore t = 600$$

After 600 hours, or 25 days, the amount of water in the tank is 700 L.

c The tank is empty when $W(t) = 0$.

$$\text{This occurs when } 0.5t = 1000$$

$$\therefore t = 2000$$

It will take 2000 hours, or 83 days, 8 hours, for the tank to empty.

27 $f(x) = \frac{x-2}{x-3}$

a $f(-x) = \frac{-x-2}{-x-3}$

$$= \frac{-(x+2)}{-(x+3)}$$

$$= \frac{x+2}{x+3}$$

b $f(x+2) = \frac{(x+2)-2}{(x+2)-3}$

$$= \frac{x+2-2}{x+2-3}$$

$$= \frac{x}{x-1}$$

c $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-2}{\frac{1}{x}-3} \times \frac{x}{x}$

$$= \frac{1-2x}{1-3x}$$

28 From the graph: Domain is $\{x \mid -6 \leq x \leq 6 \text{ and } x \neq 3\}$

Range is $\{y \mid 0 \leq y \leq 5\}$

a $x = 0$ satisfies $-6 \leq x \leq 6$ and $x \neq 3$.

\therefore "0 is in the domain of f " is true.

b $y = 0$ satisfies $0 \leq y \leq 5$.

\therefore "0 is in the range of f " is true.

c $y = 6$ does not satisfy $0 \leq y \leq 5$.

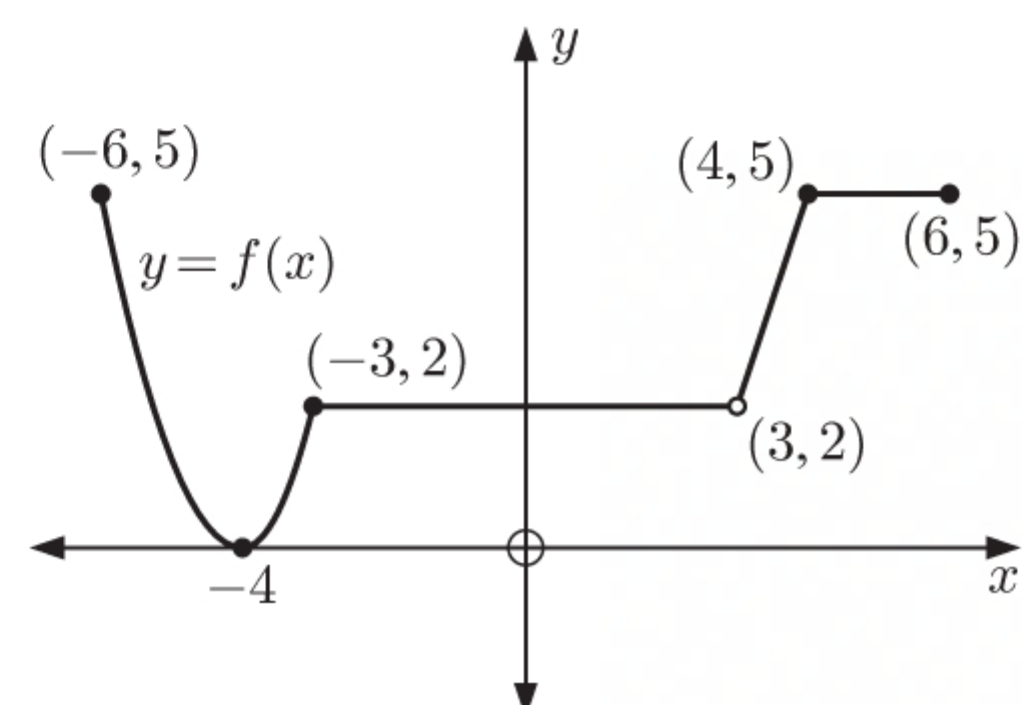
\therefore "6 is in the range of f " is false.

d $x = 3$ does not satisfy $x \neq 3$.

\therefore "3 is in the domain of f " is false.

e $y = 2$ satisfies $0 \leq y \leq 5$.

\therefore "2 is in the range of f " is true.



29 a $\sqrt{3-2x}$ is defined when $3-2x \geq 0$
 $\therefore 2x \leq 3$
 $\therefore x \leq \frac{3}{2}$

\therefore the domain is $\{x \mid x \leq \frac{3}{2}\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

b $\frac{2}{x-3}$ is defined when $x-3 \neq 0$
 $\therefore x \neq 3$

\therefore the domain is $\{x \mid x \neq 3\}$.

No matter how large or small x is, $y = f(x)$ is never zero.

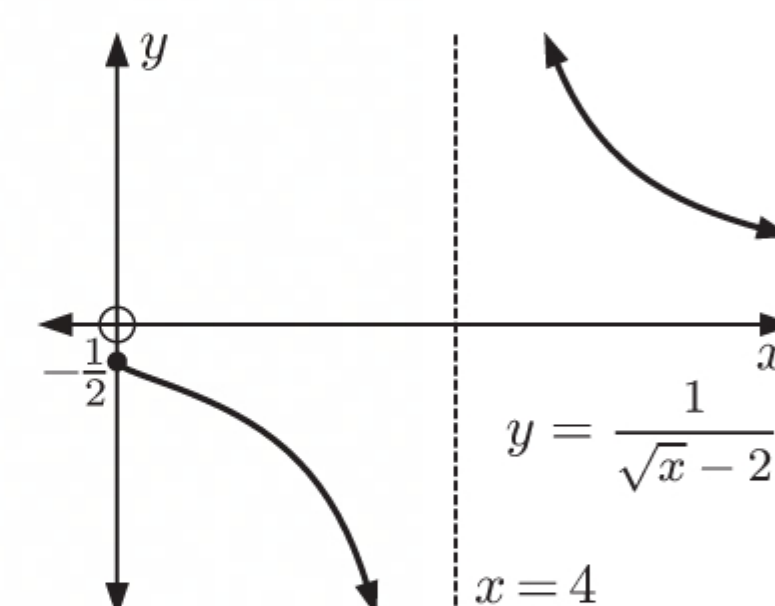
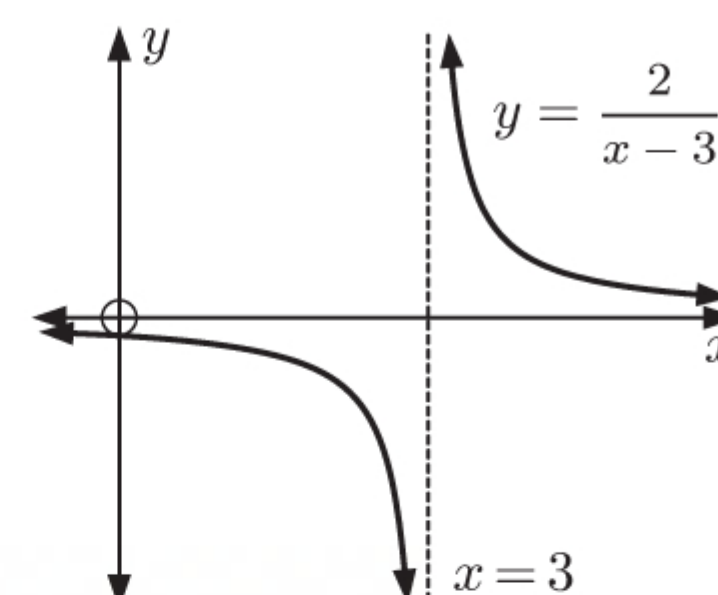
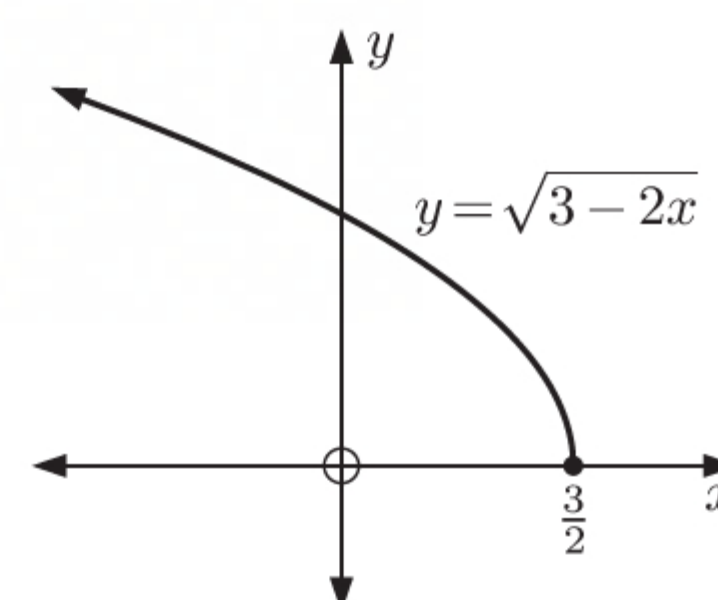
\therefore the range is $\{y \mid y \neq 0\}$.

c $\frac{1}{\sqrt{x}-2}$ is defined when $x \geq 0$ and $\sqrt{x}-2 \neq 0$
 $\therefore x \geq 0$ and $\sqrt{x} \neq 2$
 $\therefore x \geq 0$ and $x \neq 4$

\therefore the domain is $\{x \mid x \geq 0 \text{ and } x \neq 4\}$.

For $0 \leq x \leq 4$, $y \leq -\frac{1}{2}$, and for $x > 4$, $y > 0$.

\therefore the range is $\{y \mid y \leq -\frac{1}{2} \text{ or } y > 0\}$.



30 $k(t) = 2t - 4$ for $0 \leq t < 4$, $t \in \mathbb{Z}$

a The function is defined for t such that $0 \leq t < 4$ and $t \in \mathbb{Z}$.

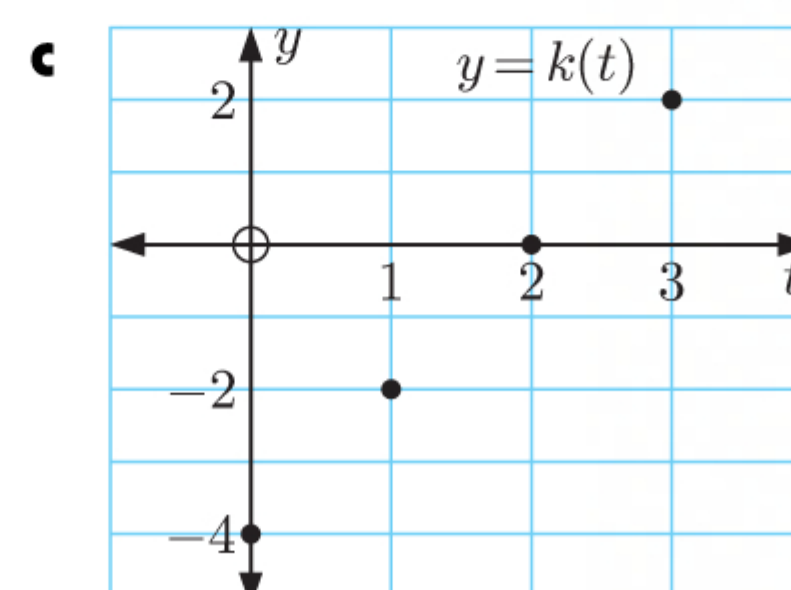
\therefore the domain is $\{0, 1, 2, 3\}$.

b $k(0) = -4$

$k(1) = -2$

$k(2) = 0$

$k(3) = 2 \therefore$ the range is $\{-4, -2, 0, 2\}$.



31 $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$

a The domain is $\{x \mid x \neq 1\}$.

The range is $\{y \mid y \neq 1\}$.

c $f(0) = \frac{2}{-1} = -2$, so the y -intercept is -2 .

$f(x) = 0$ when $x+2 = 0$
 $\therefore x = -2$

\therefore the x -intercept is -2 .

e As $x \rightarrow 1^-$, $f(x) \rightarrow -\infty$

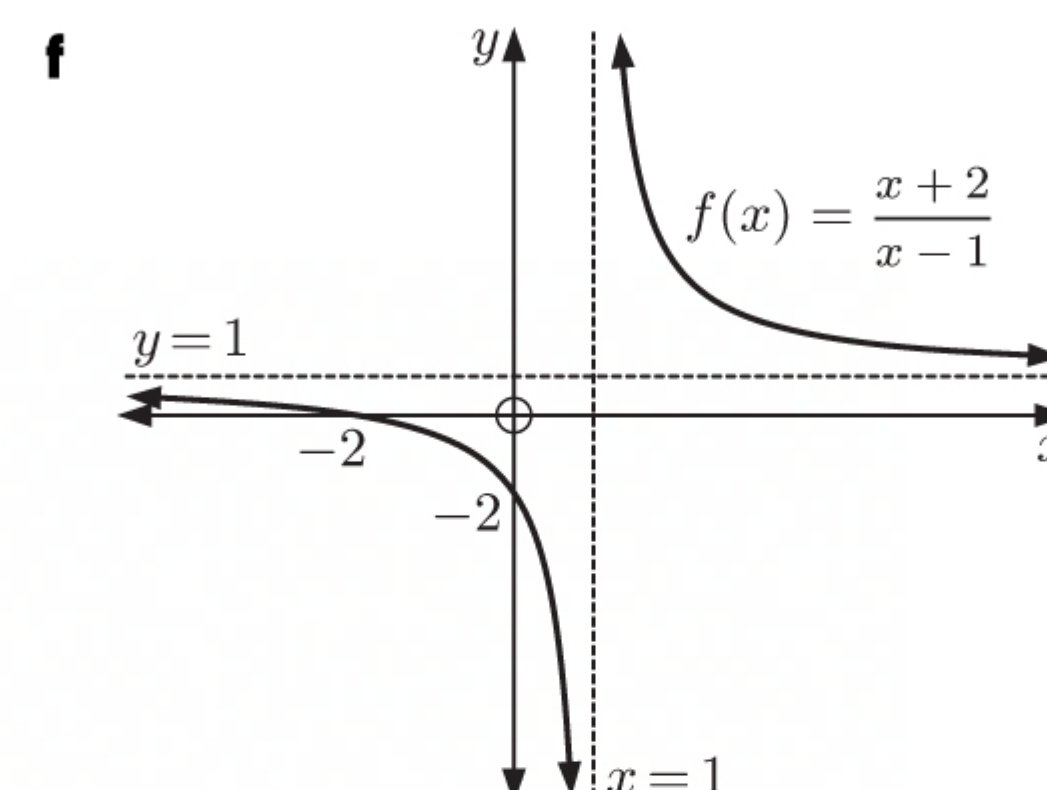
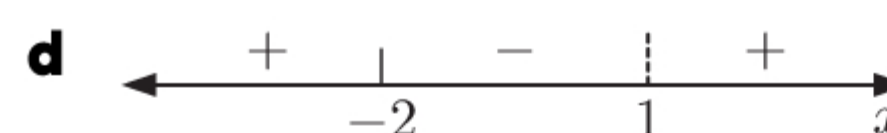
As $x \rightarrow 1^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 1^-$

As $x \rightarrow \infty$, $f(x) \rightarrow 1^+$

b The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = 1$.



32 $f(x) = 2 + \frac{4}{x+1}$

a $f(0) = 2 + \frac{4}{1} = 6$, so the y -intercept is 6.

$f(x) = 0$ when $2 + \frac{4}{x+1} = 0$

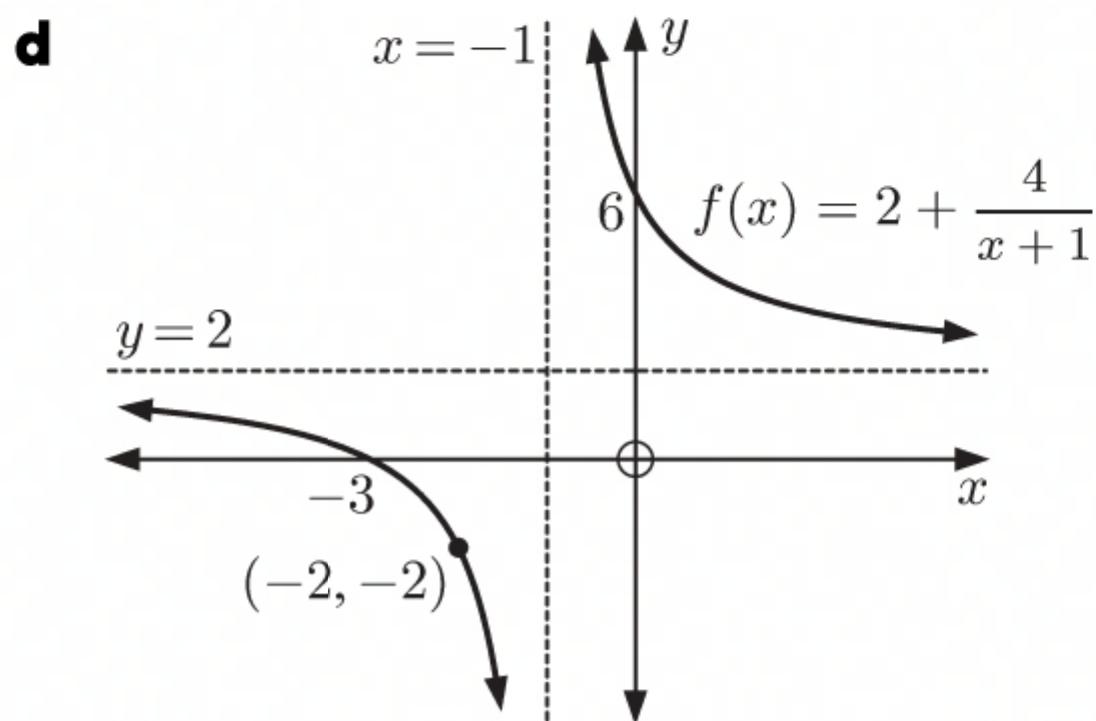
$\therefore 2(x+1) + 4 = 0$

$\therefore 2(x+1) = -4$

$\therefore x+1 = -2$

$\therefore x = -3$

\therefore the x -intercept is -3 .



b $f(-2) = 2 + \frac{4}{-2+1}$
 $= 2 + \frac{4}{-1}$
 $= -2$

c i The horizontal asymptote is $y = 2$.

ii The vertical asymptote is $x = -1$.

33 $f(x) = \frac{k}{x}$

a When $x = 5$, $y = 2$

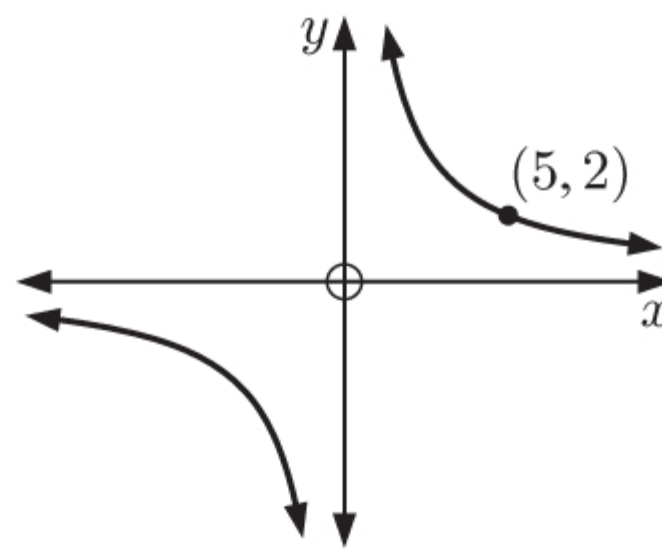
$\therefore 2 = \frac{k}{5}$

$\therefore k = 10$

b The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \neq 0\}$.

c $f(-\frac{1}{2}) = \frac{10}{-\frac{1}{2}} = -20$



d f is $y = \frac{10}{x}$

$\therefore f^{-1}$ is $x = \frac{10}{y}$

$\therefore y = \frac{10}{x}$

So, $f^{-1}(x) = \frac{10}{x} = f(x)$.

$\therefore f$ is self-inverse.

34 $f(x) = \frac{1}{x-1} + \sqrt{x+1}$, $g(x) = x^2$

a $\frac{1}{x-1}$ is defined when $x-1 \neq 0$

$\therefore x \neq 1$

$\sqrt{x+1}$ is defined when $x+1 \geq 0$

$\therefore x \geq -1$

\therefore the domain of f is $\{x \mid x \geq -1 \text{ and } x \neq 1\}$.

c The domain of g is $\{x \mid x \in \mathbb{R}\}$.

Now $\frac{1}{x^2-1}$ is defined when $x^2-1 \neq 0$

$\therefore (x+1)(x-1) \neq 0$

$\therefore x \neq -1 \text{ or } 1$

and $x^2+1 \geq 0$, so $\sqrt{x^2+1}$ is defined for every value of x .

\therefore the domain of $(f \circ g)$ is $\{x \mid x \neq -1 \text{ and } x \neq 1\}$.

This is different to the domain of f and g since $(f \circ g)$ is defined using g whose range is $\{y \mid y \geq 0\}$.

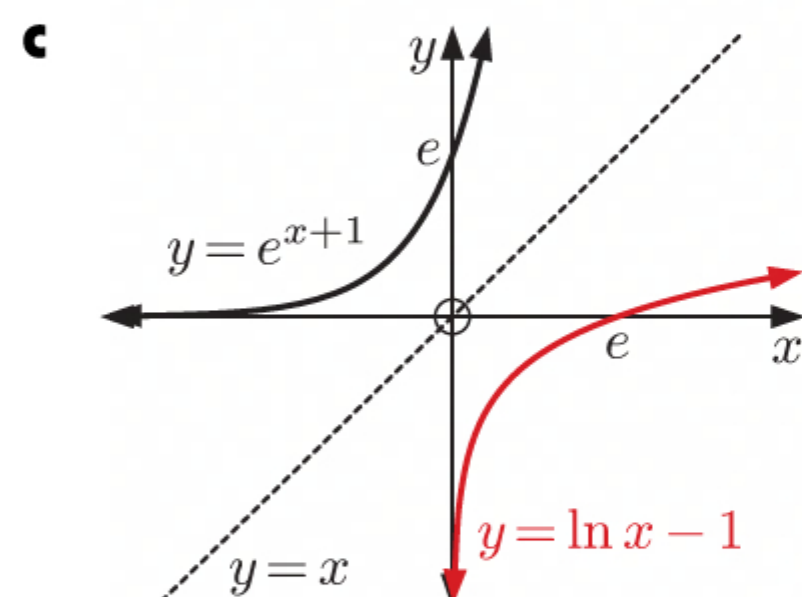
b $(f \circ g)(x) = f(g(x))$
 $= f(x^2)$
 $= \frac{1}{x^2-1} + \sqrt{x^2+1}$

$$35 \quad f: x \mapsto e^{x+1}, \quad g: x \mapsto \ln x - 1$$

$$\begin{aligned} \mathbf{a} \quad (f \circ g)(x) &= f(g(x)) \\ &= f(\ln x - 1) \\ &= e^{\ln x - 1 + 1} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

The domain is $\{x \mid x > 0\}$.

The range is $\{y \mid y > 0\}$.



$$\begin{aligned} \mathbf{b} \quad (g \circ f)(x) &= g(f(x)) \\ &= g(e^{x+1}) \\ &= \ln(e^{x+1}) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \in \mathbb{R}\}$.

d The graph of $y = g(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

$\therefore f$ and g are inverses of one another.

$$36 \quad f(x) = \sqrt[3]{x}$$

$$\begin{aligned} f \text{ is } y &= \sqrt[3]{x} \\ \therefore f^{-1} \text{ is } x &= \sqrt[3]{y} \\ \therefore y &= x^3 \\ \therefore f^{-1}(x) &= x^3 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \quad \text{If } (f \circ g)(x) &= 2x - 1, \\ \text{then } f^{-1}(f(g(x))) &= f^{-1}(2x - 1) \\ \therefore (f^{-1} \circ f)(g(x)) &= (2x - 1)^3 \\ \therefore g(x) &= (2x - 1)^3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{If } (g \circ f)(x) &= 2x - 1, \\ \text{then } (g \circ f)(f^{-1}(x)) &= 2f^{-1}(x) - 1 \\ \therefore g((f \circ f^{-1})(x)) &= 2x^3 - 1 \\ \therefore g(x) &= 2x^3 - 1 \end{aligned}$$

$$37 \quad f(x) = \sqrt{x+4}, \quad g(x) = x^2 - 3$$

$$\begin{aligned} \mathbf{a} \quad (f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 3) \\ &= \sqrt{x^2 - 3 + 4} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

$x^2 + 1 \geq 1$, so $\sqrt{x^2 + 1}$ is defined for every value of x .

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$.

$$x^2 + 1 \geq 1$$

$$\therefore \sqrt{x^2 + 1} \geq 1$$

\therefore the range is $\{y \mid y \geq 1\}$.

$$\begin{aligned} \mathbf{b} \quad (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x+4}) \\ &= (\sqrt{x+4})^2 - 3 \\ &= x + 1 \end{aligned}$$

$\sqrt{x+4}$ is defined when $x + 4 \geq 0$

$$\therefore x \geq -4$$

\therefore the domain is $\{x \mid x \geq -4\}$.

$$x \geq -4$$

$$\therefore x + 1 \geq -3$$

\therefore the range is $\{y \mid y \geq -3\}$.

$$38 \quad f: x \mapsto 3x + 1, \quad g: x \mapsto 4 - x$$

$$\begin{aligned} \mathbf{a} \quad f(g(x)) &= f(4 - x) \\ &= 3(4 - x) + 1 \\ &= 12 - 3x + 1 \\ &= 13 - 3x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (g \circ f)(-4) &= g(f(-4)) \\ &= g(3(-4) + 1) \\ &= g(-11) \\ &= 4 - (-11) \\ &= 15 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad f \text{ is } y &= 3x + 1 \\ \therefore f^{-1} \text{ is } x &= 3y + 1 \\ \therefore 3y &= x - 1 \\ \therefore y &= \frac{1}{3}x - \frac{1}{3} \\ \text{So, } f^{-1}(x) &= \frac{1}{3}x - \frac{1}{3} \\ \therefore f^{-1}\left(\frac{1}{2}\right) &= \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3} \\ &= \frac{1}{6} - \frac{1}{3} \\ &= -\frac{1}{6} \end{aligned}$$

39 $f: x \mapsto \ln x, \quad g: x \mapsto 3 + x$

a f is $y = \ln x$ g is $y = 3 + x$

$\therefore f^{-1}$ is $x = \ln y$ $\therefore g^{-1}$ is $x = 3 + y$

$\therefore y = e^x$ $\therefore y = x - 3$

So, $f^{-1}(x) = e^x$ So, $g^{-1}(x) = x - 3$

$\therefore f^{-1}(2) \times g^{-1}(2) = e^2(2 - 3) = -e^2$

b $(f \circ g)(x) = f(g(x))$ $f \circ g$ is $y = \ln(3 + x)$

$= f(3 + x)$ $\therefore (f \circ g)^{-1}$ is $x = \ln(3 + y)$

$= \ln(3 + x)$ $\therefore e^x = 3 + y$

$\therefore y = e^x - 3$

So, $(f \circ g)^{-1}(x) = e^x - 3$

$\therefore (f \circ g)^{-1}(2) = e^2 - 3$

c $(g \circ f)(x) = g(f(x))$ $g \circ f$ is $y = 3 + \ln x$

$= g(\ln x)$ $\therefore (g \circ f)^{-1}$ is $x = 3 + \ln y$

$= 3 + \ln x$ $\therefore \ln y = x - 3$

$\therefore y = e^{x-3}$

So, $(g \circ f)^{-1}(x) = e^{x-3}$

Now $(g \circ f)^{-1}(a) = \sqrt{e}$

$\therefore e^{a-3} = e^{\frac{1}{2}}$

$\therefore a - 3 = \frac{1}{2}$ {equating indices}

$\therefore a = \frac{7}{2}$

40 $f: x \mapsto x + 5, \quad g: x \mapsto 7 - 3x$

a i f is $y = x + 5$

$\therefore f^{-1}$ is $x = y + 5$

$\therefore y = x - 5$

So, $f^{-1}(x) = x - 5$

ii g is $y = 7 - 3x$

$\therefore g^{-1}$ is $x = 7 - 3y$

$\therefore 3y = 7 - x$

$\therefore y = \frac{7}{3} - \frac{1}{3}x$

So, $g^{-1}(x) = \frac{7}{3} - \frac{1}{3}x$

iii $(f \circ g)(x) = f(g(x))$

$= f(7 - 3x)$

$= (7 - 3x) + 5$

$= 12 - 3x$

b $f \circ g$ is $y = 12 - 3x$ {using **a iii**}

$\therefore (f \circ g)^{-1}$ is $x = 12 - 3y$

$\therefore 3y = 12 - x$

$\therefore y = 4 - \frac{1}{3}x$

So, $(f \circ g)^{-1}(x) = 4 - \frac{1}{3}x$

$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$= g^{-1}(x - 5)$ {using **a i**}

$= \frac{7}{3} - \frac{1}{3}(x - 5)$ {using **a ii**}

$= \frac{7}{3} - \frac{1}{3}x + \frac{5}{3}$

$= 4 - \frac{1}{3}x$

$= (f \circ g)^{-1}(x)$

41 $y = -1 + 2^{-x}$

a When $x = 0, y = -1 + 1 = 0$

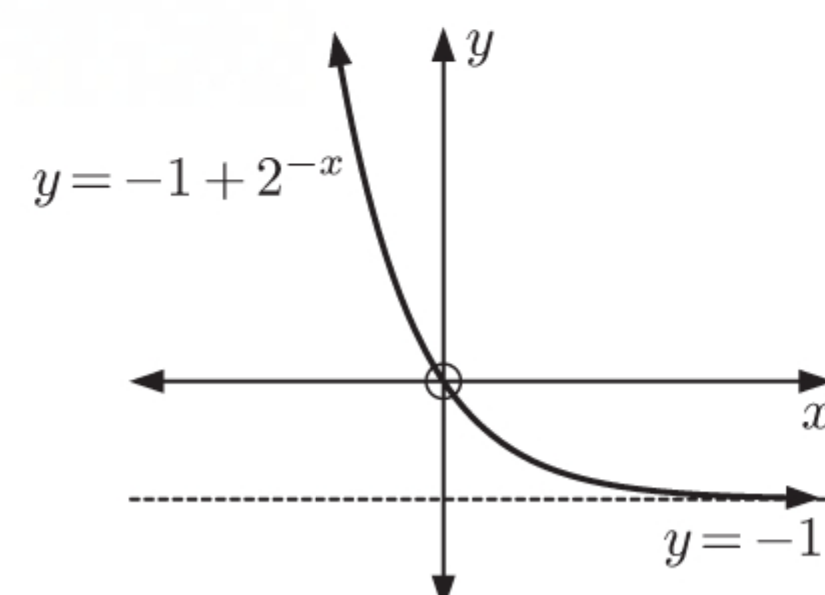
\therefore the y -intercept is 0, and the x -intercept is 0.

c The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > -1\}$.

b The horizontal asymptote is $y = -1$.

d



42 $y = a \times 2^x + b$

x	0	1	2	3
y	20	p	35	q

a When $x = 0$, $y = 20$
 $\therefore 20 = a \times 2^0 + b$
 $\therefore a + b = 20 \quad \dots (1)$

When $x = 2$, $y = 35$
 $\therefore 35 = a \times 2^2 + b$
 $\therefore 4a + b = 35 \quad \dots (2)$

b Using (1), $b = 20 - a \quad \dots (3)$

Substituting $b = 20 - a$ into (2) gives

$$4a + 20 - a = 35$$

$$\therefore 3a = 15$$

$$\therefore a = 5$$

Substituting $a = 5$ into (3) gives $b = 20 - 5 = 15$

$$\therefore a = 5 \text{ and } b = 15$$

c Using **b**, $y = 5 \times 2^x + 15$

When $x = 1$, $y = p$
 $\therefore p = 5 \times 2^1 + 15$
 $\therefore p = 25$

When $x = 3$, $y = q$
 $\therefore q = 5 \times 2^3 + 15$
 $\therefore q = 55$

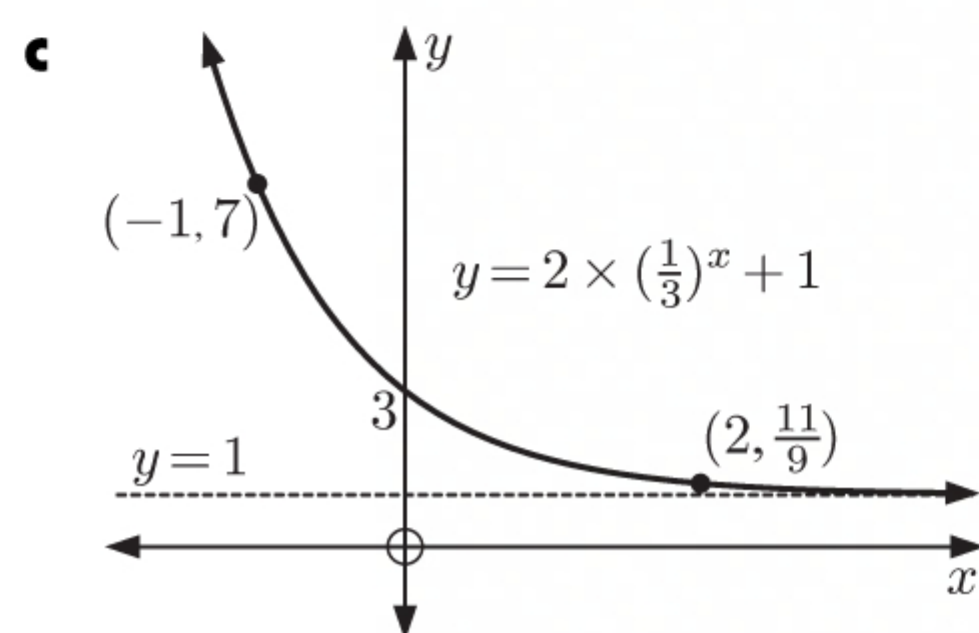
43 $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$

a i $f(0) = 2 + 1$
 $= 3$

ii $f(2) = 2 \times \left(\frac{1}{3}\right)^2 + 1$
 $= \frac{2}{9} + 1$
 $= \frac{11}{9}$

iii $f(-1) = 2 \times \left(\frac{1}{3}\right)^{-1} + 1$
 $= 2 \times 3 + 1$
 $= 7$

b The horizontal asymptote is $y = 1$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 1\}$.

44 a i When $t = 4$, $P \approx 60$

\therefore there is about 60% of Carbon-14 remaining after 4000 years.

ii When $P = 50$, $t \approx 5.5$

\therefore it will take approximately 5500 years for the percentage of Carbon-14 to fall to 50%.

b $P = 100 \times (1.1318)^{-t}$, $t \geq 0$

i When $t = 8$, $P = 100 \times (1.1318)^{-8}$
 ≈ 37.1

\therefore there is about 37.1% of Carbon-14 remaining after 8000 years.

ii When $P = 15$, $15 = 100 \times (1.1318)^{-t}$

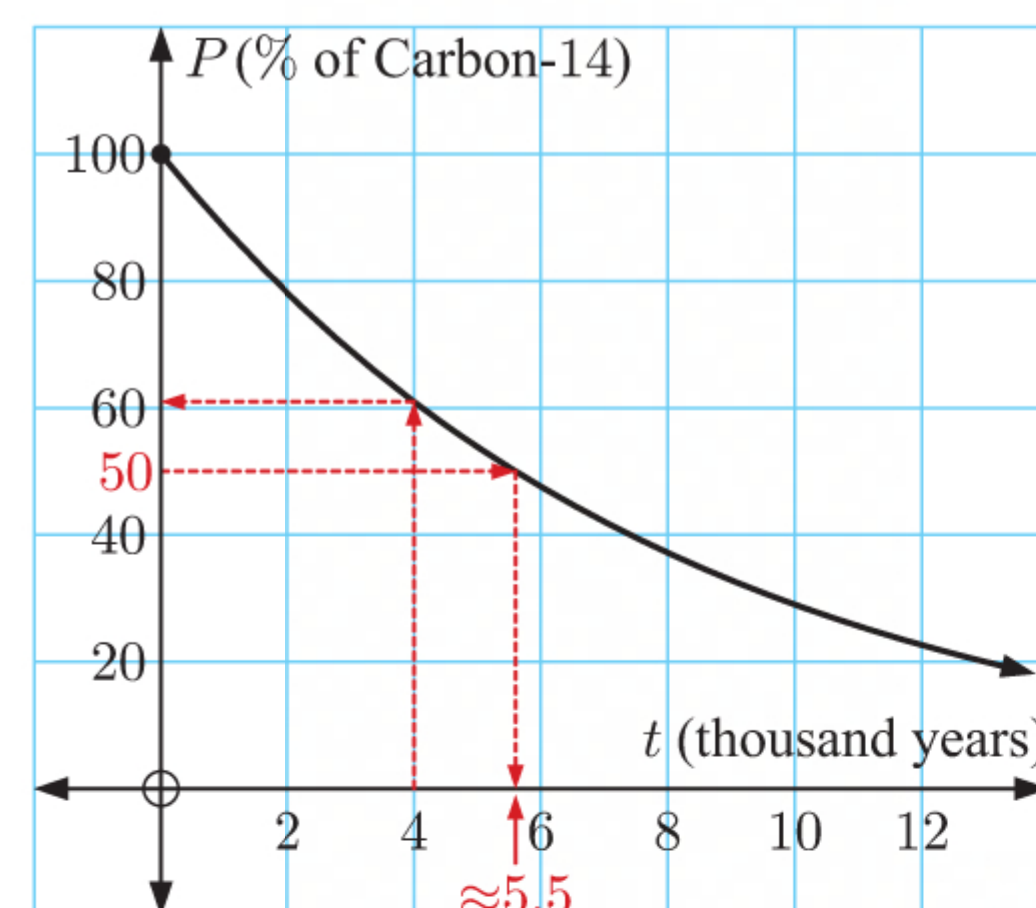
$$\therefore (1.1318)^{-t} = 0.15$$

$$\therefore -t \ln(1.1318) = \ln(0.15)$$

$$\therefore t = \frac{\ln(0.15)}{-\ln(1.1318)}$$

$$\approx 15.3$$

\therefore it will take approximately 15 300 years for the percentage of Carbon-14 to fall to 15%.



45 $N = 120 \times (1.04)^t$

a When $t = 0$, $N = 120$

\therefore there were 120 people who started the settlement.

b When $t = 4$,

$$N = 120 \times (1.04)^4 \\ \approx 140$$

\therefore there were about 140 people on the island after 4 years.

c When $N = 120 \times 2 = 240$,

$$240 = 120 \times (1.04)^t$$

$$\therefore (1.04)^t = 2$$

$$\therefore t \ln(1.04) = \ln 2$$

$$\therefore t = \frac{\ln 2}{\ln(1.04)}$$

$$\approx 17.7$$

\therefore it will take about 17.7 years for the number of people to double.

46 $T(t) = A \times B^{-t} + 3$

a i The initial internal temperature of the refrigerator was 27°C .

$$\text{So, } T(0) = 27$$

$$\therefore 27 = A + 3$$

$$\therefore A = 24$$

ii After 3 hours, the internal temperature was 6°C .

$$\text{So, } T(3) = 6$$

$$\therefore 6 = 24 \times B^{-3} + 3 \quad \{\text{using i}\}$$

$$\therefore 24 \times B^{-3} = 3$$

$$\therefore B^{-3} = \frac{1}{8}$$

$$\therefore B^3 = 8$$

$$\therefore B = 2$$

b $T(t) = 24 \times 2^{-t} + 3$ {using a}

$$\therefore T(5) = 24 \times 2^{-5} + 3 \\ = 3.75$$

\therefore the internal temperature is 3.75°C after 5 hours.

c As $t \rightarrow \infty$, $2^{-t} \rightarrow 0$

$$\therefore T(t) \rightarrow 24 \times 0 + 3 = 3$$

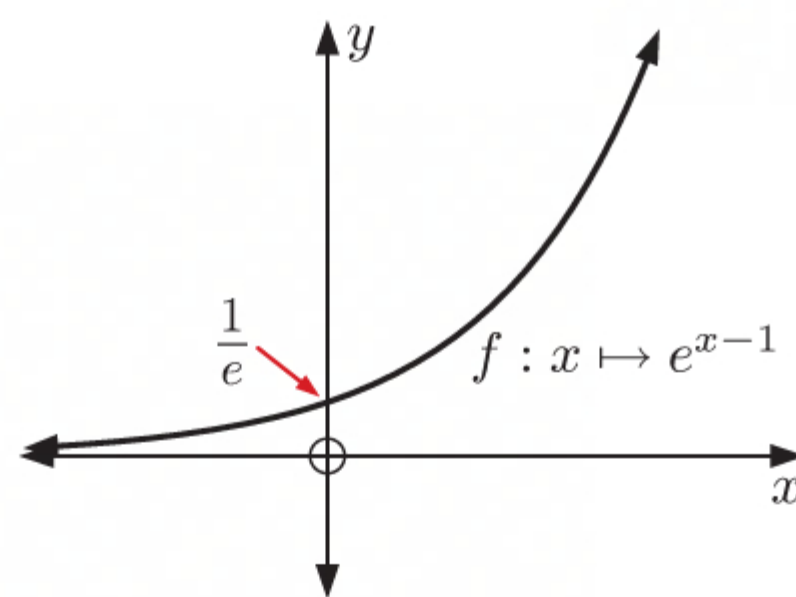
\therefore the minimum temperature that the refrigerator could be expected to reach is 3°C .

47 $f : x \mapsto e^{x-1}$

a $f(0) = e^{-1} = \frac{1}{e}$

$$f(1) = e^0 = 1$$

$$f(-1) = e^{-2} = \frac{1}{e^2}$$



b The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 0\}$.

48 $P = 1000 + ae^{kn}$

The initial population was 2000, so when $n = 0$, $P = 2000$

$$\therefore 2000 = 1000 + ae^0$$

$$\therefore a = 1000$$

$$\therefore P = 1000 + 1000e^{kn}$$

After 1 year, the population was 4000, so when $n = 1 \times 12 = 12$, $P = 4000$

$$\therefore 4000 = 1000 + 1000e^{12k}$$

$$\therefore 1000e^{12k} = 3000$$

$$\therefore e^{12k} = 3$$

$$\therefore e^k = 3^{\frac{1}{12}}$$

$$\therefore P = 1000 + 1000 \times 3^{\frac{n}{12}}$$

Now, when $P = 10\,000$, $10\,000 = 1000 + 1000 \times 3^{\frac{n}{12}}$

$$\therefore 9000 = 1000 \times 3^{\frac{n}{12}}$$

$$\therefore 3^{\frac{n}{12}} = 9$$

$$\therefore 3^{\frac{n}{12}} = 3^2$$

$$\therefore \frac{n}{12} = 2 \quad \{\text{equating indices}\}$$

$$\therefore n = 24$$

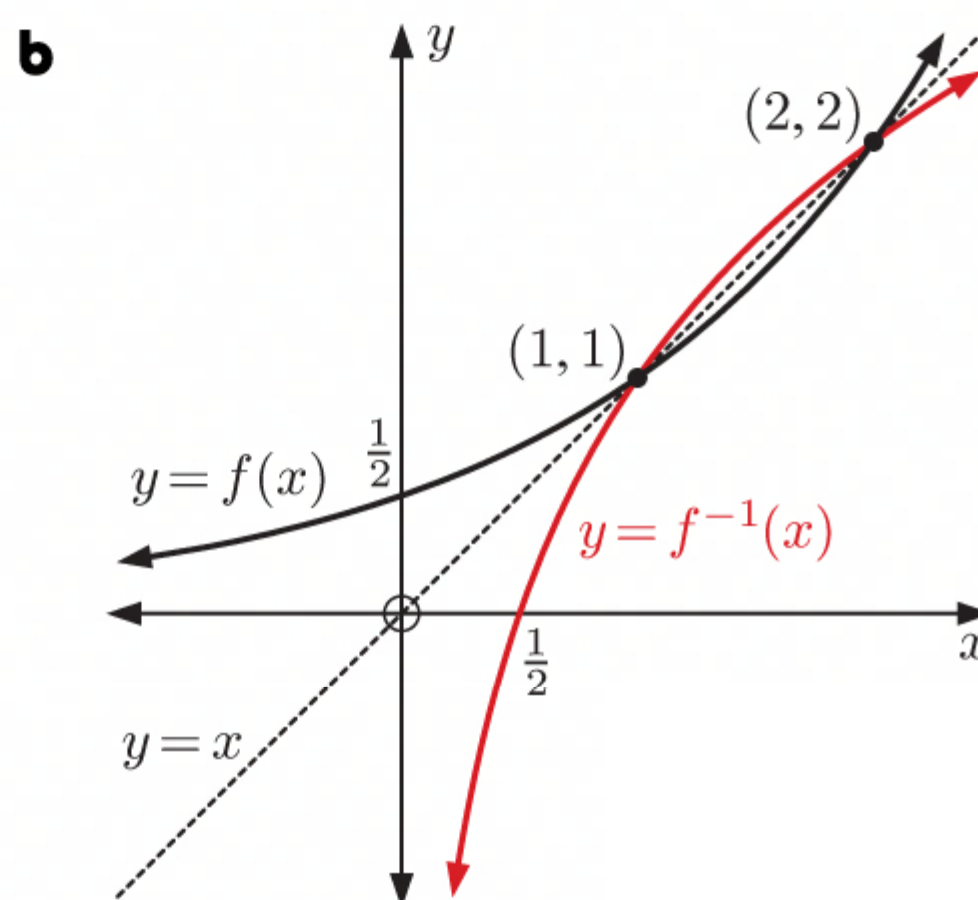
\therefore it will take 24 months, or 2 years, for the population to reach 10 000.

$$\begin{aligned} 49 \quad \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 &= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} \\ &= \frac{4}{4} \\ &= 1 \end{aligned}$$

$$50 \quad f(x) = 2^{x-1}$$

$$a \quad f(1) = 2^0 = 1$$

$$f(2) = 2^1 = 2$$



The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

$$\begin{aligned} c \quad f \text{ is } y &= 2^{x-1} \\ \therefore f^{-1} \text{ is } x &= 2^{y-1} \\ \therefore y - 1 &= \log_2 x \\ \therefore y &= \log_2 x + 1 \end{aligned}$$

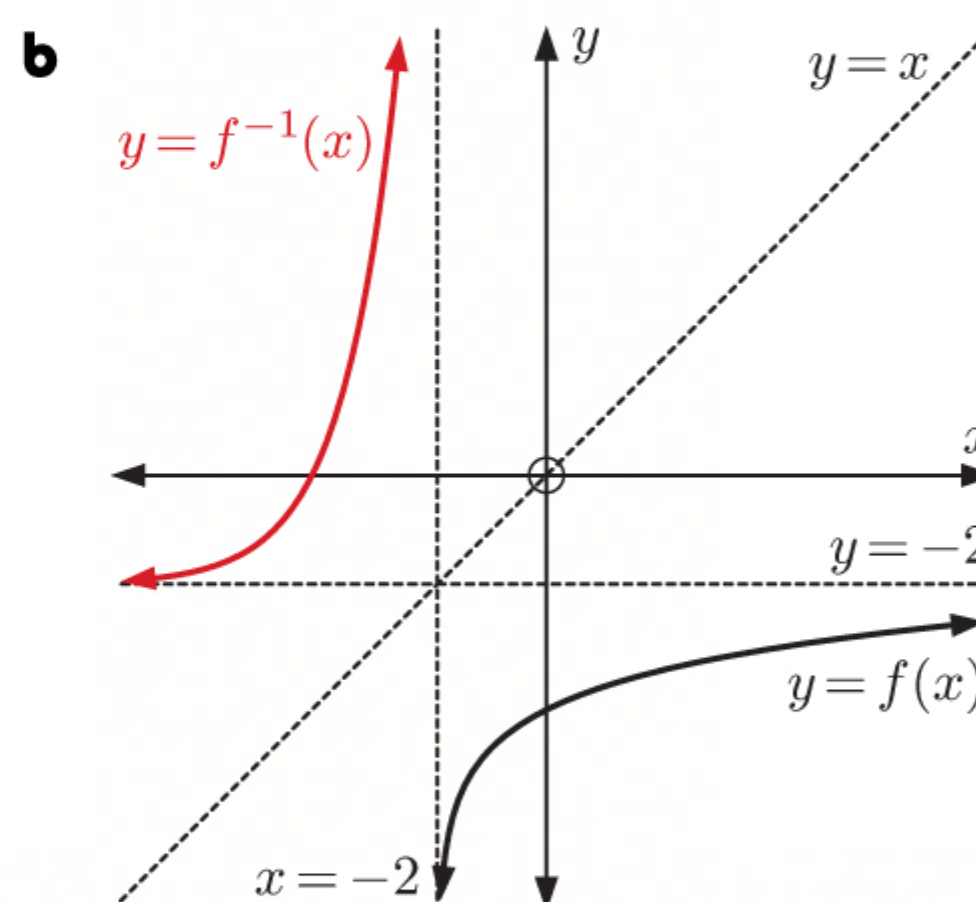
$$\text{So, } f^{-1}(x) = \log_2 x + 1$$

- d The domain of $f(x)$ is $\{x \mid x \in \mathbb{R}\}$.
The range of $f(x)$ is $\{y \mid y > 0\}$.
The domain of $f^{-1}(x)$ is $\{x \mid x > 0\}$.
The range of $f^{-1}(x)$ is $\{y \mid y \in \mathbb{R}\}$.

$$51 \quad f: x \mapsto \ln(x+2) - 5, \quad x > -2$$

$$\begin{aligned} a \quad f \text{ is } y &= \ln(x+2) - 5 \\ \therefore f^{-1} \text{ is } x &= \ln(y+2) - 5 \\ \therefore x + 5 &= \ln(y+2) \\ \therefore y + 2 &= e^{x+5} \\ \therefore y &= e^{x+5} - 2 \end{aligned}$$

$$\text{So, } f^{-1}(x) = e^{x+5} - 2$$



The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

- c The domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$.
The range of f^{-1} is $\{y \mid y > -2\}$.

52 a $f(x) = x^2 - 5x + 6$

The graph of $y = g(x)$ is found by translating $y = f(x)$ 8 units upwards.

$$\therefore g(x) = f(x) + 8$$

$$\therefore g(x) = (x^2 - 5x + 6) + 8$$

$$\therefore g(x) = x^2 - 5x + 14$$

b $f(x) = -2x^2 + x + 3$

The graph of $y = g(x)$ is found by translating $y = f(x)$ 1 unit to the right.

$$\therefore g(x) = f(x - 1)$$

$$\therefore g(x) = -2(x - 1)^2 + (x - 1) + 3$$

$$\therefore g(x) = -2(x^2 - 2x + 1) + x - 1 + 3$$

$$\therefore g(x) = -2x^2 + 4x - 2 + x + 2$$

$$\therefore g(x) = -2x^2 + 5x$$

53 $f(x) = \frac{5}{x+2}$

The graph of $y = g(x)$ is found by translating $y = f(x)$ through $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$.

$$\therefore g(x) = f(x + 4) + 6$$

$$\therefore g(x) = \frac{5}{(x + 4) + 2} + 6$$

$$\therefore g(x) = \frac{5}{x + 6} + \frac{6(x + 6)}{x + 6}$$

$$\therefore g(x) = \frac{5 + 6x + 36}{x + 6}$$

$$\therefore g(x) = \frac{6x + 41}{x + 6}$$

54 a $f(x) = \frac{1}{\sqrt{x-4}} + 3$ is defined when $\sqrt{x-4} > 0$
 $\therefore x - 4 > 0$
 $\therefore x > 4$

\therefore the domain is $\{x \mid x > 4\}$.

Now, $\frac{1}{\sqrt{x-4}} > 0$ {for $x > 4$ }

$$\therefore \frac{1}{\sqrt{x-4}} + 3 > 3$$

$$\therefore f(x) > 3$$

\therefore the range is $\{y \mid y > 3\}$.

b A translation through $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ maps $y = \frac{1}{\sqrt{x}}$ onto f .

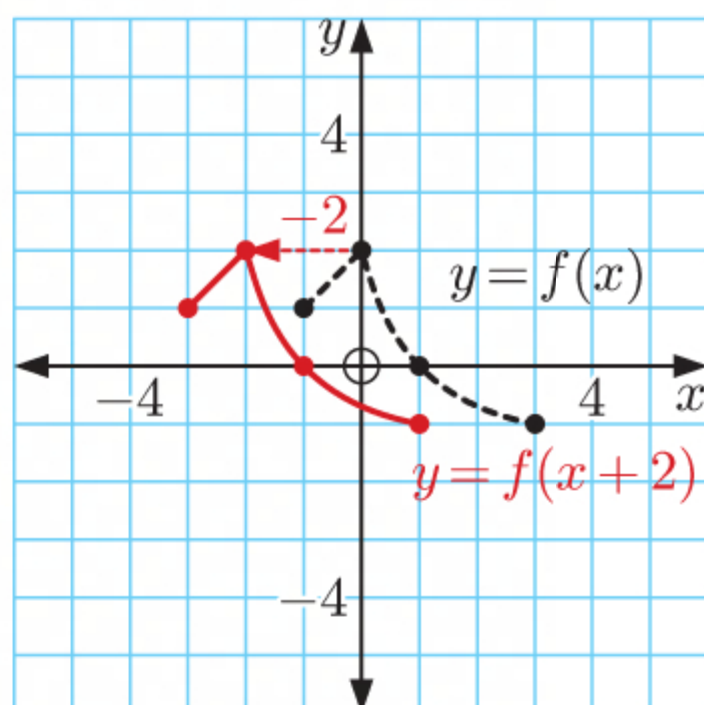
c $y = \frac{1}{\sqrt{x}}$ has vertical asymptote $x = 0$

and horizontal asymptote $y = 0$

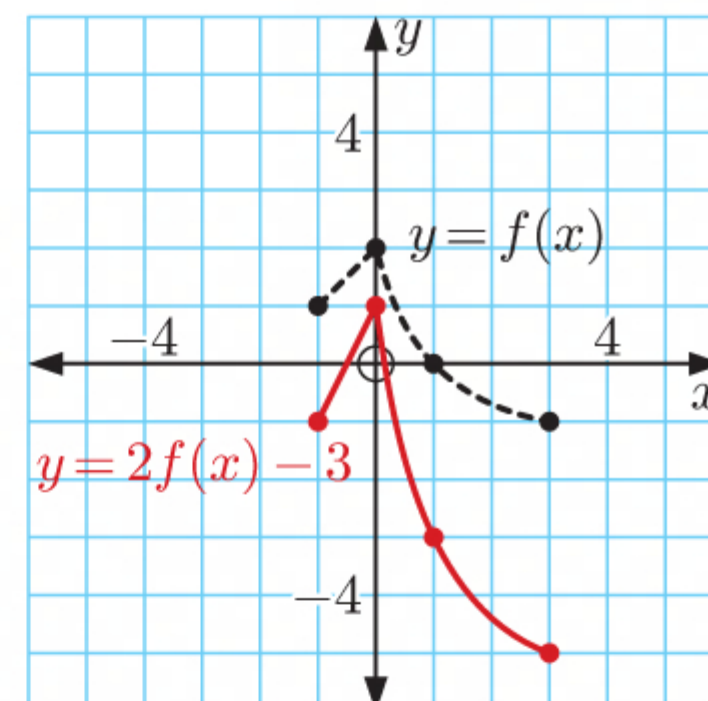
$\therefore y = f(x)$ has vertical asymptote $x = 4$

and horizontal asymptote $y = 3$.

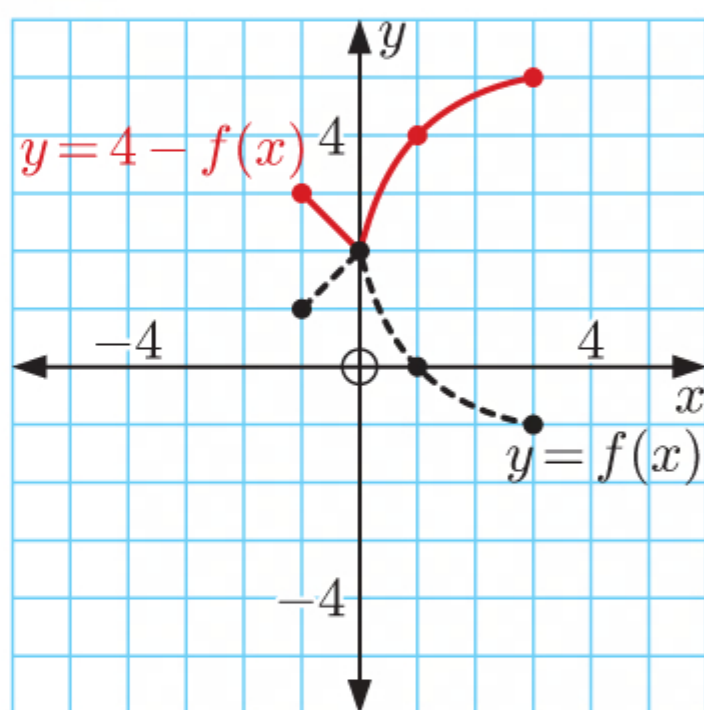
55 a The graph of $y = f(x + 2)$ is found by translating $y = f(x)$ 2 units to the left.



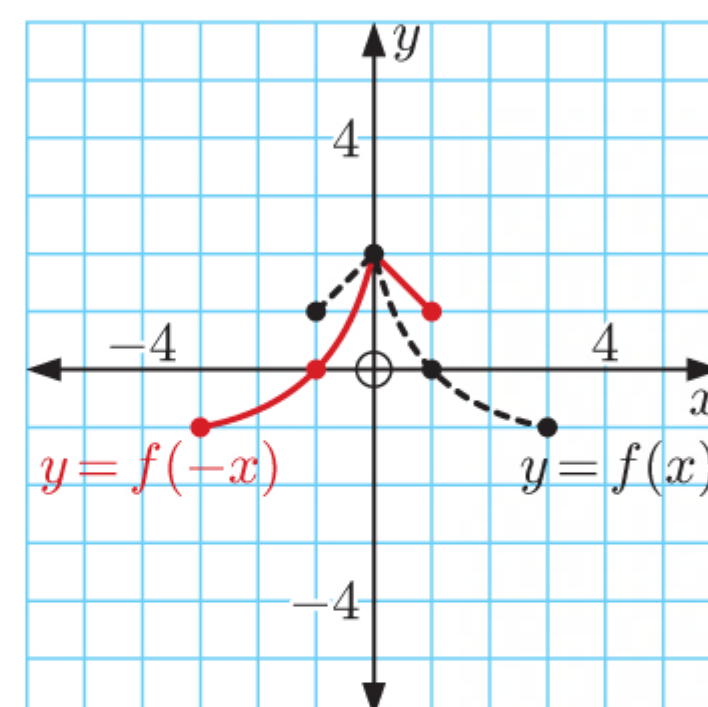
b The graph of $y = 2f(x) - 3$ is a vertical stretch of $y = f(x)$ with scale factor 2, followed by a translation 3 units downwards.



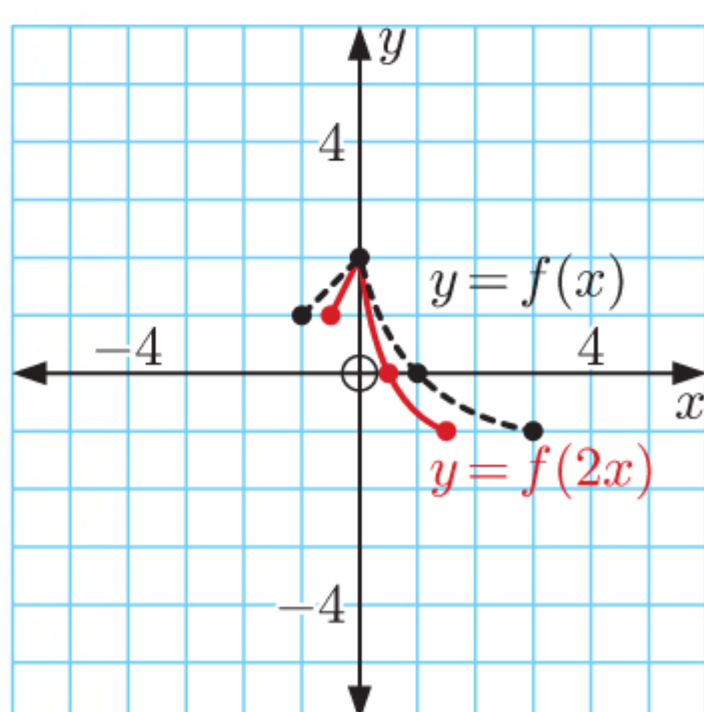
- c** The graph of $y = 4 - f(x)$ is found by reflecting $y = f(x)$ in the x -axis, then translating 4 units upwards.



- d** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



- e** The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.



56 $g : x \mapsto 4 - \ln(x - 2)$

- a** $\ln(x - 2)$ is defined when $x - 2 > 0$
 $\therefore x > 2$

So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

- b** As $x \rightarrow 2^+$, $y \rightarrow \infty$, so the vertical asymptote is $x = 2$.

As $x \rightarrow \infty$, $y \rightarrow -\infty$, so there is no horizontal asymptote.

- c** The graph of $y = h(x)$ is a horizontal stretch of $y = g(x)$ with scale factor $\frac{1}{2}$.

$$\therefore h(x) = g(2x)$$

$$\therefore h(x) = 4 - \ln(2x - 2)$$

- d** $y = g(x)$ has vertical asymptote $x = 2$ {using **b**}

$$\therefore y = h(x) \text{ has vertical asymptote } x = \frac{1}{2}(2) = 1.$$

57 a $f(x) \xrightarrow[\text{scale factor 2}]{\text{horizontal stretch}} f\left(\frac{1}{2}x\right) \xrightarrow[\text{scale factor 3}]{\text{vertical stretch}} 3f\left(\frac{1}{2}x\right) = g(x)$

A horizontal stretch with scale factor 2, then a vertical stretch with scale factor 3 maps $y = f(x)$ onto $y = g(x)$.

- b** Each point on $y = g(x)$ is 2 times their distance that $y = f(x)$ is from the y -axis, and 3 times their distance that $y = f(x)$ is from the x -axis.

The point $(-6, 3)$ on $y = f(x)$ is 6 units from the y -axis, and 3 units from the x -axis. The corresponding point on $y = g(x)$, which is $2 \times 6 = 12$ units from the y -axis and $3 \times 3 = 9$ units from the x -axis, is $(-12, 9)$.

- c** Each point on $y = f(x)$ is $\frac{1}{2}$ times their distance that $y = g(x)$ is from the y -axis, and $\frac{1}{3}$ times their distance that $y = g(x)$ is from the x -axis.

The point $(4, -9)$ on $y = g(x)$ is 4 units from the y -axis, and 9 units from the x -axis. The corresponding point on $y = f(x)$, which is $\frac{1}{2} \times 4 = 2$ units from the y -axis and $\frac{1}{3} \times 9 = 3$ units from the x -axis, is $(2, -3)$.

58 Let $y = f(x) = \frac{2}{x}$

- a** The image when $y = f(x)$ is reflected in the y -axis has equation

$$y = f(-x)$$

$$\therefore y = \frac{2}{-x}$$

$$\therefore y = -\frac{2}{x}$$

- c** The image when $y = f(x)$ is stretched horizontally with scale factor 3 has equation

$$y = f\left(\frac{1}{3}x\right)$$

$$\therefore y = \frac{2}{\frac{1}{3}x}$$

$$\therefore y = \frac{6}{x}$$

- b** The image when $y = f(x)$ is translated through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ has equation

$$y = f(x+1) + 2$$

$$\therefore y = \frac{2}{x+1} + 2$$

59 $f(x)$ has domain $\{x \mid x < 0, x \geq 2\}$ and range $\{y \mid y \geq 3\}$.

- a** $g(x) = f(x-3) + 2$ translates every point on $y = f(x)$ 3 units to the right and 2 units upwards.

$$\therefore g(x) \text{ has domain } \{x \mid x < 3, x \geq 5\} \text{ and range } \{y \mid y \geq 5\}.$$

- b** $g(x) = 4 - \frac{1}{2}f(5x)$ transforms every point by a horizontal stretch with scale factor $\frac{1}{5}$, a vertical stretch with scale factor $\frac{1}{2}$, a reflection in the x -axis, then a translation 4 units upwards.

$$\therefore g(x) \text{ has domain } \{x \mid x < 0, x \geq \frac{2}{5}\} \text{ and range } \{y \mid y \leq \frac{5}{2}\}.$$

60 T_A is a horizontal translation 4 units to the right.

T_B is a reflection in the x -axis.

T_C is a translation through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a $f(x) \xrightarrow{T_A} f(x-4) \xrightarrow{T_B} -f(x-4)$

The resulting function is $-f(x-4)$.

c $f(x) \xrightarrow{T_B} -f(x) \xrightarrow{T_C} -f(x-1) + 1$

The resulting function is $-f(x-1) + 1$.

b $f(x) \xrightarrow{T_B} -f(x) \xrightarrow{T_A} -f(x-4)$

The resulting function is $-f(x-4)$.

d $f(x) \xrightarrow{T_C} f(x-1) + 1 \xrightarrow{T_B} -f(x-1) - 1$

The resulting function is $-f(x-1) - 1$.

61 $f: x \mapsto \ln(x-2)$

a $\ln(x-2)$ is defined when $x-2 > 0$

$$\therefore x > 2$$

\therefore the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b As $x \rightarrow 2^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 2$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

c $f(x) \xrightarrow{\text{vertical stretch scale factor 3}} 3f(x) \xrightarrow{\text{reflection in the } y\text{-axis}} 3f(-x)$

The resulting function has equation $y = 3f(-x)$

which is $y = 3\ln(-x-2)$.

TOPIC 3 SKILL BUILDER QUESTIONS

- 1 a Let the height of the triangles be h cm.

$$\text{Now } h^2 = 20^2 + 5^2$$

$$\therefore h = \sqrt{20^2 + 5^2} = 5\sqrt{17}$$

Surface area

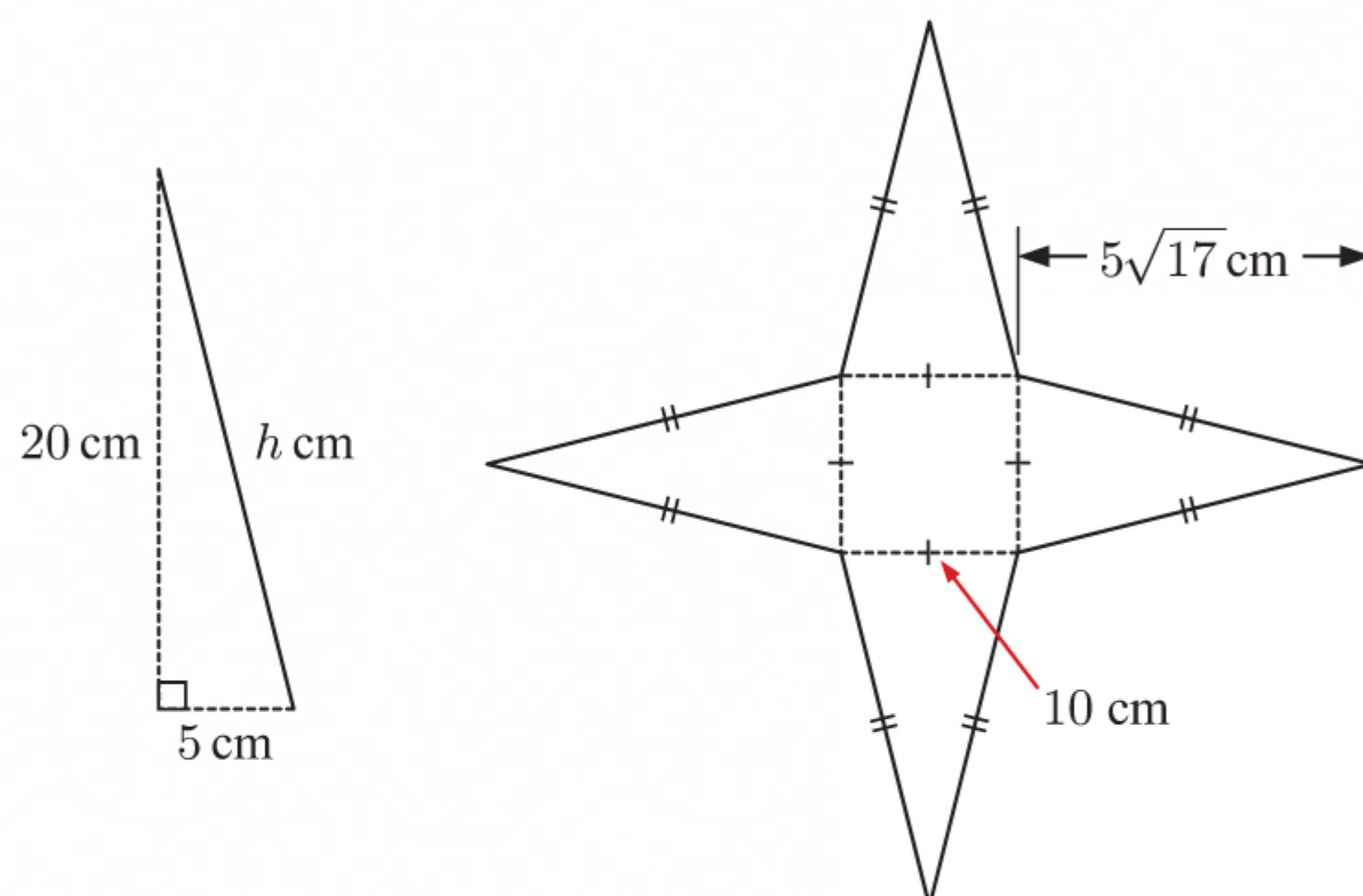
$$= \text{area of square} + 4 \times \text{area of triangle}$$

$$= 10^2 + 4\left(\frac{1}{2} \times 10 \times 5\sqrt{17}\right)$$

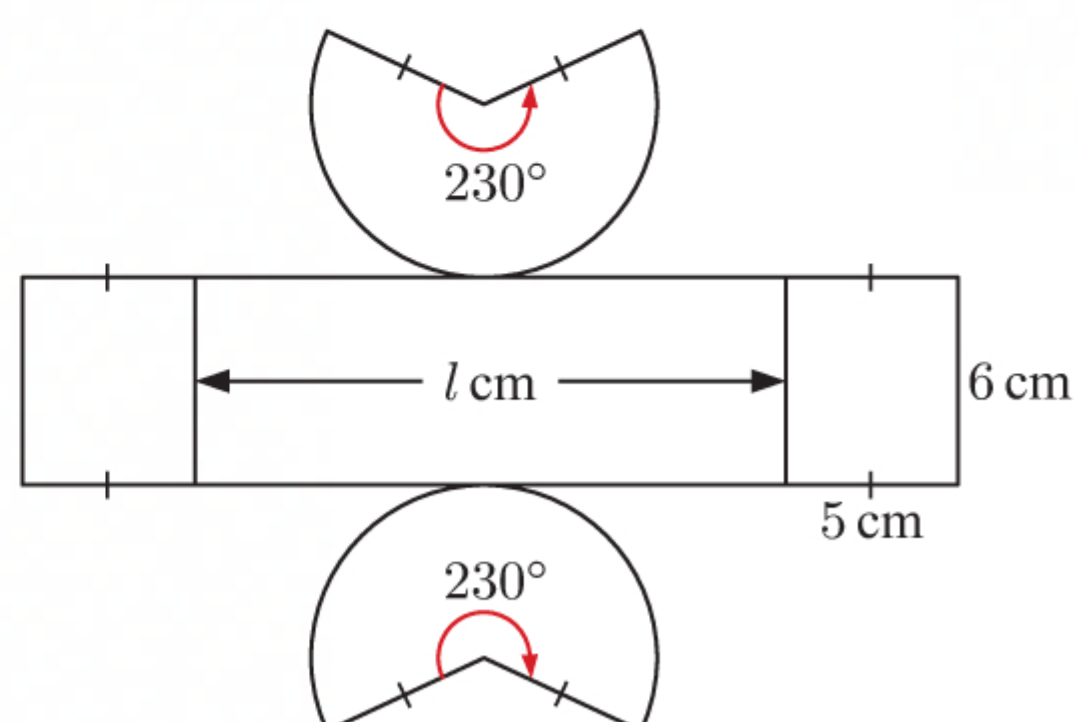
$$= 100 + 4(25\sqrt{17})$$

$$= 100 + 100\sqrt{17} \text{ cm}^2$$

$$\approx 512 \text{ cm}^2$$



b



$$l = 2\pi r \times \frac{230}{360}$$

$$= 2\pi(5) \times \frac{23}{36}$$

$$= \frac{115\pi}{18}$$

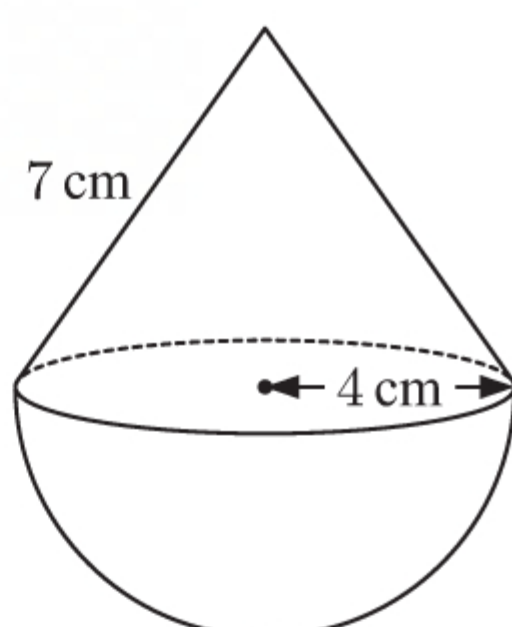
$$\text{Surface area} = 2 \times \text{area of sector} + \text{area of curved surface} + 2 \times \text{area of rectangle}$$

$$= 2 \times \frac{230}{360} \times \pi(5)^2 + \frac{115\pi}{18} \times 6 + 2 \times 6 \times 5$$

$$= \frac{575\pi}{18} + \frac{115\pi}{3} + 60$$

$$\approx 281 \text{ cm}^2$$

c



$$\text{Surface area} = \frac{1}{2}4\pi r^2 + \pi rs$$

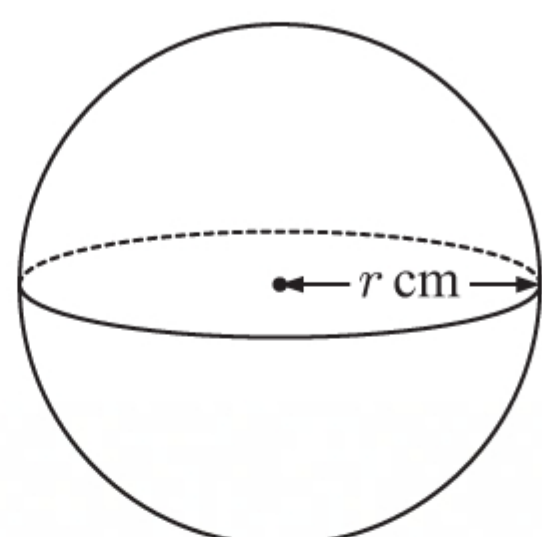
$$= \frac{1}{2} \times 4\pi(4)^2 + \pi(4)(7)$$

$$= 32\pi + 28\pi$$

$$= 60\pi \text{ cm}^2$$

$$\approx 188 \text{ cm}^2$$

2



Let the beach ball have radius r cm.

$$\text{Surface area} = 2800 \text{ cm}^2$$

$$\therefore 4\pi r^2 = 2800$$

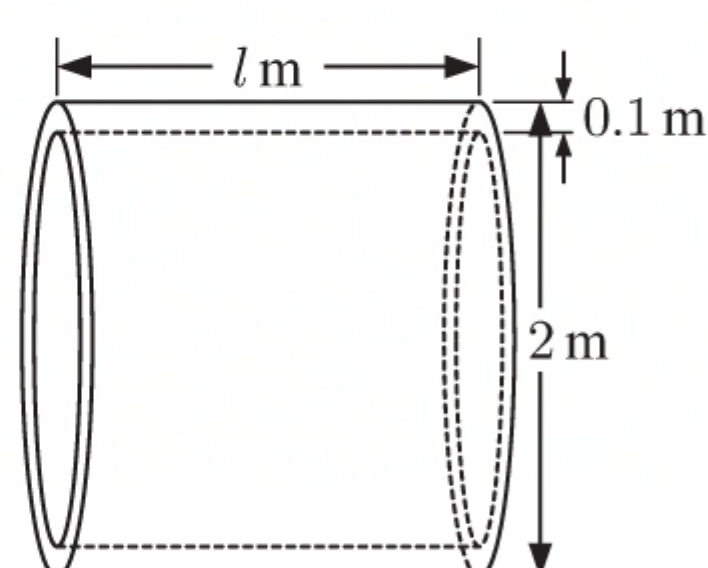
$$\therefore r^2 = \frac{700}{\pi}$$

$$\therefore r = \sqrt{\frac{700}{\pi}} \quad \{r > 0\}$$

$$\therefore r \approx 14.9$$

\therefore the radius of the beach ball is approximately 14.9 cm.

3



Let the pipe have length l m.

Now volume of concrete = volume of whole cylinder - volume of hollow section

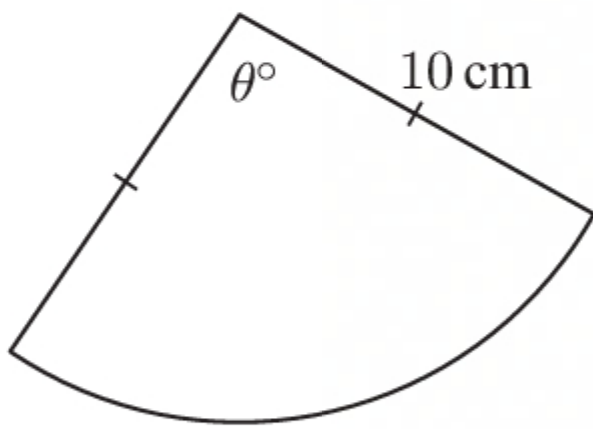
$$\therefore 3 = \pi(1)^2 \times l - \pi(0.9)^2 \times l$$

$$\therefore 3 = \pi l - 0.81\pi l$$

$$\therefore 3 = 0.19\pi l$$

$$\therefore l = \frac{3}{0.19\pi} \approx 5.03$$

\therefore the pipe is approximately 5.03 m long.

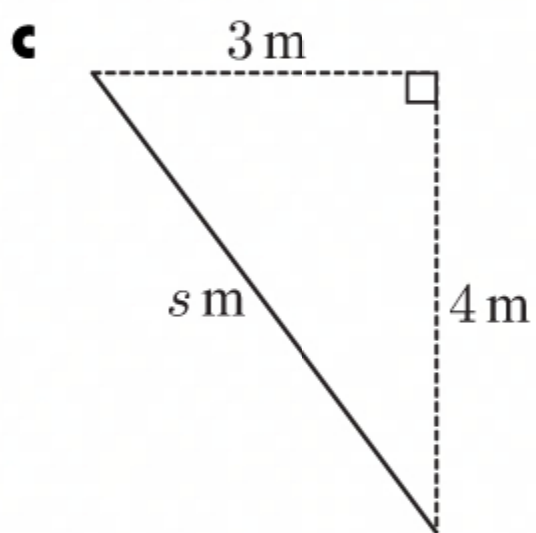
4

a Perimeter = $2r + \text{arc length}$
 $\therefore 40 = 2(10) + \text{arc length}$
 $\therefore \text{arc length} = 20 \text{ cm}$

b Now, $\text{arc length} = \frac{\theta}{360} \times 2\pi r$
 $\therefore 20 = \frac{\theta}{360} \times 2\pi(10) \quad \{\text{from a}\}$
 $\therefore 20 = \frac{\theta\pi}{18}$
 $\therefore \theta = \frac{360}{\pi}$
 $\therefore \text{area} = \frac{\theta}{360} \times \pi r^2$
 $= \frac{\left(\frac{360}{\pi}\right)}{360} \times \pi(10)^2$
 $= 100 \text{ cm}^2$

5 a Total height = hemisphere radius + cone height
 $\therefore 7 = \text{hemisphere radius} + 4$
 $\therefore \text{hemisphere radius} = 3 \text{ m}$
 $\therefore \text{cone radius} = 3 \text{ m} \quad \{\text{hemisphere radius} = \text{cone radius}\}$

b Volume = volume of hemisphere + volume of cone
 $= \frac{1}{2} \times \frac{4}{3}\pi r^3 + \frac{1}{3} \times \pi r^2 \times h$
 $= \frac{1}{2} \times \frac{4}{3}\pi(3)^3 + \frac{1}{3} \times \pi(3)^2 \times 4$
 $= 18\pi + 12\pi$
 $= 30\pi \approx 94.2 \text{ m}^3$



Let the slant height of the cone be $s \text{ m}$.

$\therefore s^2 = 4^2 + 3^2 \quad \{\text{Pythagoras}\}$
 $\therefore s^2 = 16 + 9$
 $\therefore s^2 = 25$
 $\therefore s = 5 \quad \{s > 0\}$

\therefore the slant height of the cone is 5 m .

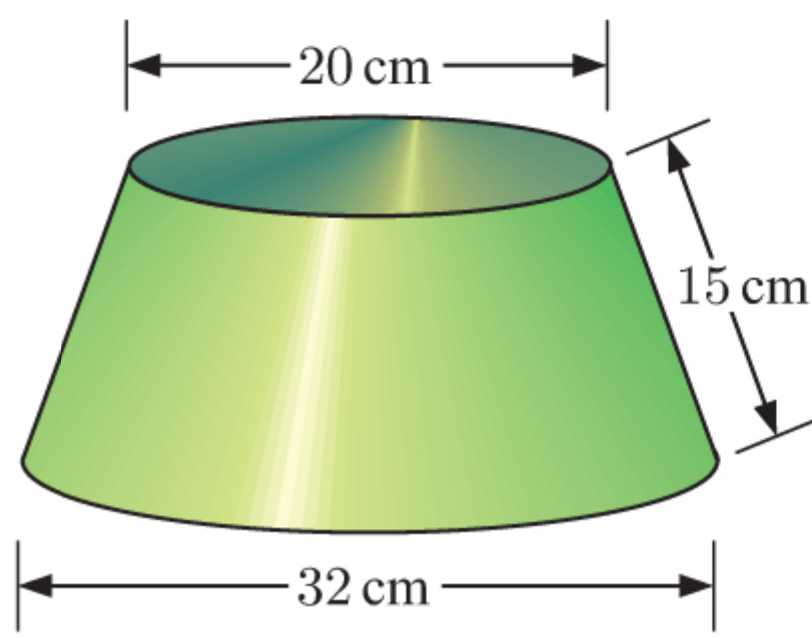
d Surface area = $\frac{1}{2} \times 4\pi r^2 + \pi r s$
 $= 2\pi(3)^2 + \pi(3)(5)$
 $= 18\pi + 15\pi$
 $= 33\pi \approx 104 \text{ m}^2$

e Weight = surface area \times weight of polymer per m^2
 $= 33\pi \times 1.23$
 $\approx 128 \text{ kg}$

6 a Arc length = $\frac{\theta}{360} \times 2\pi r$
 $\therefore \pi = \frac{\theta}{360} \times 2\pi(3)$
 $\therefore \pi = \frac{\pi\theta}{60}$
 $\therefore \theta = 60^\circ$

b Shaded area = $\frac{360 - 60}{360} \times \pi(3)^2$
 $= \frac{300}{360} \times 9\pi$
 $= \frac{15}{2}\pi \approx 23.6 \text{ cm}^2$

7


 The shorter arc length $= 2\pi\left(\frac{20}{2}\right) = 20\pi$ cm

 The longer arc length $= 2\pi\left(\frac{32}{2}\right) = 32\pi$ cm

 Now, the shorter arc length $= \theta r$

$$\therefore 20\pi = \theta r \quad \dots (*)$$

 and the longer arc length $= \theta(r + 15)$

$$\therefore 32\pi = \theta r + 15\theta$$

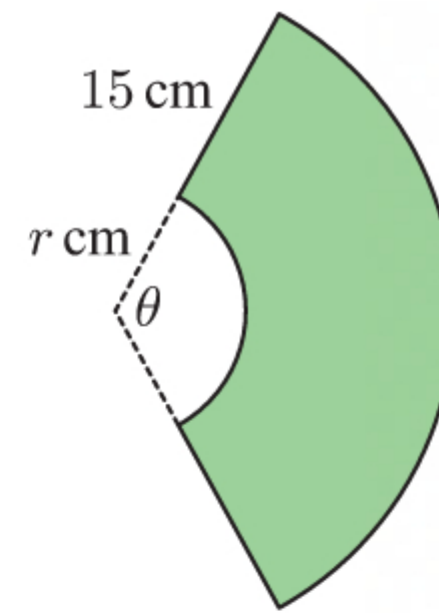
$$\therefore 32\pi = 20\pi + 15\theta \quad \{\text{using } (*)\}$$

$$\therefore 15\theta = 12\pi$$

$$\therefore \theta = \frac{4\pi}{5}$$

 Substituting $\theta = \frac{4\pi}{5}$ into $(*)$ gives $20\pi = \frac{4\pi}{5}r$

$$\therefore r = 25$$



8 a A(2, 4, 1) and B(4, 0, 7)

$$\begin{aligned} \text{i } AB &= \sqrt{(4-2)^2 + (0-4)^2 + (7-1)^2} \\ &= \sqrt{2^2 + (-4)^2 + 6^2} \\ &= \sqrt{4 + 16 + 36} \\ &= \sqrt{56} \text{ units} \end{aligned}$$

b A(3, -5, 2) and B(-1, 2, -3)

$$\begin{aligned} \text{i } AB &= \sqrt{(-1-3)^2 + (2-(-5))^2 + (-3-2)^2} \\ &= \sqrt{(-4)^2 + 7^2 + (-5)^2} \\ &= \sqrt{16 + 49 + 25} \\ &= \sqrt{90} \text{ units} \end{aligned}$$

c A(-6, 0, 5) and B(-3, -3, 1)

$$\begin{aligned} \text{i } AB &= \sqrt{(-3-(-6))^2 + (-3-0)^2 + (1-5)^2} \\ &= \sqrt{3^2 + (-3)^2 + (-4)^2} \\ &= \sqrt{9 + 9 + 16} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

 ii The midpoint is $\left(\frac{2+4}{2}, \frac{4+0}{2}, \frac{1+7}{2}\right)$,
which is (3, 2, 4).

 ii The midpoint is $\left(\frac{3+(-1)}{2}, \frac{-5+2}{2}, \frac{2+(-3)}{2}\right)$,
which is $\left(1, -\frac{3}{2}, -\frac{1}{2}\right)$.

 ii The midpoint is $\left(\frac{-6+(-3)}{2}, \frac{0+(-3)}{2}, \frac{5+1}{2}\right)$,
which is $\left(-\frac{9}{2}, -\frac{3}{2}, 3\right)$.

9 P(k, 6, -5) and Q(2, -1, -8)

$$\begin{aligned} PQ &= \sqrt{(2-k)^2 + (-1-6)^2 + (-8-(-5))^2} \\ &= \sqrt{(2-k)^2 + (-7)^2 + (-3)^2} \\ &= \sqrt{4 - 4k + k^2 + 49 + 9} \\ &= \sqrt{k^2 - 4k + 62} \end{aligned}$$

 Now $PQ = 9$ units

$$\therefore \sqrt{k^2 - 4k + 62} = 9$$

$$\therefore k^2 - 4k + 62 = 81$$

$$\therefore k^2 - 4k - 19 = 0$$

$$\therefore k = \frac{4 \pm \sqrt{16 - 4(1)(-19)}}{2}$$

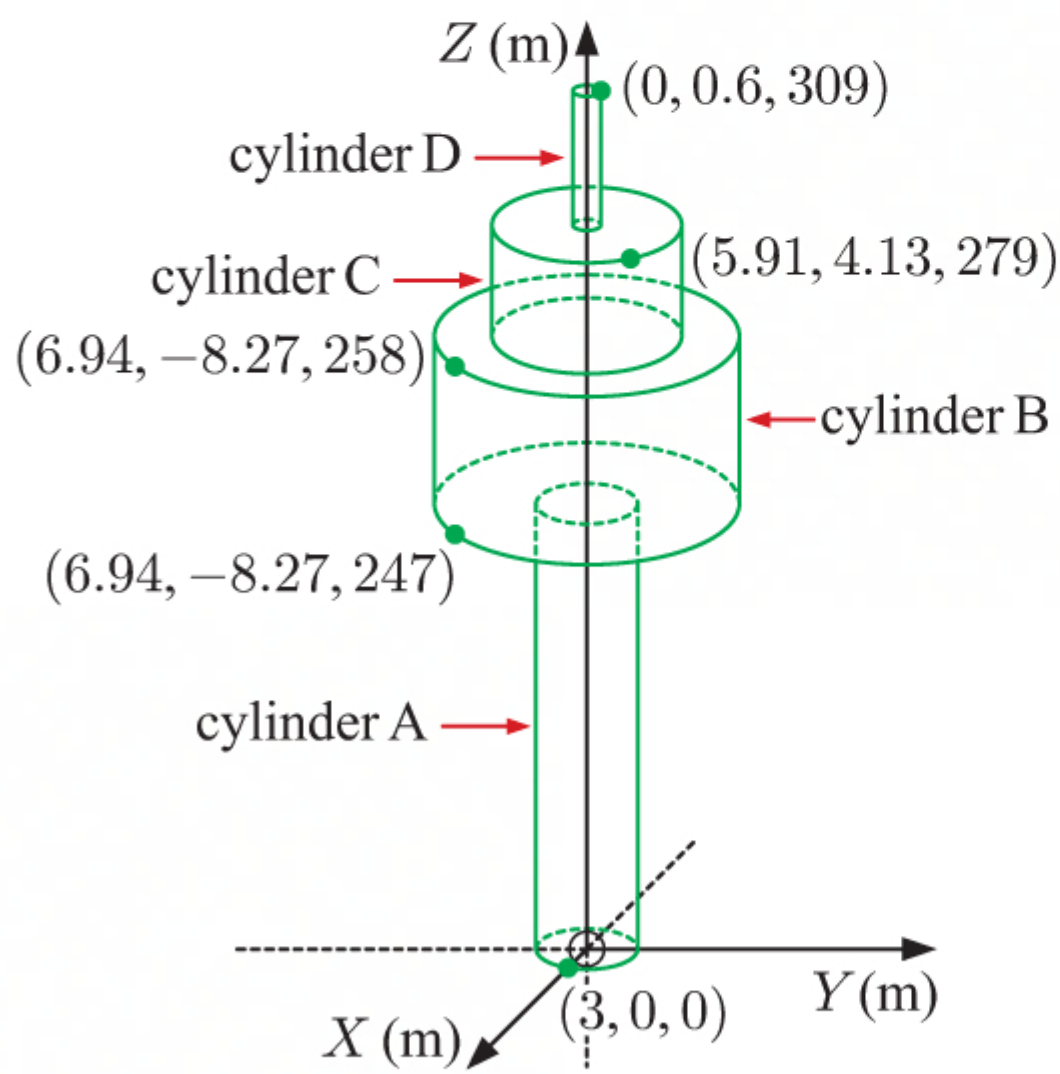
$$= \frac{4 \pm \sqrt{92}}{2}$$

$$= \frac{4 \pm 2\sqrt{23}}{2}$$

$$= 2 \pm \sqrt{23}$$

$$\therefore k = 2 - \sqrt{23} \quad \text{or} \quad 2 + \sqrt{23}$$

10



Height of cylinder A = 247 m

Radius of cylinder A = 3 m

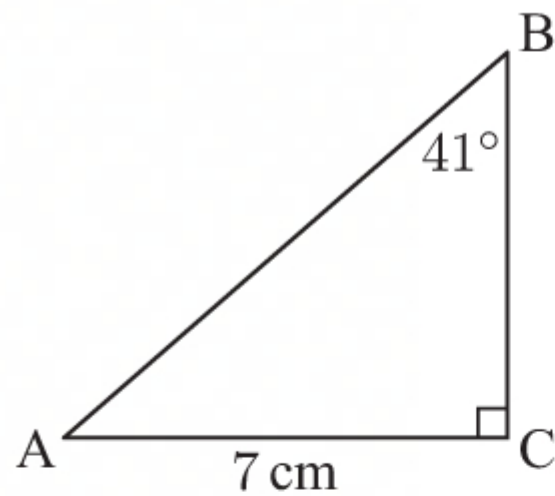
Height of cylinder B = $258 - 247 = 11$ mRadius of cylinder B = $\sqrt{6.94^2 + (-8.27)^2}$
 $= \sqrt{116.5565}$ mHeight of cylinder C = $279 - 258 = 21$ mRadius of cylinder C = $\sqrt{5.91^2 + 4.13^2}$
 $= \sqrt{51.985}$ mHeight of cylinder D = $309 - 279 = 30$ m

Radius of cylinder D = 0.6 m

Volume of tower = volume of cylinder A + volume of cylinder B + volume of cylinder C + volume of cylinder D

$$\begin{aligned}
 &= \pi(3)^2 \times 247 + \pi(\sqrt{116.5565})^2 \times 11 + \pi(\sqrt{51.985})^2 \times 21 + \pi(0.6)^2 \times 30 \\
 &= 2223\pi + 1282.1215\pi + 1091.685\pi + 10.8\pi \\
 &= 4607.6065\pi \\
 &\approx 14\,475.22 \\
 &\approx 14\,500 \text{ m}^3
 \end{aligned}$$

11 a



$$\sin 41^\circ = \frac{7}{AB}$$

$$\therefore AB = \frac{7}{\sin 41^\circ}$$

$$\text{and } \tan 41^\circ = \frac{7}{BC}$$

$$\therefore BC = \frac{7}{\tan 41^\circ}$$

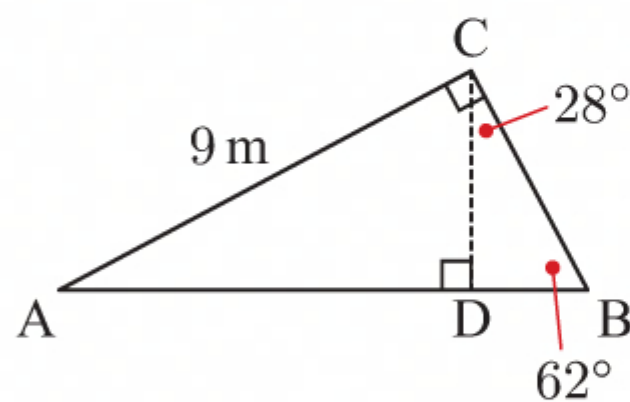
$$\text{Perimeter} = AB + BC + AC$$

$$\begin{aligned}
 &= \frac{7}{\sin 41^\circ} + \frac{7}{\tan 41^\circ} + 7 \\
 &\approx 25.7 \text{ cm}
 \end{aligned}$$

$$\text{Area} = \frac{1}{2} \times AC \times BC$$

$$\begin{aligned}
 &= \frac{1}{2} \times 7 \times \frac{7}{\tan 41^\circ} \\
 &\approx 28.2 \text{ cm}^2
 \end{aligned}$$

b



$$\hat{ABC} = 90^\circ - 28^\circ = 62^\circ \quad \{\text{angles in triangle BCD}\}$$

Now in triangle ABC:

$$\tan 62^\circ = \frac{9}{BC}$$

$$\therefore BC = \frac{9}{\tan 62^\circ}$$

$$\text{and } \sin 62^\circ = \frac{9}{AB}$$

$$\therefore AB = \frac{9}{\sin 62^\circ}$$

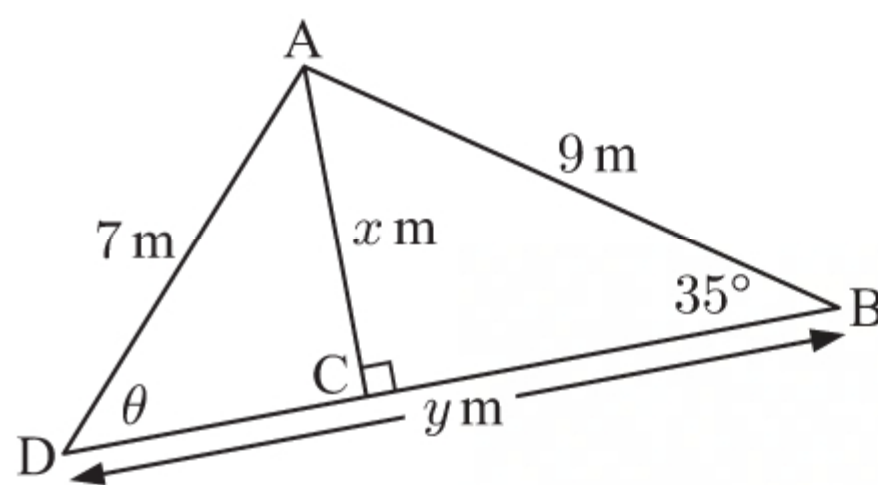
$$\text{Perimeter} = AB + BC + AC$$

$$\begin{aligned}
 &= \frac{9}{\sin 62^\circ} + \frac{9}{\tan 62^\circ} + 9 \\
 &\approx 24.0 \text{ m}
 \end{aligned}$$

$$\text{Area} = \frac{1}{2} \times AC \times BC$$

$$\begin{aligned}
 &= \frac{1}{2} \times 9 \times \frac{9}{\tan 62^\circ} \\
 &\approx 21.5 \text{ m}^2
 \end{aligned}$$

12 a



$$\text{In triangle ABC, } \sin 35^\circ = \frac{x}{9}$$

$$\therefore x = 9 \sin 35^\circ \approx 5.16$$

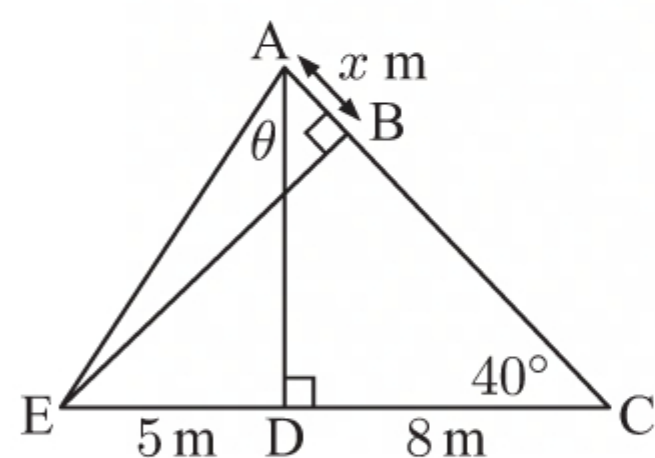
$$\text{In triangle ACD, } \sin \theta = \frac{x}{7}$$

$$= \frac{9 \sin 35^\circ}{7}$$

$$\therefore \theta = \sin^{-1}\left(\frac{9 \sin 35^\circ}{7}\right) \approx 47.5^\circ$$

$$\begin{aligned}
 \hat{DAB} &\approx 180^\circ - 35^\circ - 47.5^\circ \quad \{\text{angles in triangle DAB}\} \\
 &\approx 97.5^\circ
 \end{aligned}$$

$$\text{Using the cosine rule in triangle DAB, } y \approx \sqrt{7^2 + 9^2 - 2(7)(9) \cos 97.5^\circ} \approx 12.1$$

b


In triangle ACD, $\tan 40^\circ = \frac{AD}{8}$

$$\therefore AD = 8 \tan 40^\circ$$

In triangle ADE, $\tan \theta = \frac{5}{AD}$

$$= \frac{5}{8 \tan 40^\circ}$$

$$\therefore \theta = \tan^{-1}\left(\frac{5}{8 \tan 40^\circ}\right)$$

$$\approx 36.68^\circ$$

$$\approx 36.7^\circ$$

and $\sin 36.68^\circ \approx \frac{5}{AE}$

$$\therefore AE \approx \frac{5}{\sin 36.68^\circ}$$

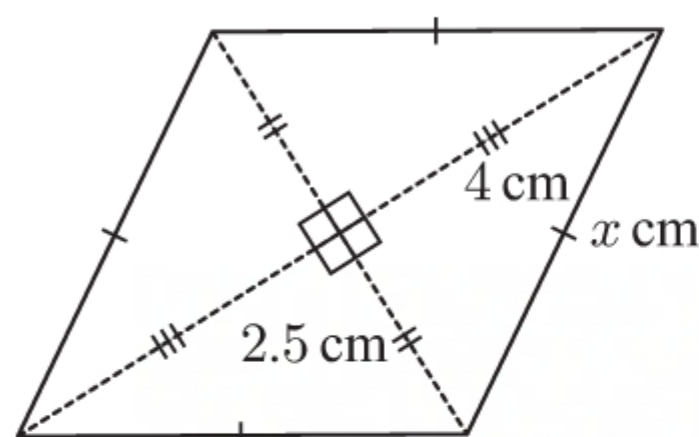
Now $\widehat{DAC} = 90^\circ - 40^\circ$ {angles in triangle ACD}
 $= 50^\circ$

$$\therefore \widehat{EAB} \approx 50^\circ + 36.68^\circ \approx 86.68^\circ$$

In triangle ABE, $\cos \widehat{EAB} = \frac{x}{AE}$

$$\therefore \cos 86.68^\circ \approx \frac{x}{\left(\frac{5}{\sin 36.68^\circ}\right)}$$

$$\therefore x \approx \frac{5 \cos 86.68^\circ}{\sin 36.68^\circ} \approx 0.485$$

13 a


b Let x cm be the side length of the rhombus.

$$\therefore x^2 = 4^2 + (2.5)^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 22.25$$

$$\therefore x = \sqrt{22.25} \quad \{x > 0\}$$

$$\therefore x \approx 4.72$$

\therefore the length of the rhombus' sides are approximately 4.72 cm.

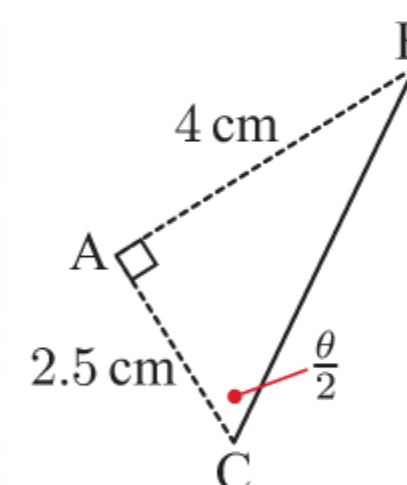
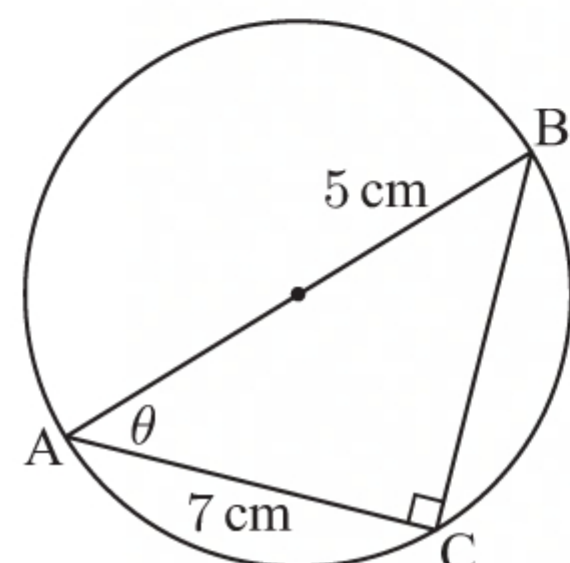
c Let θ be the larger angle in the rhombus.

$$\therefore \widehat{ACB} = \frac{\theta}{2} \quad \{\text{diagonals of a rhombus bisect its angles}\}$$

$$\therefore \tan \frac{\theta}{2} = \frac{4}{2.5}$$

$$\therefore \frac{\theta}{2} = \tan^{-1}\left(\frac{4}{2.5}\right)$$

$$\therefore \theta = 2 \tan^{-1}\left(\frac{4}{2.5}\right) \approx 116^\circ$$


14 a


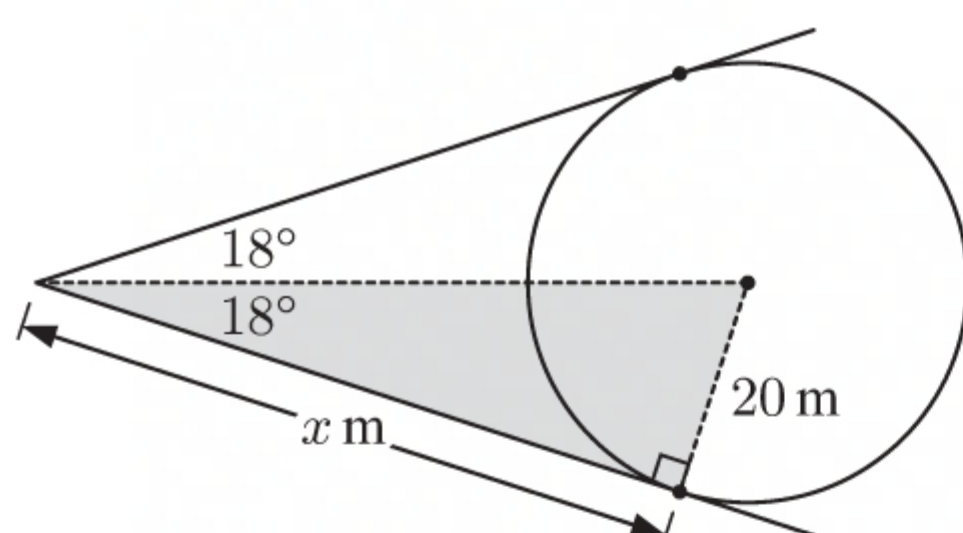
$$\widehat{ACB} = 90^\circ \quad \{\text{angle in a semi-circle}\}$$

$$\therefore \triangle ABC \text{ is right angled at C.}$$

$$\therefore \cos \theta = \frac{7}{AB}$$

$$\therefore \cos \theta = \frac{7}{10}$$

$$\therefore \theta = \cos^{-1}\left(\frac{7}{10}\right) \approx 45.6^\circ$$

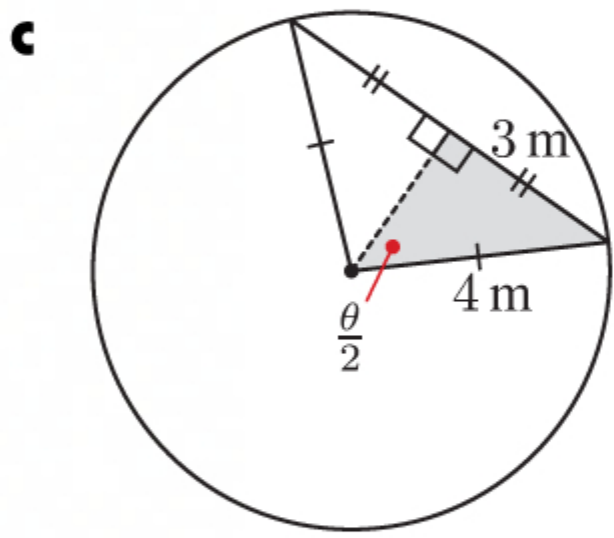
b


We construct the right angled triangle as shown.

For the shaded triangle, $\tan 18^\circ = \frac{20}{x}$

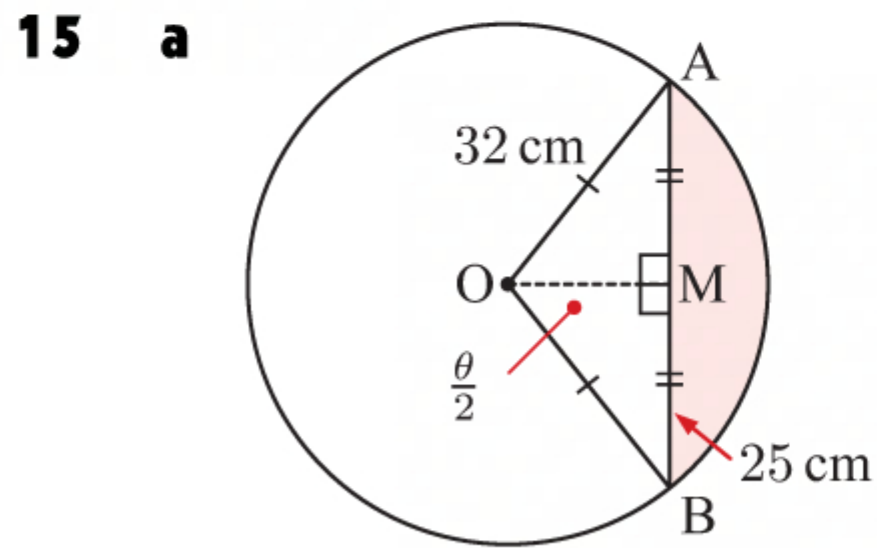
$$\therefore x = \frac{20}{\tan 18^\circ}$$

$$\therefore x \approx 61.6$$



We construct the altitude as shown.

For the shaded triangle, $\sin \frac{\theta}{2} = \frac{3}{4}$
 $\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)$
 $\therefore \theta = 2 \sin^{-1}\left(\frac{3}{4}\right)$
 $\therefore \theta \approx 97.2^\circ$

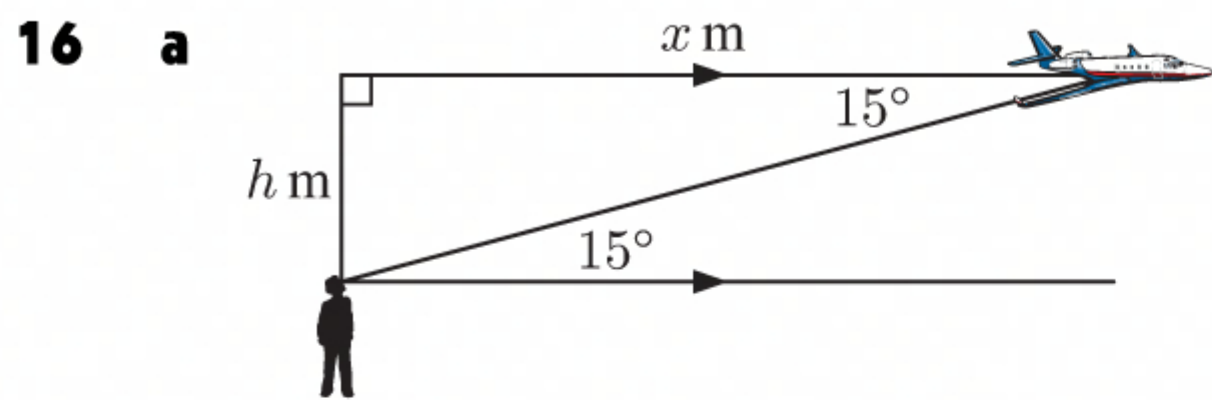


Let $\widehat{AOB} = \theta \quad \therefore \widehat{BOM} = \frac{\theta}{2}$

In triangle OMB, $\sin \frac{\theta}{2} = \frac{25}{32}$
 $\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{25}{32}\right)$
 $\therefore \theta = 2 \sin^{-1}\left(\frac{25}{32}\right)$
 $\therefore \theta \approx 1.79 \text{ radians}$

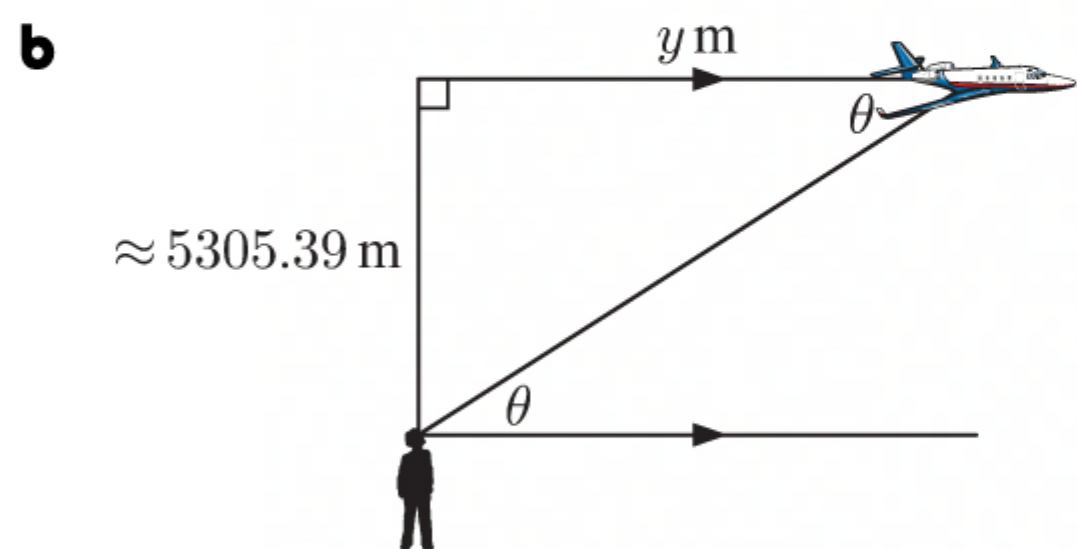
b Sector area $= \frac{1}{2}\theta r^2$ Triangle area $= \frac{1}{2}r^2 \sin \theta$
 $\approx \frac{1}{2}(1.79)(32)^2$ $\approx \frac{1}{2}(32)^2 \sin(1.79)$
 $\approx 918.19 \text{ cm}^2$ $\approx 499.37 \text{ cm}^2$

\therefore area of shaded segment = sector area – triangle area
 $\approx 918.19 - 499.37$
 ≈ 418.81
 $\approx 419 \text{ cm}^2$



$x = \text{speed} \times \text{time}$
 $= 110 \times 180 \quad \{3 \text{ minutes} = 180 \text{ seconds}\}$
 $= 19\,800$
 $\therefore \tan 15^\circ = \frac{h}{19\,800}$
 $\therefore h = 19\,800 \tan 15^\circ$
 $\therefore h \approx 5305.39 \approx 5310$

\therefore the plane is approximately $5310 \text{ m} \approx 5.31 \text{ km}$ above the ground.



$y = \text{speed} \times \text{time}$
 $= 110 \times 420 \quad \{7 \text{ minutes} = 420 \text{ seconds}\}$
 $= 46\,200$
 $\therefore \tan \theta \approx \frac{5305.39}{46\,200}$
 $\therefore \theta \approx \tan^{-1}\left(\frac{5305.39}{46\,200}\right)$
 $\therefore \theta \approx 6.55^\circ$

\therefore the angle of elevation of the plane at 2:42 pm is approximately 6.55° .

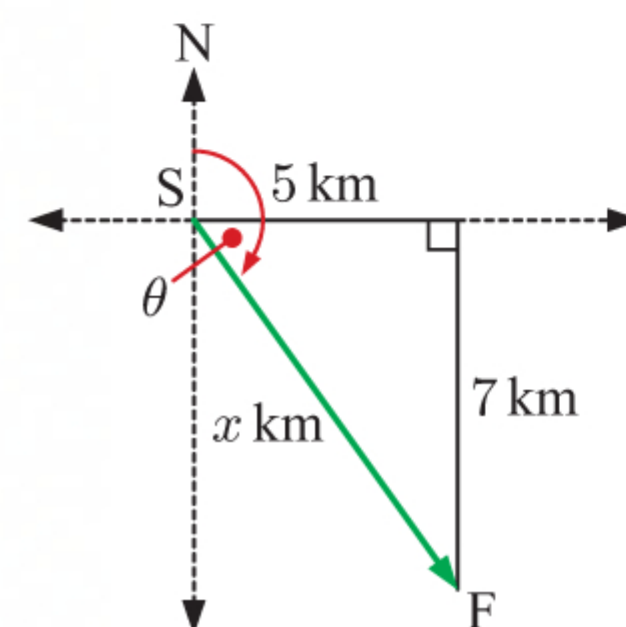
17 a Suppose the helicopter starts at S and lands at F.

Now $x^2 = 7^2 + 5^2 \quad \{\text{Pythagoras}\}$
 $= 74$
 $\therefore x = \sqrt{74} \quad \{x > 0\}$
 ≈ 8.60

\therefore the helicopter is about 8.60 km from its starting point.

b $\tan \theta = \frac{7}{5}$
 $\therefore \theta = \tan^{-1}\left(\frac{7}{5}\right) \approx 54.5^\circ$

So, the bearing $\approx 90^\circ + 54.5^\circ$
 $\approx 144.5^\circ$



- 18 a** The projection of $[AE]$ onto the base plane is $[DE]$.

\therefore the required angle is \widehat{AED} .

$$\tan \theta = \frac{4}{10}$$

$$\therefore \theta = \tan^{-1}\left(\frac{4}{10}\right) \approx 21.8^\circ$$

The angle is about 21.8° .

- b** The projection of $[BD]$ onto the base plane is $[CD]$.

\therefore the required angle is \widehat{BDC} .

$$\tan \theta = \frac{4}{11}$$

$$\therefore \theta = \tan^{-1}\left(\frac{4}{11}\right) \approx 20.0^\circ$$

The angle is about 20.0° .

- c** The projection of $[BE]$ onto the base plane is $[CE]$.

\therefore the required angle is \widehat{BEC} .

Let CE be x cm.

Using Pythagoras in $\triangle ECF$, $x^2 = 10^2 + 11^2$

$$\therefore x^2 = 221$$

$$\therefore x = \sqrt{221} \quad \{x > 0\}$$

Let \widehat{BEC} be α .

$$\therefore \tan \alpha = \frac{4}{\sqrt{221}}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{4}{\sqrt{221}}\right) \approx 15.1^\circ$$

The angle is about 15.1° .

- d** The projection of $[AM]$ onto the base plane is $[DM]$.

\therefore the required angle is \widehat{AMD} .

Let DM be x cm.

Using Pythagoras in $\triangle DCM$, $x^2 = 11^2 + 5^2$

$$\therefore x^2 = 146$$

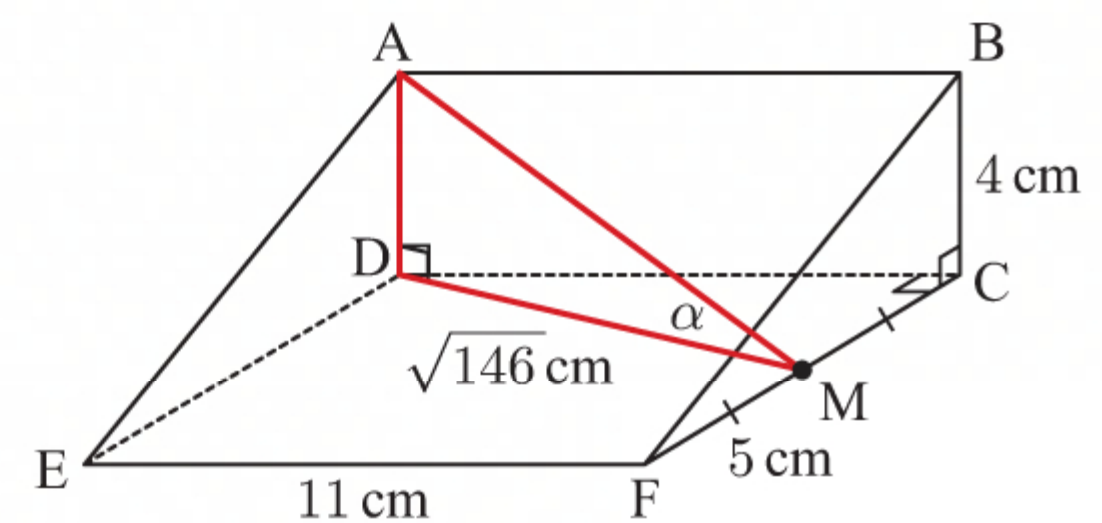
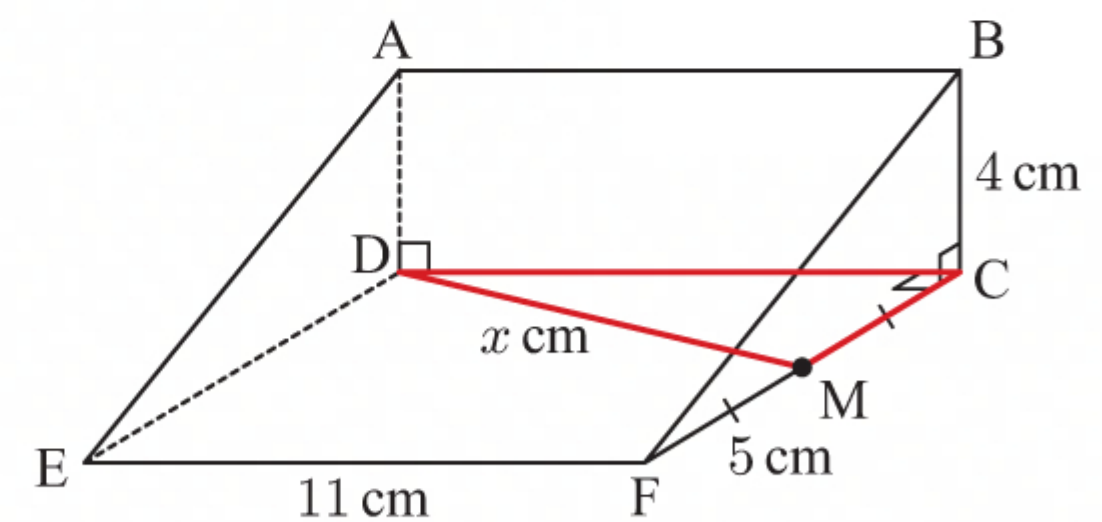
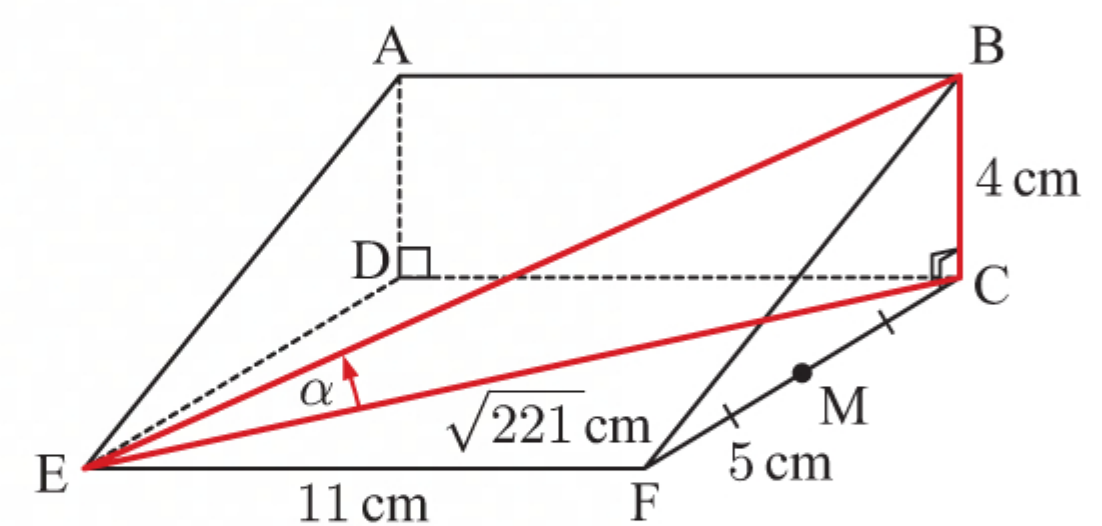
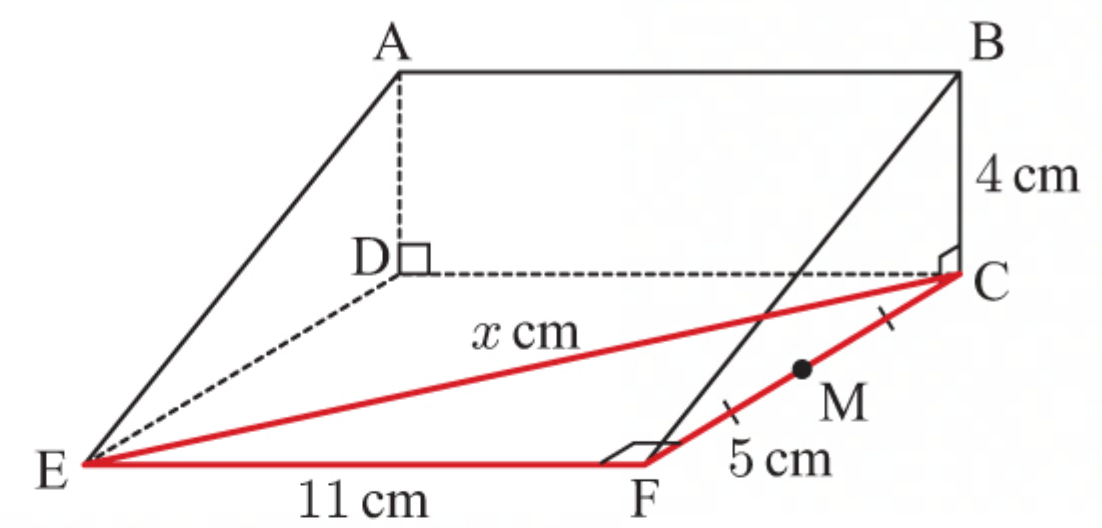
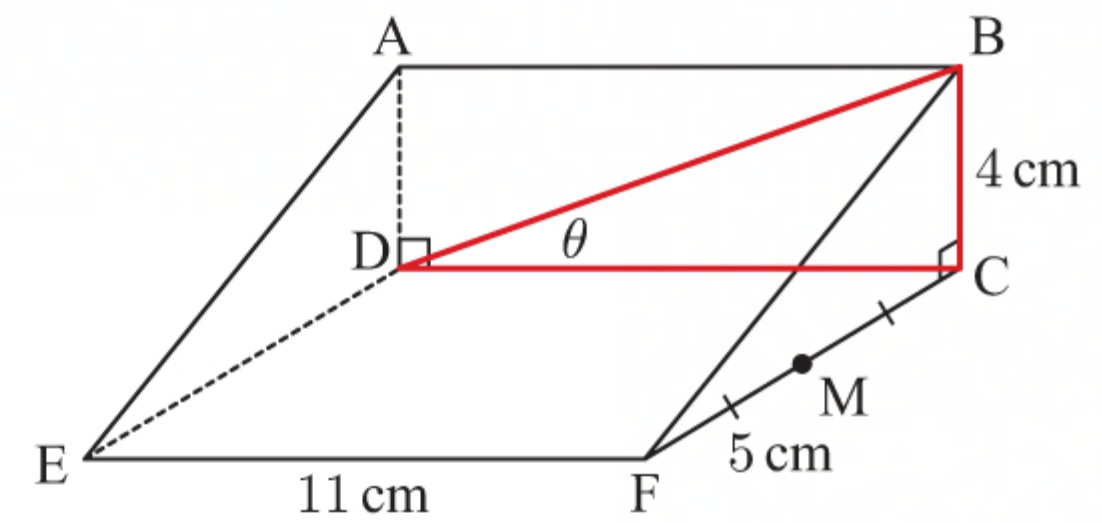
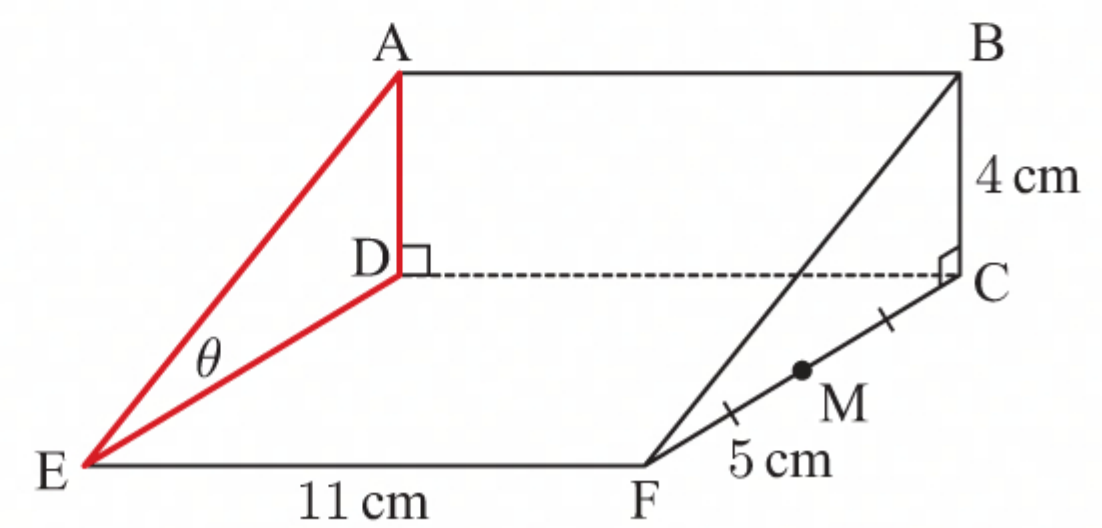
$$\therefore x = \sqrt{146} \quad \{x > 0\}$$

Let \widehat{AMD} be α .

$$\therefore \tan \alpha = \frac{4}{\sqrt{146}}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{4}{\sqrt{146}}\right) \approx 18.3^\circ$$

The angle is about 18.3° .



- 19 a** M is $(-5, 4, 6)$

- b** Let \widehat{CMD} be θ .

Now $DM = 5$ units

$$\text{and } CD = \sqrt{(-10 - (-10))^2 + (4 - 0)^2 + (6 - 0)^2}$$

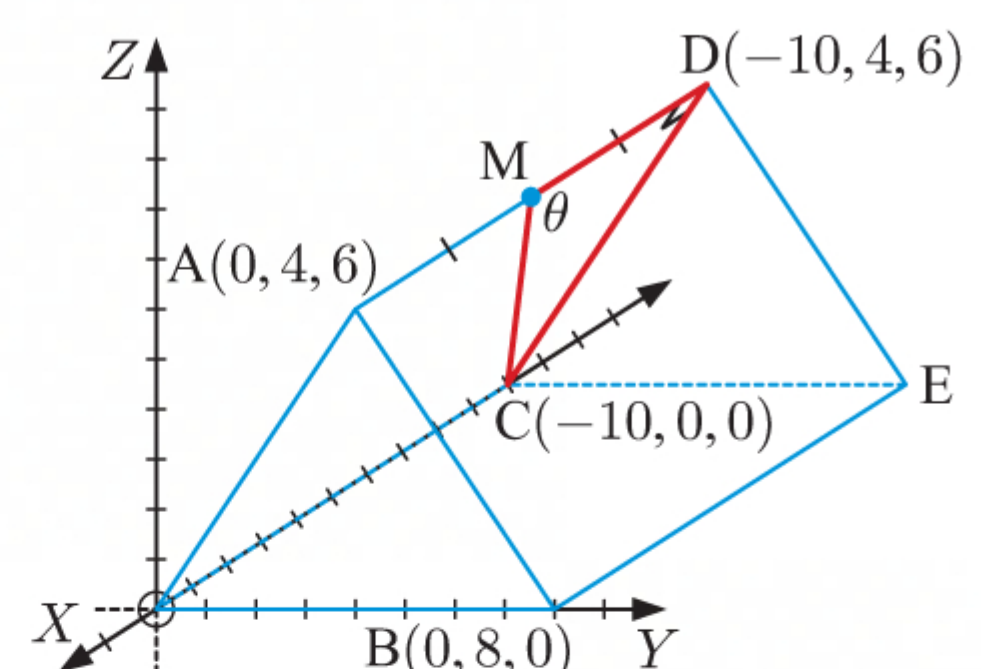
$$= \sqrt{0^2 + 4^2 + 6^2}$$

$$= \sqrt{52} \text{ units}$$

$$\therefore \tan \theta = \frac{\sqrt{52}}{5}$$

$$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{52}}{5}\right) \approx 55.3^\circ$$

$$\therefore \widehat{CMD} \approx 55.3^\circ$$



- c i** The required angle is \widehat{DON} , where N has coordinates $(-10, 4, 0)$.

Now $DN = 6$ units

$$\begin{aligned}\text{and } NO &= \sqrt{(-10 - 0)^2 + (4 - 0)^2 + (0 - 0)^2} \\ &= \sqrt{(-10)^2 + 4^2 + 0^2} \\ &= \sqrt{116} \text{ units}\end{aligned}$$

$$\therefore \tan \theta = \frac{6}{\sqrt{116}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{116}}\right) \approx 29.1^\circ$$

The angle is about 29.1° .

- ii** The required angle is \widehat{MEP} , where P has coordinates $(-5, 4, 0)$.

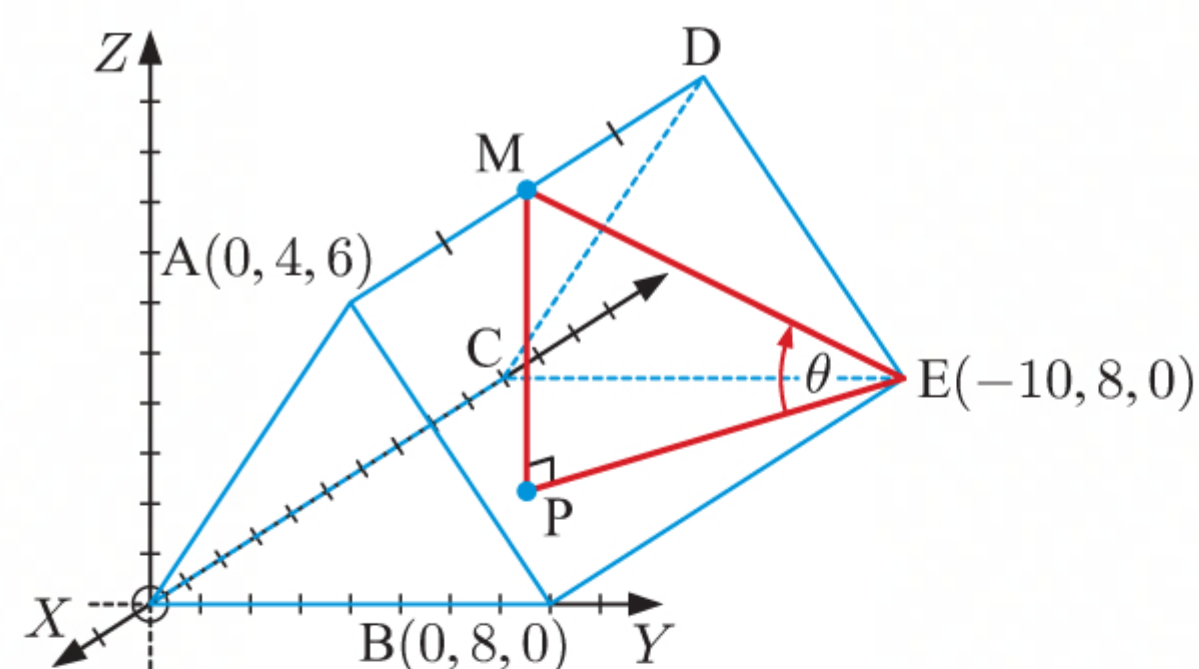
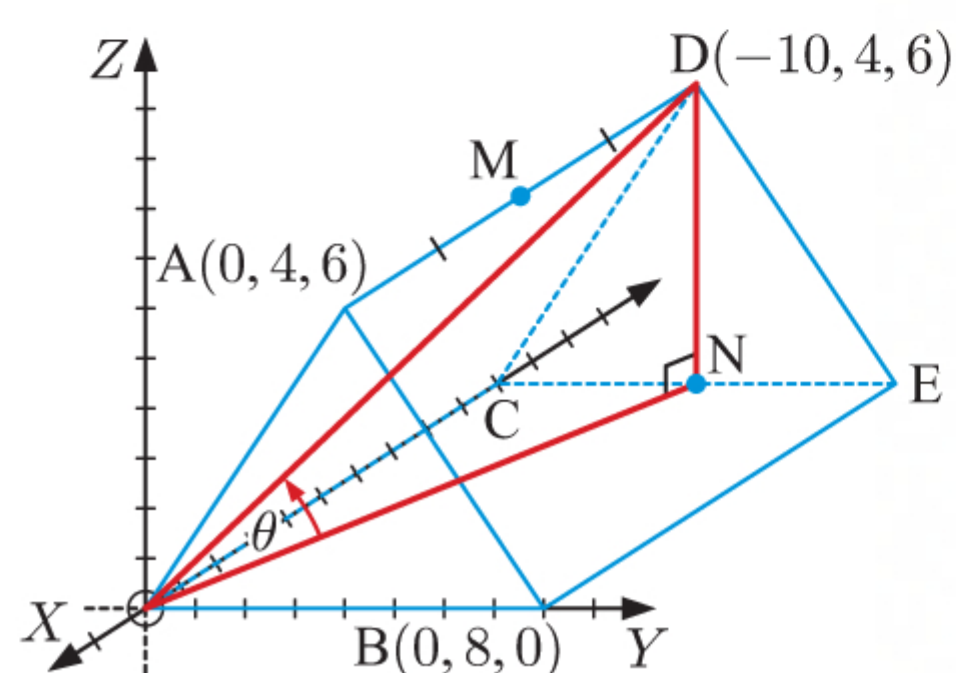
Now $MP = 6$ units

$$\begin{aligned}\text{and } PE &= \sqrt{(-10 - (-5))^2 + (8 - 4)^2 + (0 - 0)^2} \\ &= \sqrt{(-5)^2 + 4^2 + 0^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

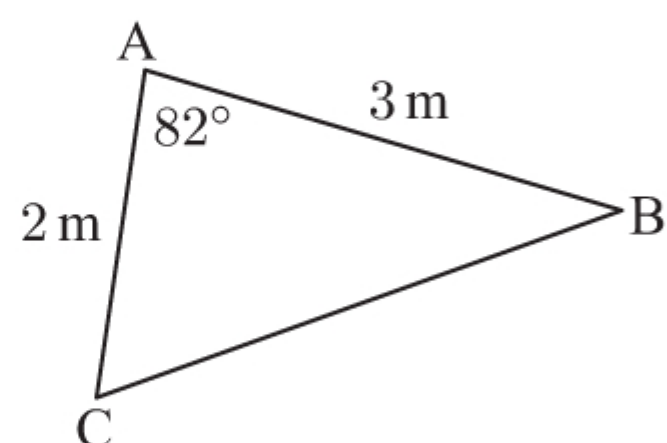
$$\therefore \tan \theta = \frac{6}{\sqrt{41}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{41}}\right) \approx 43.1^\circ$$

The angle is about 43.1° .

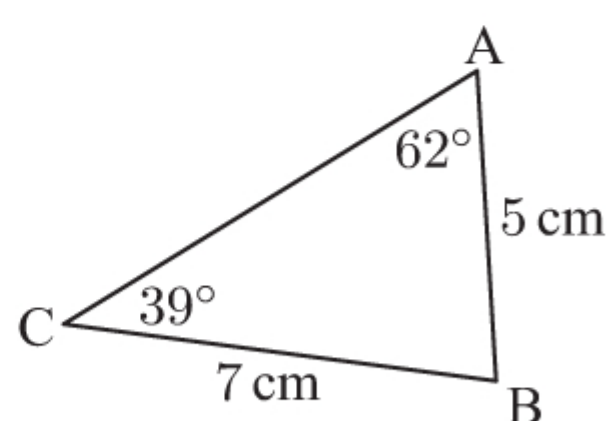


20 a



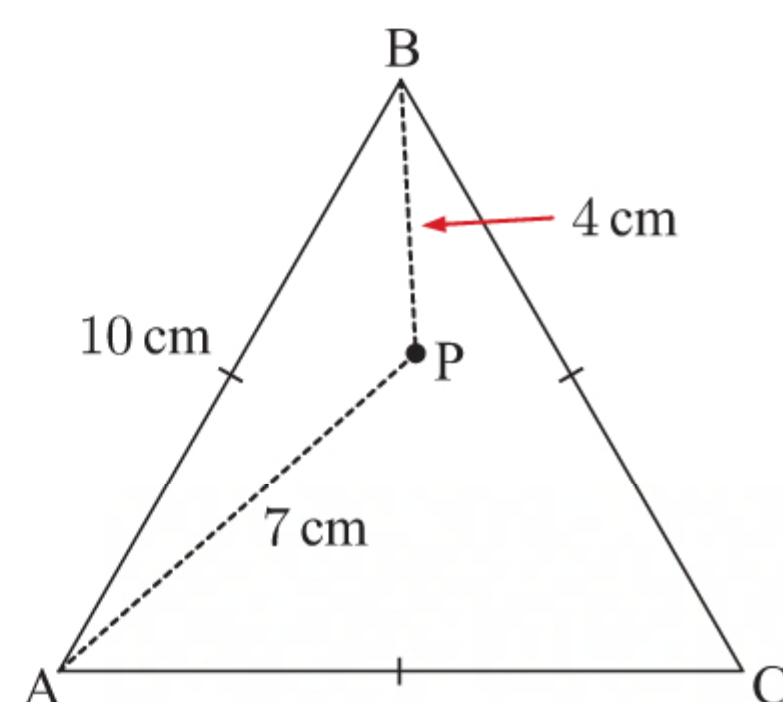
$$\begin{aligned}\text{Area} &= \frac{1}{2}bc \sin A \\ &= \frac{1}{2} \times 2 \times 3 \times \sin 82^\circ \\ &\approx 2.97 \text{ m}^2\end{aligned}$$

b



$$\begin{aligned}\widehat{ABC} &= 180^\circ - 62^\circ - 39^\circ \quad \{\text{angles in a triangle}\} \\ &= 79^\circ \\ \therefore \text{area} &= \frac{1}{2}ac \sin B \\ &= \frac{1}{2} \times 7 \times 5 \times \sin 79^\circ \\ &\approx 17.2 \text{ cm}^2\end{aligned}$$

21 a

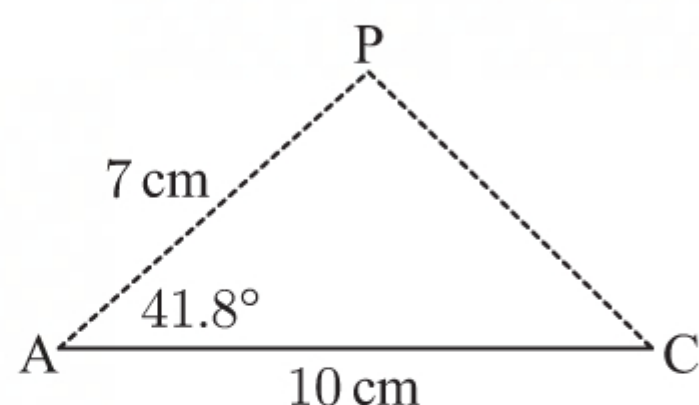


- b i** By the cosine rule in $\triangle BAP$:

$$\begin{aligned}\cos \widehat{BAP} &= \frac{10^2 + 7^2 - 4^2}{2 \times 7 \times 10} \\ \therefore \cos \widehat{BAP} &= \frac{133}{140} \\ \therefore \widehat{BAP} &= \cos^{-1}\left(\frac{133}{140}\right) \approx 18.2^\circ\end{aligned}$$

- ii** $\widehat{BAC} = 60^\circ$ {angles in an equilateral triangle}
 $\therefore \widehat{CAP} = 60^\circ - \widehat{BAP}$
 $\approx 60^\circ - 18.2^\circ$
 $\approx 41.8^\circ$

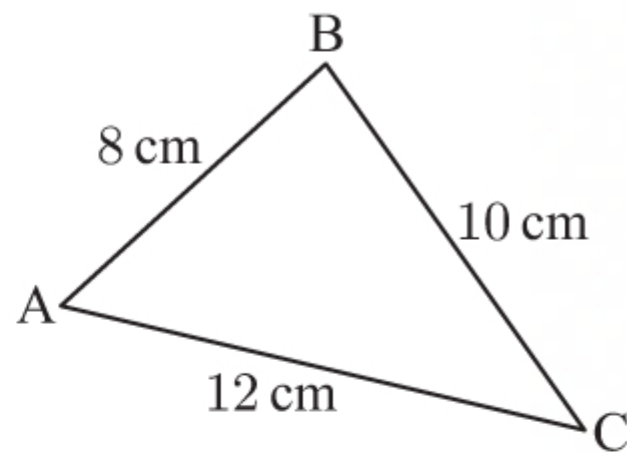
c



By the cosine rule in $\triangle APC$:

$$\begin{aligned}CP^2 &\approx 10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ \\ \therefore CP &\approx \sqrt{10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ} \\ \therefore CP &\approx 6.68 \text{ cm}\end{aligned}$$

22 a



b The smallest angle in triangle ABC is opposite the shortest side.

 $\therefore \widehat{BCA}$ is the smallest angle.

By the cosine rule:

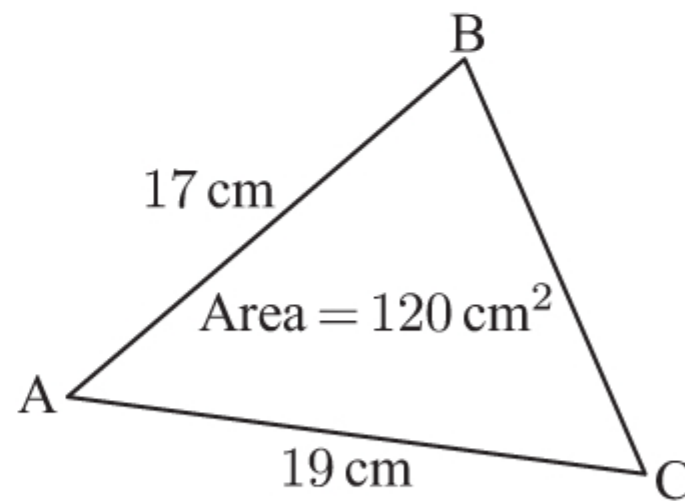
$$\cos \widehat{BCA} = \frac{12^2 + 10^2 - 8^2}{2 \times 12 \times 10}$$

$$\therefore \cos \widehat{BCA} = \frac{180}{240} = \frac{3}{4}$$

$$\therefore \widehat{BCA} = \cos^{-1}\left(\frac{3}{4}\right) \approx 41.4^\circ$$

$$\begin{aligned} \text{c Area} &= \frac{1}{2}ab \sin C \\ &\approx \frac{1}{2} \times 10 \times 12 \times \sin 41.4^\circ \\ &\approx 39.7 \text{ cm}^2 \end{aligned}$$

23 a



b

$$\text{Area} = 120 \text{ cm}^2$$

$$\therefore \frac{1}{2} \times 17 \times 19 \times \sin \widehat{BAC} = 120$$

$$\therefore \sin \widehat{BAC} = \frac{120 \times 2}{17 \times 19}$$

$$\therefore \sin \widehat{BAC} = \frac{240}{323}$$

$$\therefore \widehat{BAC} = \sin^{-1}\left(\frac{240}{323}\right) \approx 48.0^\circ$$

c By the cosine rule:

$$CP^2 \approx 17^2 + 19^2 - 2(17)(19) \cos 48.0^\circ$$

$$\therefore CP \approx \sqrt{17^2 + 19^2 - 2(17)(19) \cos 48.0^\circ}$$

$$\therefore CP \approx 14.8 \text{ cm}$$

 d Volume = cross-sectional area \times length

$$= 120 \times 13.5$$

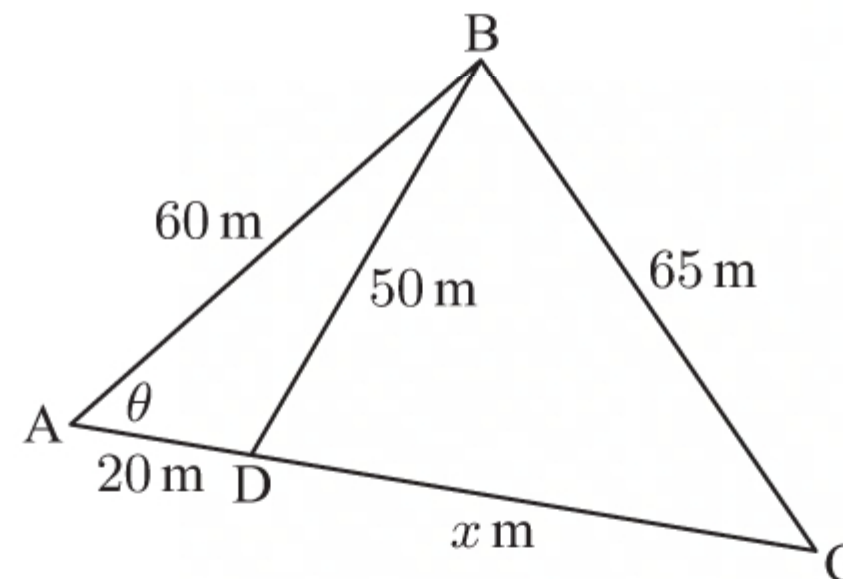
$$= 1620 \text{ cm}^3$$

 24 a By the cosine rule in $\triangle BAD$:

$$\cos \theta = \frac{60^2 + 20^2 - 50^2}{2 \times 60 \times 20}$$

$$\therefore \cos \theta = \frac{1500}{2400}$$

$$\therefore \cos \theta = \frac{5}{8}$$


 b By the cosine rule in $\triangle ABC$:

$$65^2 = 60^2 + (20 + x)^2 - 2(60)(20 + x) \cos \theta$$

$$\therefore 65^2 = 60^2 + (20 + x)^2 - 2(60)(20 + x)\left(\frac{5}{8}\right) \quad \{\text{using a}\}$$

$$\therefore 4225 = 3600 + 400 + 40x + x^2 - 1500 - 75x$$

$$\therefore x^2 - 35x - 1725 = 0$$

$$\therefore x = \frac{35 \pm \sqrt{1225 - 4(1)(-1725)}}{2(1)} \quad \{\text{quadratic formula}\}$$

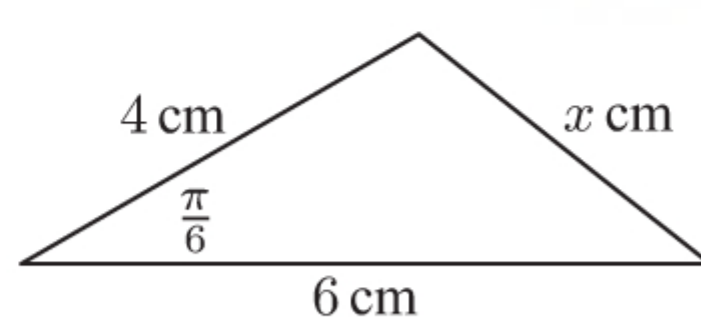
$$\therefore x = \frac{35 \pm \sqrt{8125}}{2}$$

$$\therefore x = \frac{35 \pm 25\sqrt{13}}{2}$$

$$\therefore x = \frac{35 + 25\sqrt{13}}{2} \quad \{x > 0\}$$

$$\therefore x \approx 62.6$$

25 a



By the cosine rule:

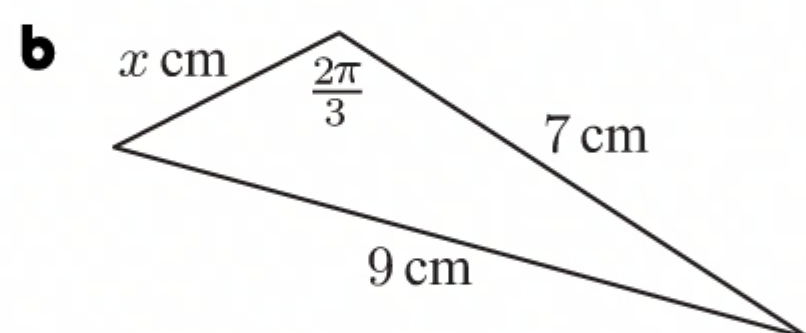
$$x^2 = 4^2 + 6^2 - 2(4)(6) \cos \frac{\pi}{6}$$

$$\therefore x^2 = 16 + 36 - 48\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore x^2 = 52 - 24\sqrt{3}$$

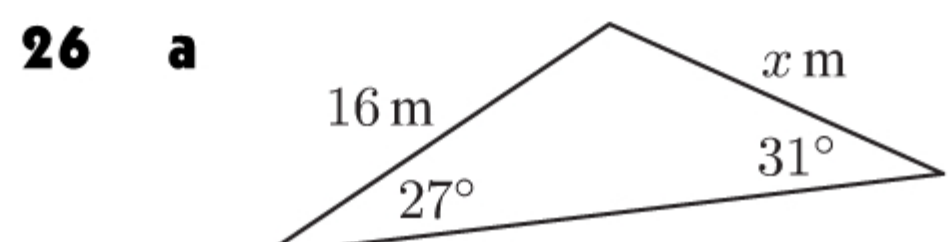
$$\therefore x = \sqrt{52 - 24\sqrt{3}} \quad \{x > 0\}$$

$$\therefore x \approx 3.23$$



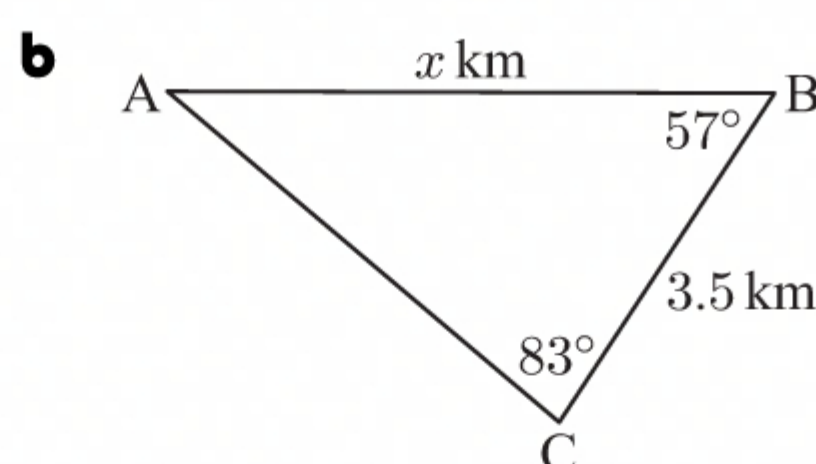
By the cosine rule:

$$\begin{aligned}
 9^2 &= x^2 + 7^2 - 2(x)(7) \cos \frac{2\pi}{3} \\
 \therefore 81 &= x^2 + 49 - 14\left(-\frac{1}{2}\right)x \\
 \therefore 81 &= x^2 + 49 + 7x \\
 \therefore x^2 + 7x - 32 &= 0 \\
 \therefore x &= \frac{-7 \pm \sqrt{49 - 4(1)(-32)}}{2(1)} \quad \{\text{quadratic formula}\} \\
 \therefore x &= \frac{-7 \pm \sqrt{177}}{2} \\
 \therefore x &= \frac{-7 + \sqrt{177}}{2} \quad \{x > 0\} \\
 \therefore x &\approx 3.15
 \end{aligned}$$



By the sine rule:

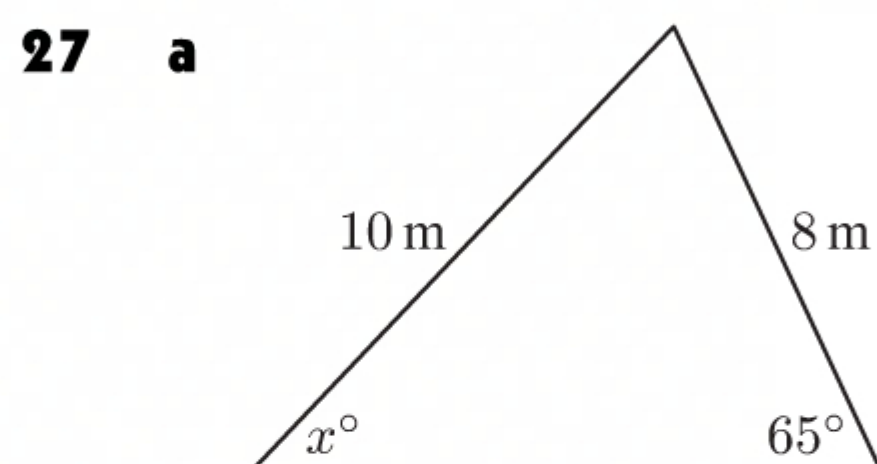
$$\begin{aligned}
 \frac{x}{\sin 27^\circ} &= \frac{16}{\sin 31^\circ} \\
 \therefore x &= \frac{16 \sin 27^\circ}{\sin 31^\circ} \\
 \therefore x &\approx 14.1
 \end{aligned}$$



$$\begin{aligned}
 \widehat{BAC} &= 180^\circ - 57^\circ - 83^\circ \quad \{\text{angles in a triangle}\} \\
 &= 40^\circ
 \end{aligned}$$

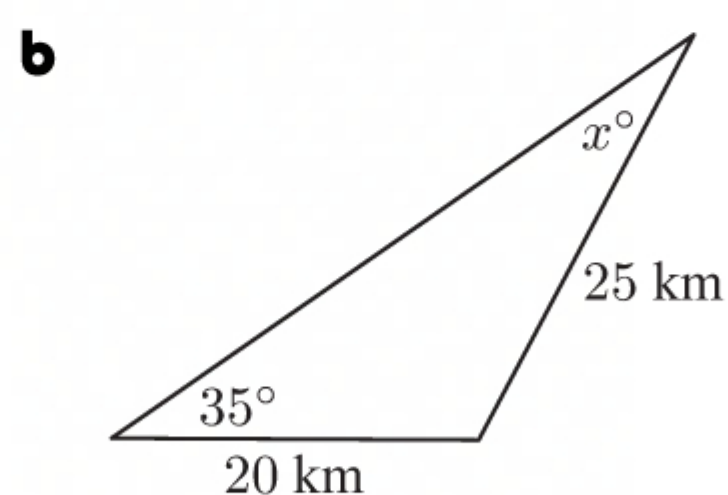
Now by the sine rule:

$$\begin{aligned}
 \frac{x}{\sin 83^\circ} &= \frac{3.5}{\sin 40^\circ} \\
 \therefore x &= \frac{3.5 \sin 83^\circ}{\sin 40^\circ} \\
 \therefore x &\approx 5.40
 \end{aligned}$$



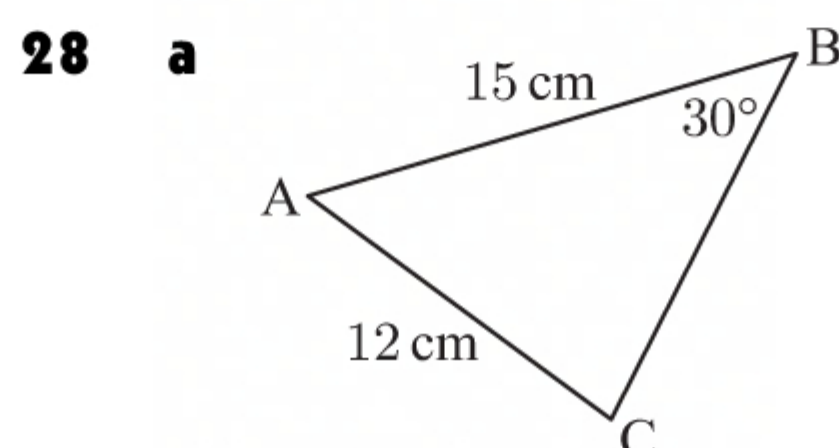
By the sine rule:

$$\begin{aligned}
 \frac{\sin x^\circ}{8} &= \frac{\sin 65^\circ}{10} \\
 \therefore \sin x^\circ &= \frac{8 \sin 65^\circ}{10} \\
 \therefore x &= \sin^{-1}\left(\frac{8 \sin 65^\circ}{10}\right) \\
 \therefore x &\approx 46.5
 \end{aligned}$$



By the sine rule:

$$\begin{aligned}
 \frac{\sin x^\circ}{20} &= \frac{\sin 35^\circ}{25} \\
 \therefore \sin x^\circ &= \frac{20 \sin 35^\circ}{25} \\
 \therefore x &= \sin^{-1}\left(\frac{20 \sin 35^\circ}{25}\right) \\
 \therefore x &\approx 27.3
 \end{aligned}$$

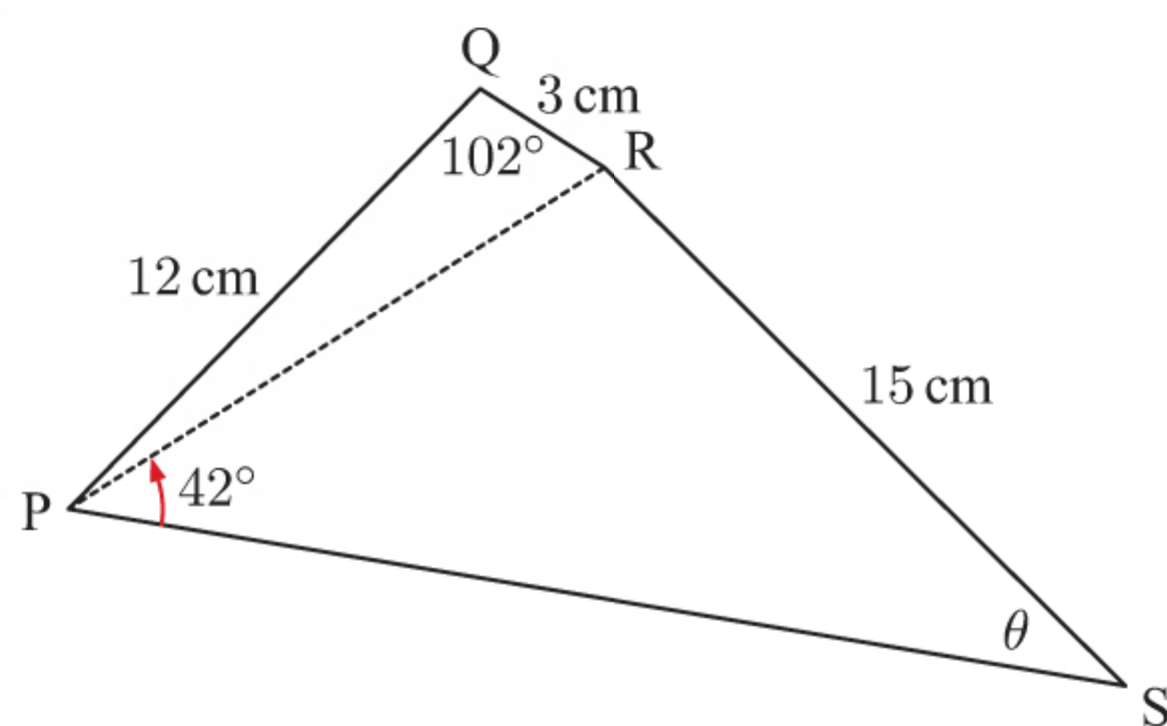


$$\begin{aligned}
 \frac{\sin \widehat{ACB}}{15} &= \frac{\sin 30^\circ}{12} \quad \{\text{sine rule}\} \\
 \therefore \sin \widehat{ACB} &= \frac{15 \sin 30^\circ}{12} \\
 \therefore \widehat{ACB} &= \sin^{-1}\left(\frac{15 \sin 30^\circ}{12}\right) \quad \text{or its supplement} \\
 \therefore \widehat{ACB} &\approx 38.7^\circ \text{ or } 180^\circ - 38.7^\circ \\
 \therefore \widehat{ACB} &\approx 38.7^\circ \text{ or } 141.3^\circ
 \end{aligned}$$

b \widehat{BAC} is acute if \widehat{ACB} is obtuse.

$$\begin{aligned}
 \therefore \widehat{BAC} &\approx 180^\circ - 30^\circ - 141.3^\circ \quad \{\text{using a}\} \\
 &\approx 8.7^\circ
 \end{aligned}$$

29 a

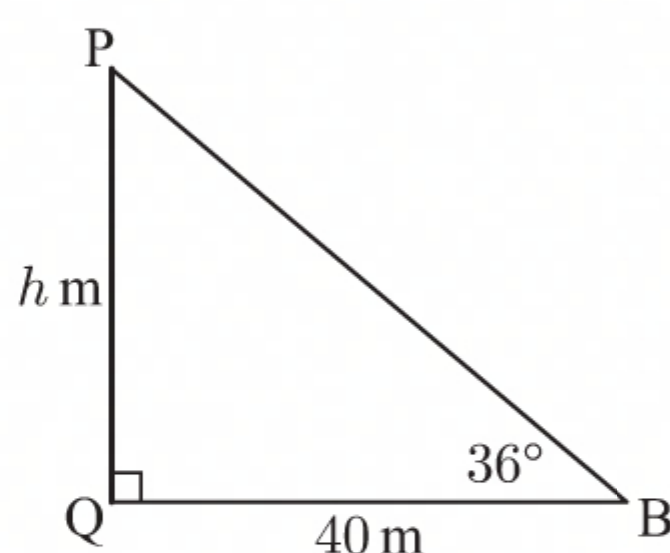

 By the cosine rule in $\triangle PQR$:

$$\begin{aligned} PR^2 &= 3^2 + 12^2 - 2(3)(12) \cos 102^\circ \\ \therefore PR &= \sqrt{3^2 + 12^2 - 2(3)(12) \cos 102^\circ} \\ \therefore PR &\approx 12.96 \text{ cm} \approx 13.0 \text{ cm} \end{aligned}$$

 b By the sine rule in $\triangle PRS$:

$$\begin{aligned} \frac{\sin \theta}{PR} &= \frac{\sin 42^\circ}{15} \\ \therefore \sin \theta &= \frac{PR \sin 42^\circ}{15} \\ \therefore \sin \theta &\approx \frac{12.96 \sin 42^\circ}{15} \quad \{\text{from a}\} \\ \therefore \theta &\approx \sin^{-1} \left(\frac{12.96 \sin 42^\circ}{15} \right) \approx 35.3^\circ \end{aligned}$$

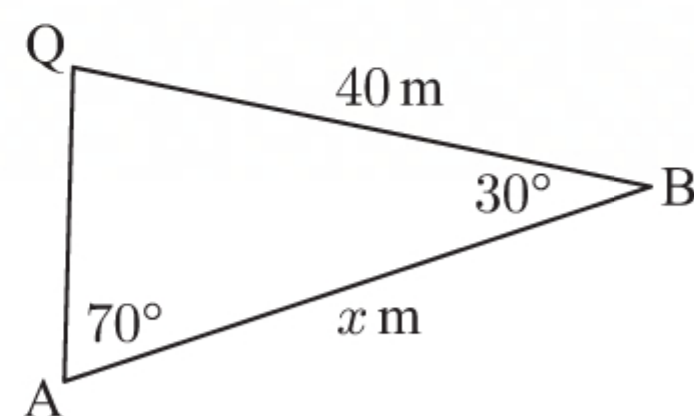
30 a


 Let PQ be h m.

$$\begin{aligned} \therefore \tan 36^\circ &= \frac{h}{40} \\ \therefore h &= 40 \tan 36^\circ \approx 29.1 \end{aligned}$$

The height of the pole is about 29.1 m.

b

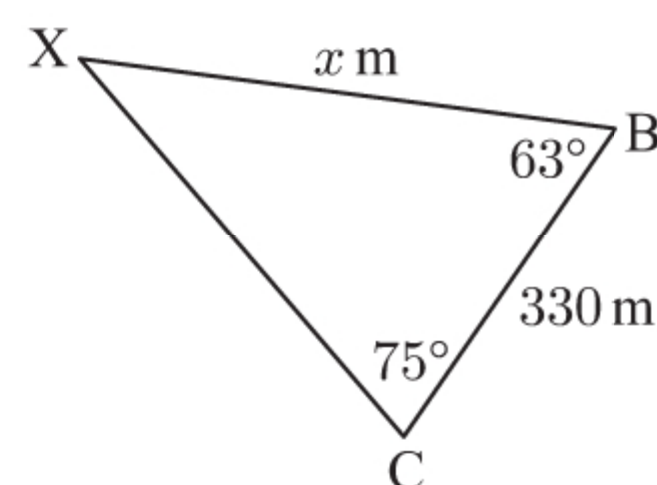

 Let AB be x m.

$$\begin{aligned} \angle AQB &= 180^\circ - 70^\circ - 30^\circ \quad \{\text{angles in a triangle}\} \\ &= 80^\circ \end{aligned}$$

$$\begin{aligned} \therefore \frac{x}{\sin 80^\circ} &= \frac{40}{\sin 70^\circ} \quad \{\text{sine rule}\} \\ \therefore x &= \frac{40 \sin 80^\circ}{\sin 70^\circ} \\ \therefore x &\approx 41.9 \end{aligned}$$

The distance between A and B is about 41.9 m.

31 a

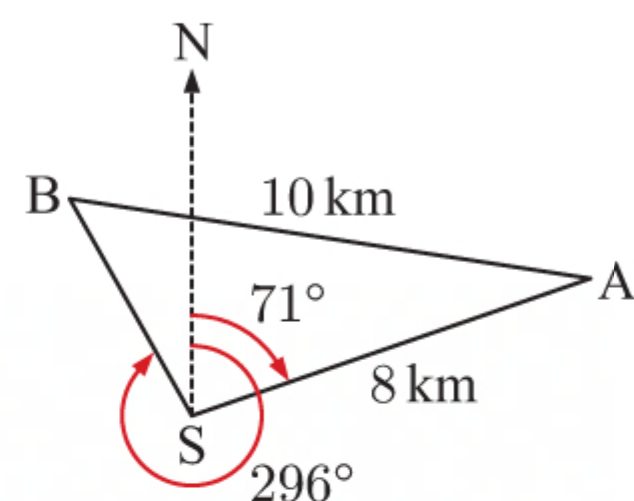

 b Let XB be x m.

$$\begin{aligned} \angle CXB &= 180^\circ - 63^\circ - 75^\circ \quad \{\text{angles in a triangle}\} \\ &= 42^\circ \end{aligned}$$

$$\begin{aligned} \therefore \frac{x}{\sin 75^\circ} &= \frac{330}{\sin 42^\circ} \quad \{\text{sine rule}\} \\ \therefore x &= \frac{330 \sin 75^\circ}{\sin 42^\circ} \\ \therefore x &\approx 476 \end{aligned}$$

The distance between the monument and B is about 476 m.

32 a



$$\begin{aligned} \mathbf{b} \quad N_1\hat{S}B &= 360^\circ - 296^\circ \quad \{\text{angles at a point}\} \\ &= 64^\circ \end{aligned}$$

$$\therefore B\hat{S}A = 64^\circ + 71^\circ = 135^\circ$$

$$\therefore \frac{\sin \alpha}{8} = \frac{\sin 135^\circ}{10} \quad \{\text{sine rule}\}$$

$$\therefore \sin \alpha = \frac{8 \sin 135^\circ}{10}$$

$$\therefore \alpha = \sin^{-1}\left(\frac{8 \sin 135^\circ}{10}\right) \approx 34.4^\circ$$

$$\begin{aligned} \text{Now } \theta &= 180^\circ - 135^\circ - \alpha \quad \{\text{angles in a triangle}\} \\ &\approx 180^\circ - 135^\circ - 34.4^\circ \\ &\approx 10.6^\circ \end{aligned}$$

$$\begin{aligned} N_2\hat{A}S &= 180^\circ - 71^\circ \quad \{\text{co-interior angles}\} \\ &= 109^\circ \end{aligned}$$

$$\therefore \text{bearing of B from A} \approx 360^\circ - 109^\circ + 10.6^\circ \approx 262^\circ$$

c By the cosine rule:

$$BS^2 = 10^2 + 8^2 - 2(10)(8) \cos \theta$$

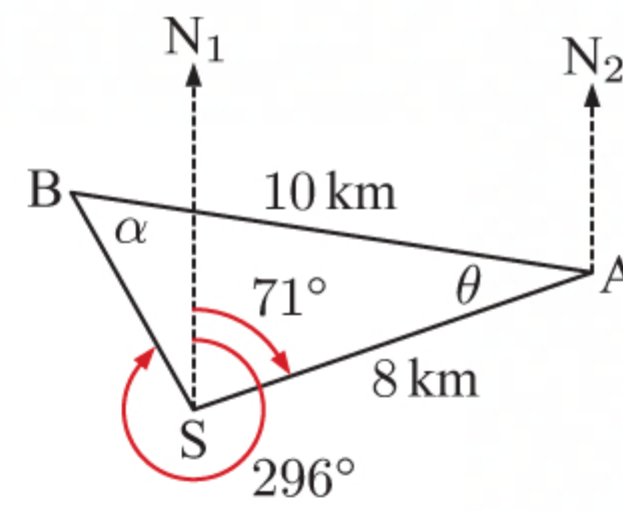
$$\therefore BS = \sqrt{10^2 + 8^2 - 2(10)(8) \cos \theta}$$

$$\therefore BS \approx \sqrt{10^2 + 8^2 - 2(10)(8) \cos 10.6^\circ} \quad \{\text{from b}\}$$

$$\therefore BS \approx 2.59 \text{ km} \approx 2590 \text{ m}$$

$$\begin{aligned} \text{Now time} &= \frac{\text{distance}}{\text{speed}} \\ &\approx \frac{2590}{7} \\ &\approx 370 \text{ seconds} \end{aligned}$$

It will take about 370 seconds or 6 minutes 10 seconds for train B to reach the train station.



$$\mathbf{33} \quad a = \sin 20^\circ, \quad b = \tan 50^\circ$$

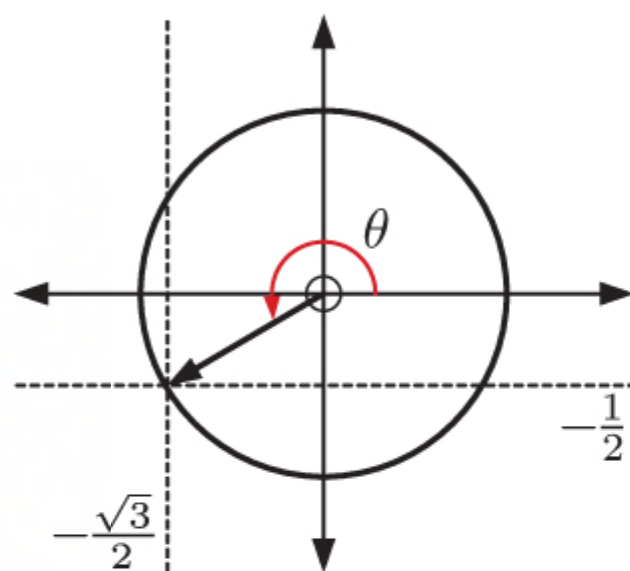
$$\begin{aligned} \mathbf{a} \quad \sin 160^\circ &= \sin(180^\circ - 20^\circ) \\ &= \sin 20^\circ \\ &= a \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan(-50^\circ) &= \frac{\sin(-50^\circ)}{\cos(-50^\circ)} \\ &= \frac{-\sin 50^\circ}{\cos 50^\circ} \\ &= -\tan 50^\circ \\ &= -b \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \cos 70^\circ &= \cos(90^\circ - 20^\circ) \\ &= \sin 20^\circ \\ &= a \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \tan 20^\circ &= \frac{\sin 20^\circ}{\cos 20^\circ} \\ &= \frac{\sin 20^\circ}{\sqrt{1 - \sin^2 20^\circ}} \quad \{0^\circ < 20^\circ < 90^\circ, \text{ so } \cos 20^\circ > 0\} \\ &= \frac{a}{\sqrt{1 - a^2}} \end{aligned}$$

34 a



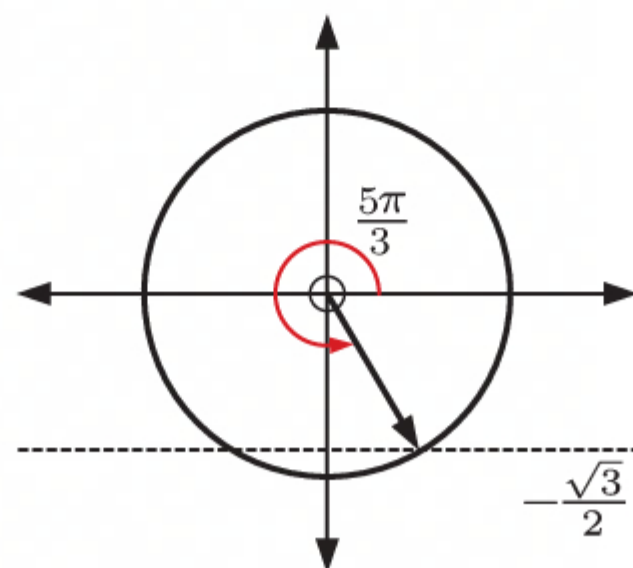
$$\sin \theta = -\frac{1}{2} \quad \text{and} \quad \cos \theta = -\frac{\sqrt{3}}{2}$$

$$\therefore \theta = 210^\circ$$

$$\begin{aligned} \text{b } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

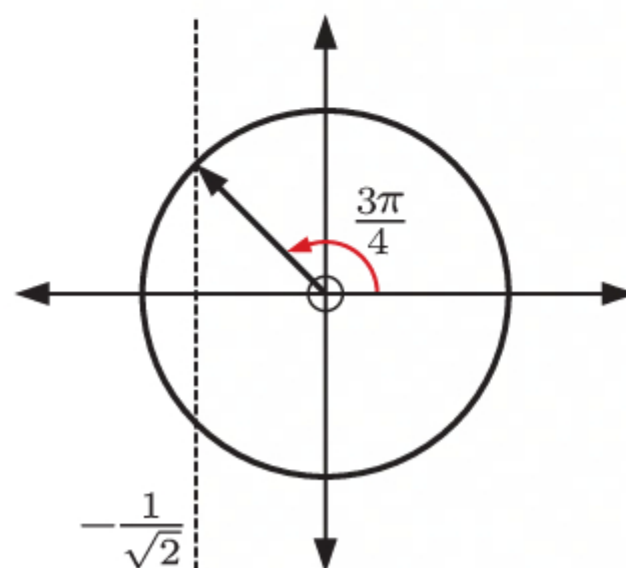
$$\begin{aligned} \text{c } \tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} \quad \{\text{double angle formulae}\} \\ &= \frac{2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right)^2} \\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= \sqrt{3} \end{aligned}$$

35 a



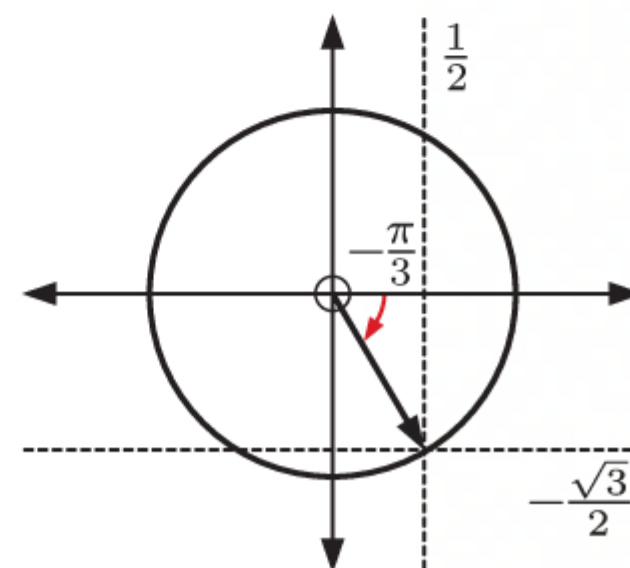
$$\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

b



$$\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

c



$$\begin{aligned} \tan\left(-\frac{\pi}{3}\right) &= \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} \\ &= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= -\sqrt{3} \end{aligned}$$

36 a

$$\begin{aligned} \sin \frac{\pi}{3} \cos \frac{\pi}{4} &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{4} \end{aligned}$$

b

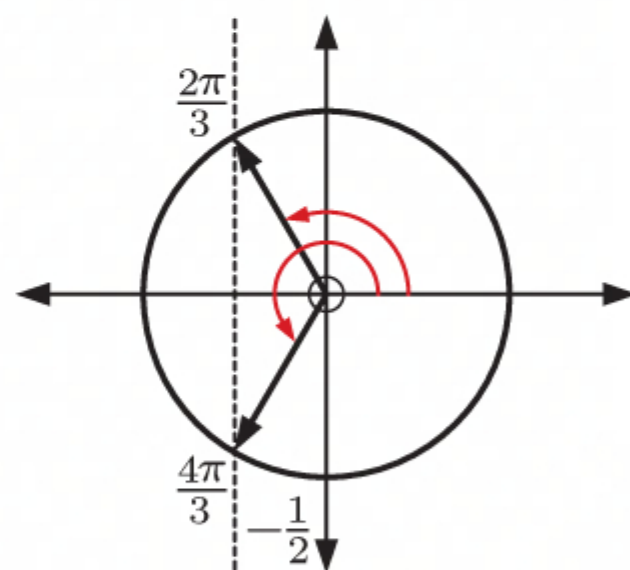
$$\begin{aligned} 2 \tan^2\left(\frac{2\pi}{3}\right) + 1 &= 2\left(\frac{\sin \frac{2\pi}{3}}{\cos \frac{2\pi}{3}}\right)^2 + 1 \\ &= 2\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)^2 + 1 \\ &= 2(-\sqrt{3})^2 + 1 \\ &= 2(3) + 1 \\ &= 7 \end{aligned}$$

c

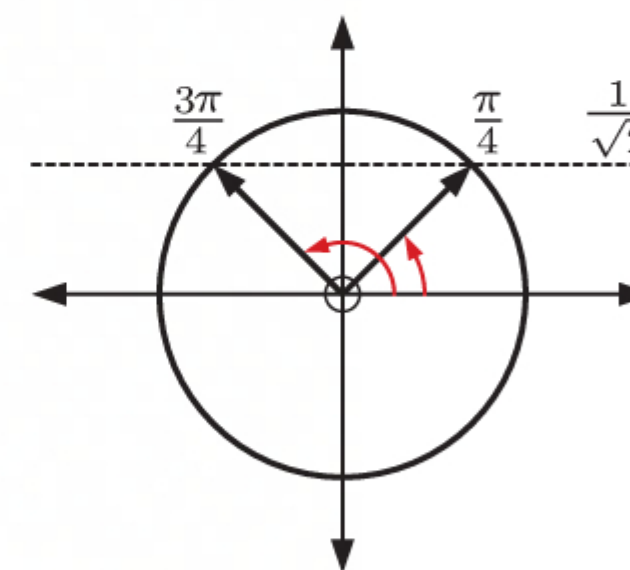
$$\begin{aligned} &\frac{\cos \frac{5\pi}{6} \tan^2\left(\frac{3\pi}{4}\right)}{\sin\left(-\frac{\pi}{3}\right)} \\ &= \frac{\cos \frac{5\pi}{6} \left(\frac{\sin \frac{3\pi}{4}}{\cos \frac{3\pi}{4}}\right)^2}{\sin\left(-\frac{\pi}{3}\right)} \\ &= \frac{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)} \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

37 a

$$\begin{aligned} \cos \theta &= -\frac{1}{2} \\ \therefore \theta &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \end{aligned}$$

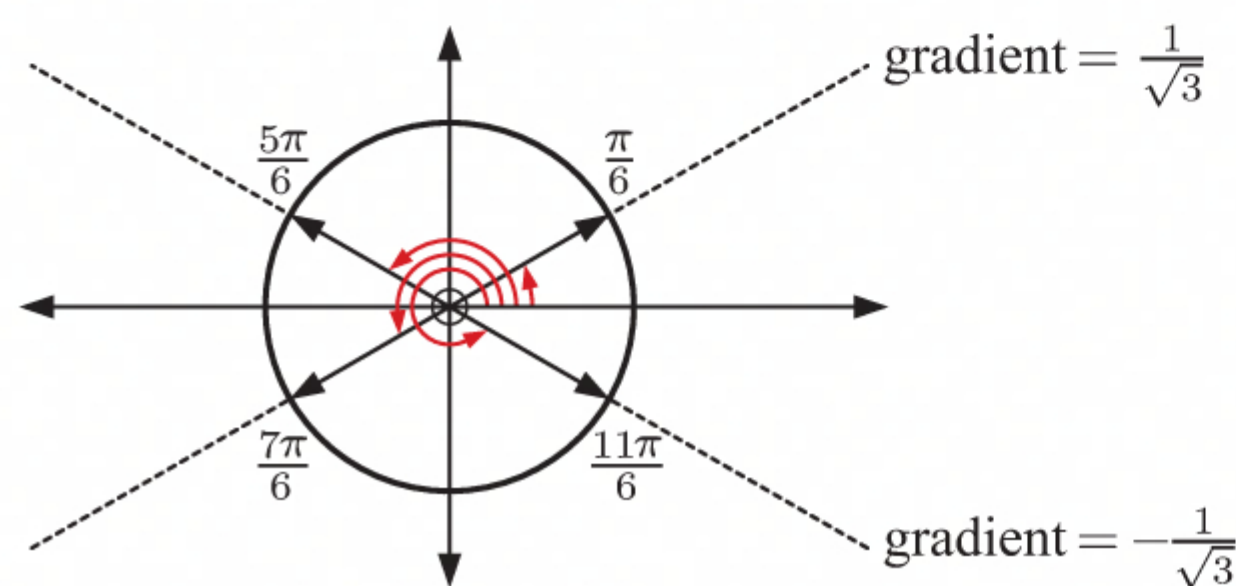


$$\begin{aligned} \text{b } \sin \theta &= \frac{1}{\sqrt{2}} \\ \therefore \theta &= \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \end{aligned}$$



c

$$\begin{aligned} \tan^2 \theta &= \frac{1}{3} \\ \therefore \tan \theta &= \pm \frac{1}{\sqrt{3}} \\ \therefore \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6} \end{aligned}$$



38 a $\sin \theta = \frac{4}{5}$

Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(\frac{4}{5}\right)^2 = 1$$

$$\therefore \cos^2 \theta + \frac{16}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{9}{25}$$

$$\therefore \cos \theta = \pm \frac{3}{5}$$

b $\cos \theta = -\frac{2}{7}$

Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(-\frac{2}{7}\right)^2 + \sin^2 \theta = 1$$

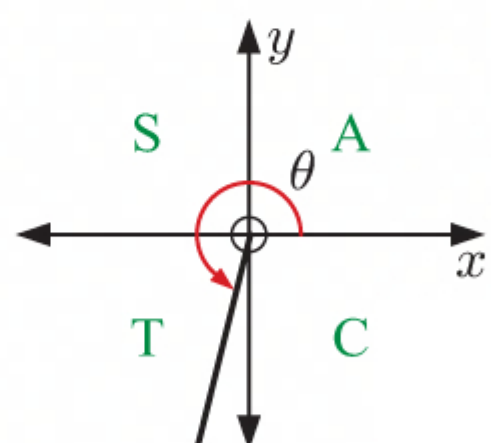
$$\therefore \frac{4}{49} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{45}{49}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{45}}{7}$$

39 a $\cos \theta = -\frac{1}{4}$ and $\pi < \theta < \frac{3\pi}{2}$

θ is in quadrant 3, so $\sin \theta$ is negative.



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(-\frac{1}{4}\right)^2 + \sin^2 \theta = 1$$

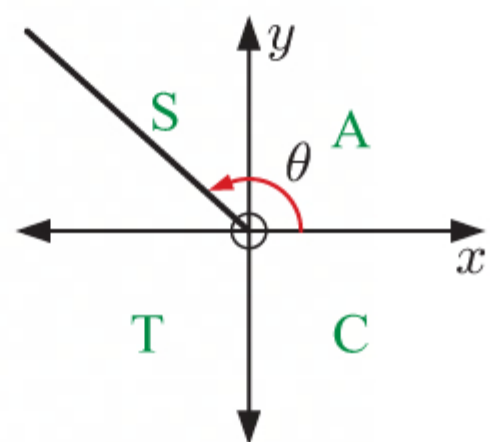
$$\therefore \frac{1}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{15}{16}$$

$$\therefore \sin \theta = -\frac{\sqrt{15}}{4} \quad \{\sin \theta < 0\}$$

b $\sin \theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$

θ is in quadrant 2, so $\cos \theta$ is negative.



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(\frac{2}{3}\right)^2 = 1$$

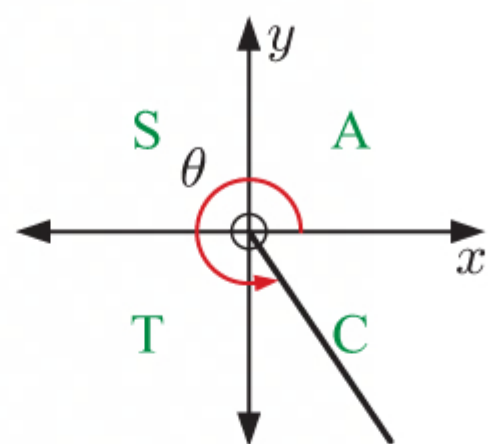
$$\therefore \cos^2 \theta + \frac{4}{9} = 1$$

$$\therefore \cos^2 \theta = \frac{5}{9}$$

$$\therefore \cos \theta = -\frac{\sqrt{5}}{3} \quad \{\cos \theta < 0\}$$

c $\sin \theta = -\frac{5}{6}$ and $\frac{3\pi}{2} < \theta < 2\pi$

θ is in quadrant 4, so $\cos \theta$ is positive.



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(-\frac{5}{6}\right)^2 = 1$$

$$\therefore \cos^2 \theta + \frac{25}{36} = 1$$

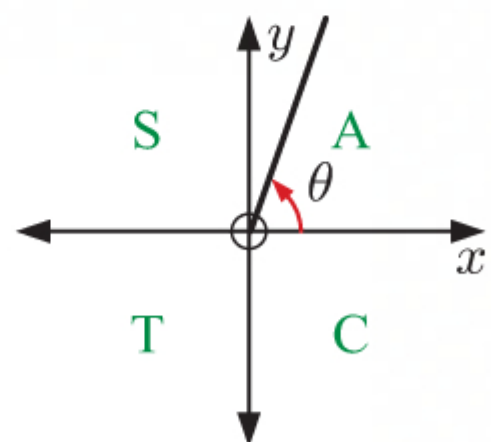
$$\therefore \cos^2 \theta = \frac{11}{36}$$

$$\therefore \cos \theta = \frac{\sqrt{11}}{6} \quad \{\cos \theta > 0\}$$

$$\begin{aligned} \text{So, } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{5}{6}}{\frac{\sqrt{11}}{6}} \\ &= -\frac{5}{\sqrt{11}} \end{aligned}$$

d $\cos \theta = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$

θ is in quadrant 1, so $\sin \theta$ is positive.



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(\frac{1}{3}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \frac{1}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{8}{9}$$

$$\therefore \sin \theta = \frac{\sqrt{8}}{3} \quad \{\sin \theta > 0\}$$

$$\begin{aligned} \text{So, } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\frac{\sqrt{8}}{3}}{\frac{1}{3}} \\ &= \sqrt{8} \end{aligned}$$

40 a $\tan \theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

θ is in quadrant 2, so $\sin \theta > 0$ and $\cos \theta < 0$.

Now $\tan \theta = -\frac{1}{3}$

$$\therefore \frac{\sin \theta}{\cos \theta} = -\frac{1}{3}$$

$$\therefore \cos \theta = -3 \sin \theta \quad \dots (*)$$

So, $\cos^2 \theta + \sin^2 \theta = 1$

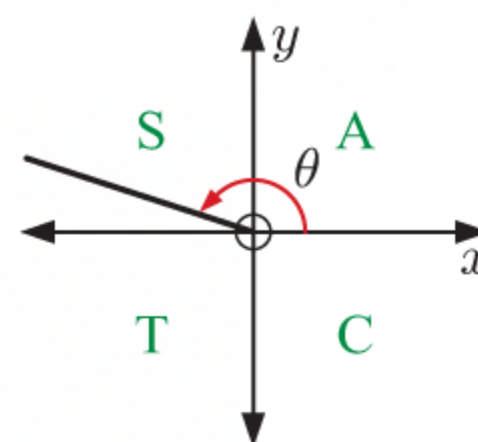
$$\therefore 9 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

$$\therefore 10 \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{1}{10}$$

$$\therefore \sin \theta = \frac{1}{\sqrt{10}} \quad \{\sin \theta > 0\}$$

$$\therefore \cos \theta = -\frac{3}{\sqrt{10}}$$



b $\tan \theta = \frac{1}{\sqrt{2}}$ and $\pi < \theta < \frac{3\pi}{2}$

θ is in quadrant 3, so $\sin \theta < 0$ and $\cos \theta < 0$.

Now $\tan \theta = \frac{1}{\sqrt{2}}$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{1}{\sqrt{2}}$$

$$\therefore \cos \theta = \sqrt{2} \sin \theta \quad \dots (*)$$

So, $\cos^2 \theta + \sin^2 \theta = 1$

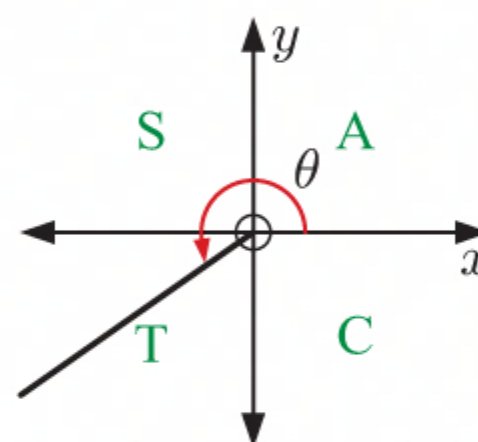
$$\therefore 2 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

$$\therefore 3 \sin^2 \theta = 1$$

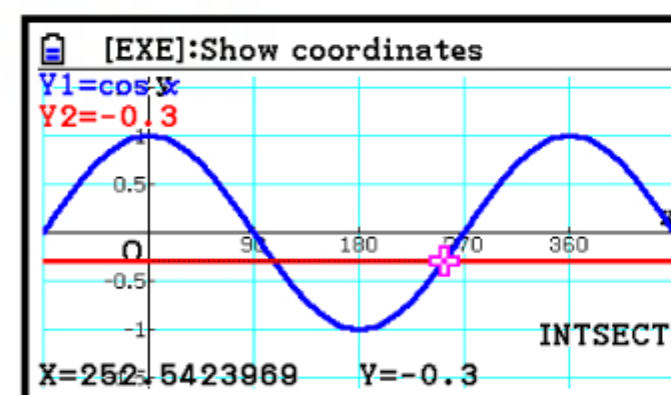
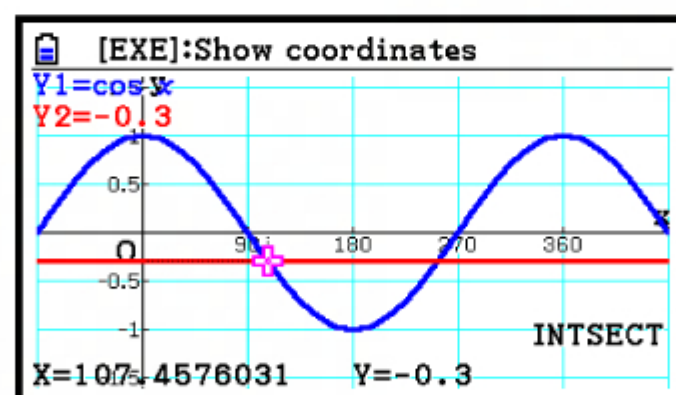
$$\therefore \sin^2 \theta = \frac{1}{3}$$

$$\therefore \sin \theta = -\frac{1}{\sqrt{3}} \quad \{\sin \theta < 0\}$$

$$\therefore \cos \theta = -\sqrt{\frac{2}{3}}$$

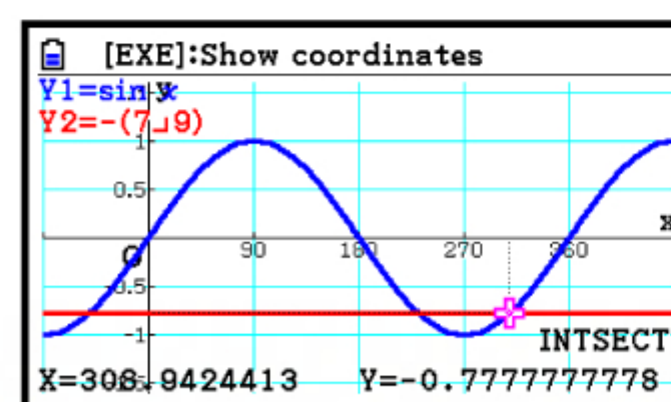
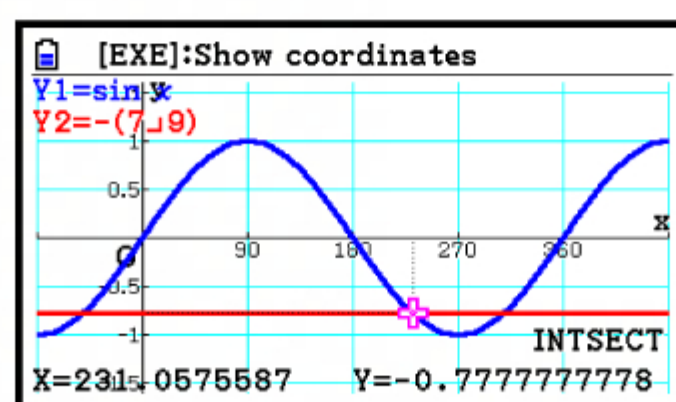


41 a We graph the functions $Y_1 = \cos X$ and $Y_2 = -0.3$ on the same set of axes.



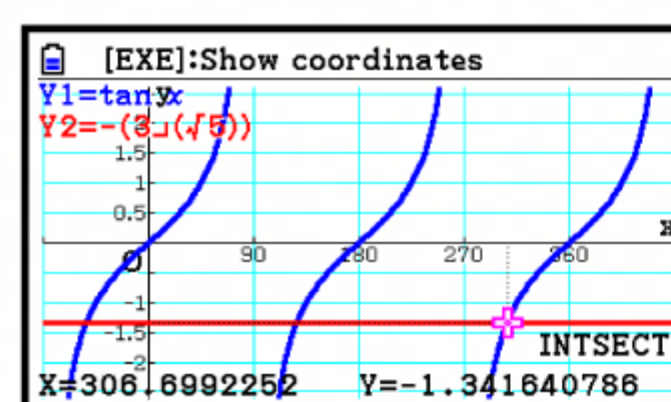
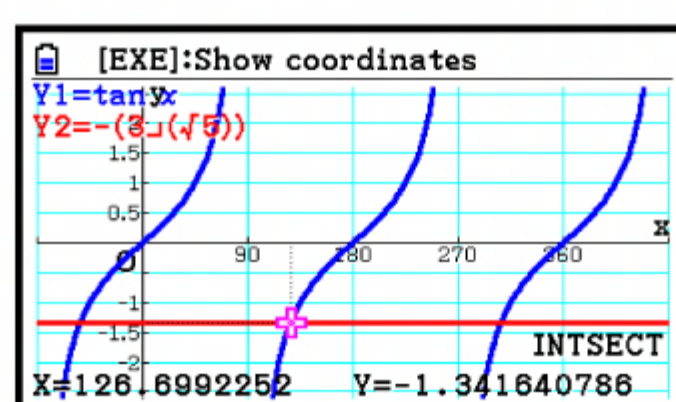
The solutions are $\theta \approx 107^\circ, 253^\circ$.

b We graph the functions $Y_1 = \sin X$ and $Y_2 = -\frac{7}{9}$ on the same set of axes.

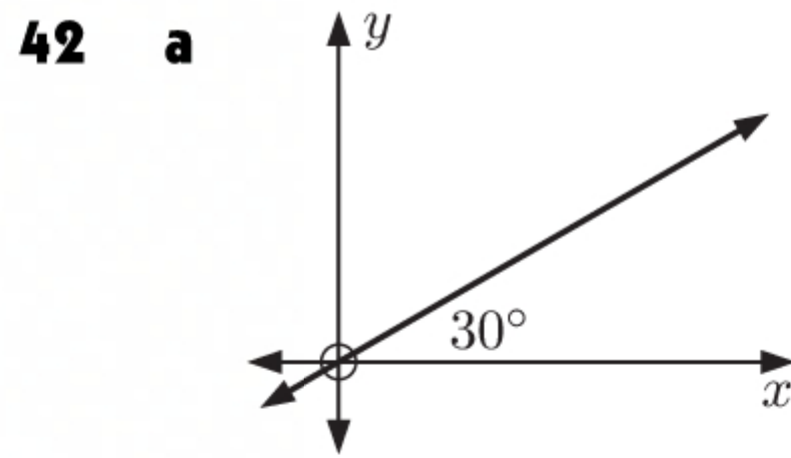


The solutions are $\theta \approx 231^\circ, 309^\circ$.

c We graph the functions $Y_1 = \tan X$ and $Y_2 = -\frac{3}{\sqrt{5}}$ on the same set of axes.

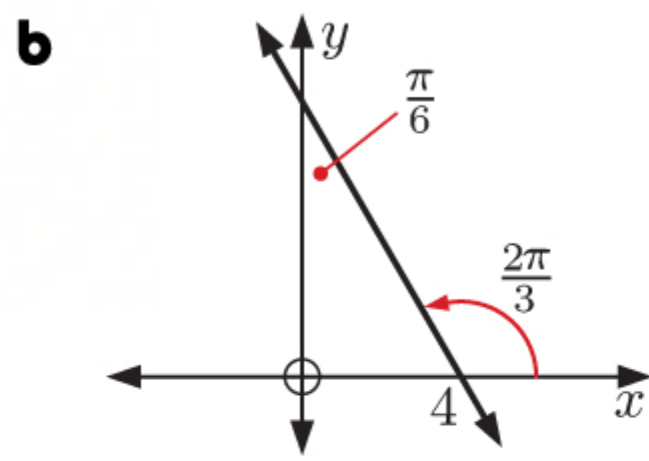


The solutions are $\theta \approx 127^\circ, 307^\circ$.



The line has gradient $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$ and y -intercept 0.

\therefore the line has equation $y = \frac{1}{\sqrt{3}}x$.



The line makes an angle of $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ with the positive x -axis.
{exterior angle of a triangle theorem}

\therefore the line has gradient $m = \tan \frac{2\pi}{3} = -\sqrt{3}$.

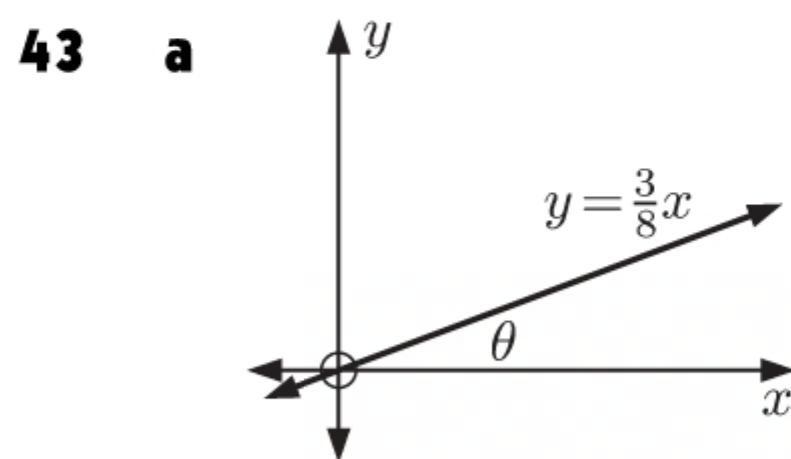
\therefore the line has equation $y = -\sqrt{3}x + c$.

When $x = 4$, $y = 0$, $\therefore 0 = -\sqrt{3}(4) + c$

$$\therefore 0 = -4\sqrt{3} + c$$

$$\therefore c = 4\sqrt{3}$$

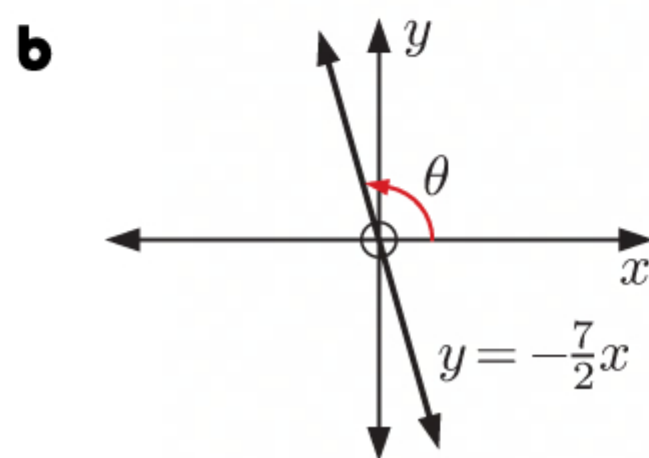
\therefore the equation of the line is $y = -\sqrt{3}x + 4\sqrt{3}$.



The line has gradient $\frac{3}{8}$, so $\tan \theta = \frac{3}{8}$.

Using technology, $\tan^{-1}(\frac{3}{8}) \approx 0.359$

$\therefore \theta \approx 0.359$



The line has gradient $-\frac{7}{2}$, so $\tan \theta = -\frac{7}{2}$.

Using technology, $\tan^{-1}(-\frac{7}{2}) \approx -1.29$

But $0 < \theta < \pi$, so $\theta \approx \pi - 1.29 \approx 1.85$

44 a For the function $f(x) = \sin 4x$:

- the amplitude is $|a| = 1$
- the principal axis is $y = 0$
- the period is $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$.

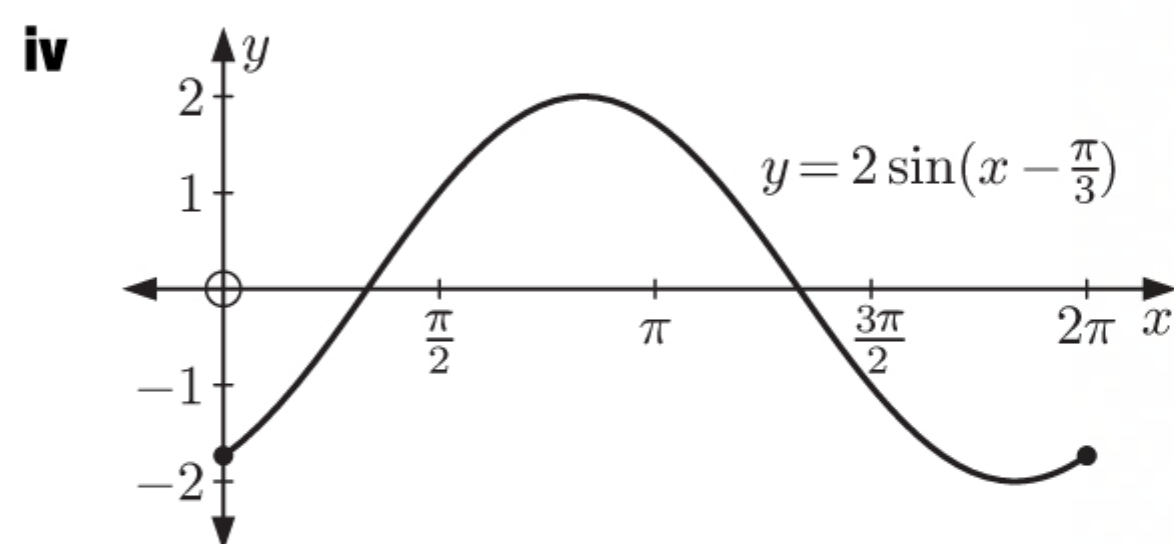
b For the function $f(x) = -2 \sin \frac{x}{2} - 1$:

- the amplitude is $|a| = |-2| = 2$
- the principal axis is $y = -1$
- the period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

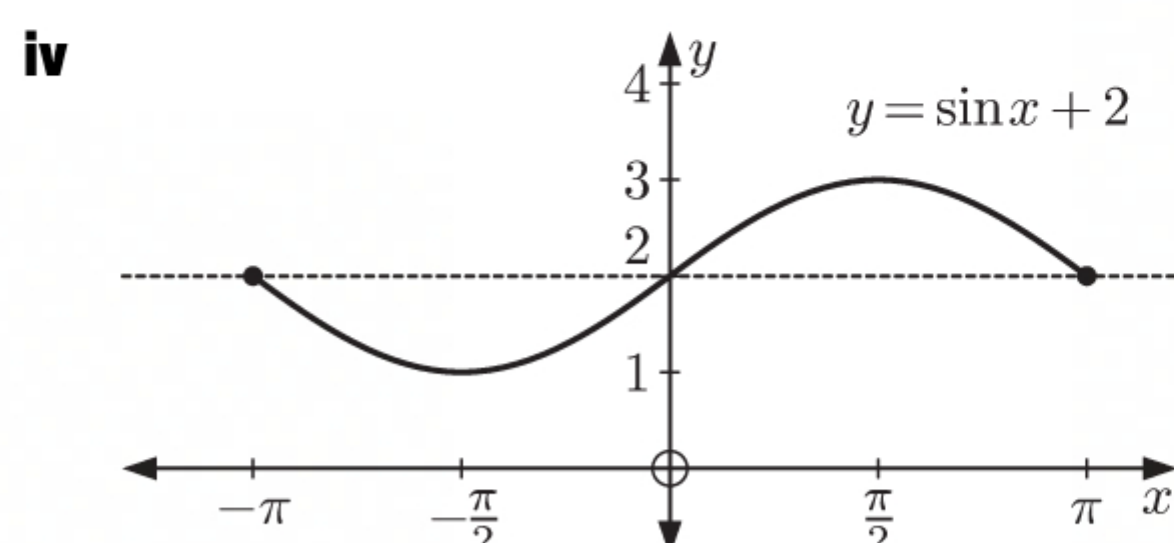
45 a i The amplitude is $|a| = 2$.

iii The period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

ii The principal axis is $y = 0$.



ii The principal axis is $y = 2$.



b i The amplitude is $|a| = 1$.

iii The period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

- c** **i** The amplitude is $|a| = 3$.
iii The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

- d** **i** The amplitude is $|a| = 1$.
iii The period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

- e** **i** The amplitude is $|a| = 1$.
iii The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

- f** **i** The amplitude is $|a| = |-6| = 6$.
iii The period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

46 a **i** $y = \tan \frac{x}{2}$ has period $\frac{\pi}{b} = \frac{\pi}{(\frac{1}{2})} = 2\pi$.

- ii** $y = \tan \frac{x}{2}$ is undefined when

$$\frac{x}{2} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

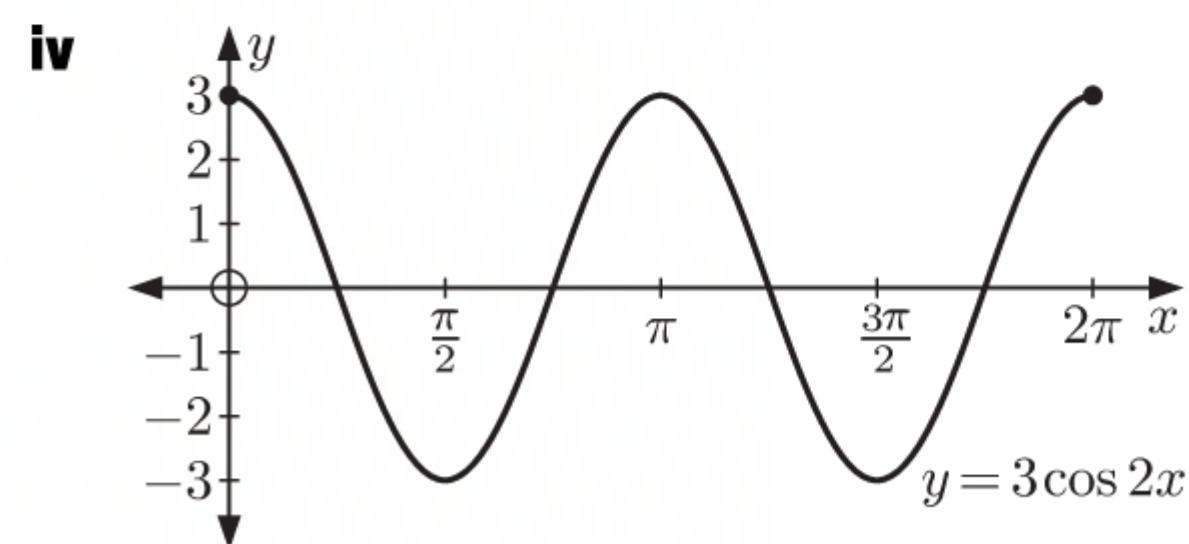
b **i** $y = 5 \tan 3x$ has period $\frac{\pi}{b} = \frac{\pi}{3}$.

- ii** $y = 5 \tan 3x$ is undefined when

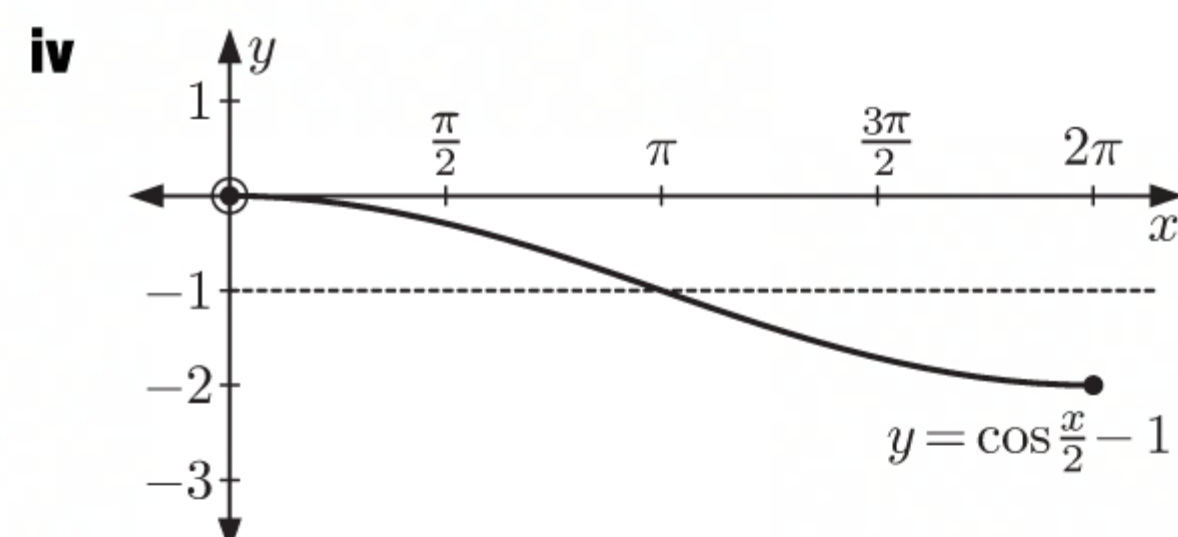
$$3x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3}, \quad k \in \mathbb{Z}$$

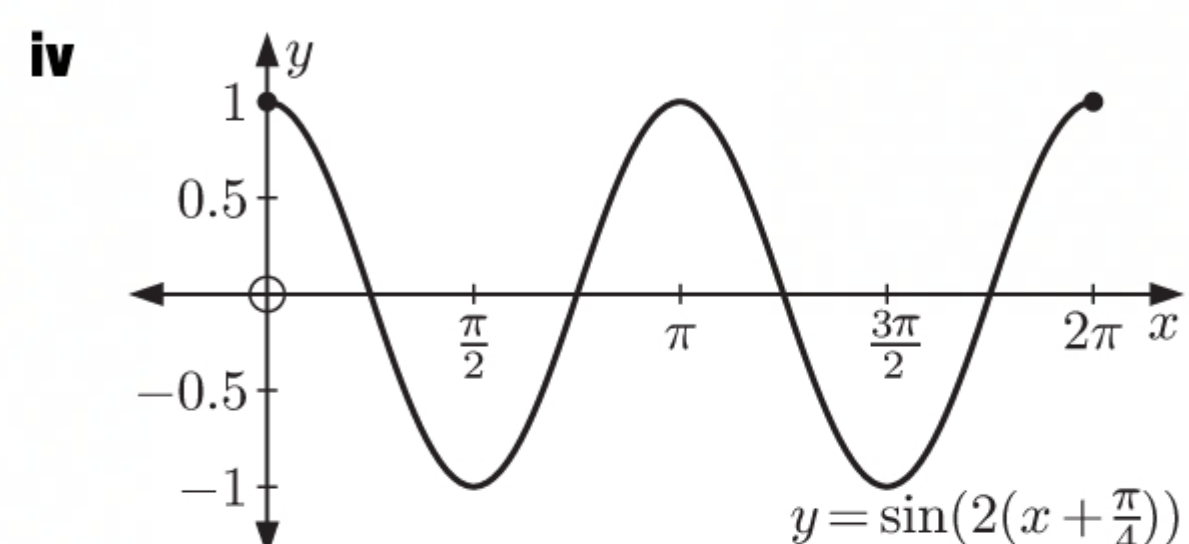
- ii** The principal axis is $y = 0$.



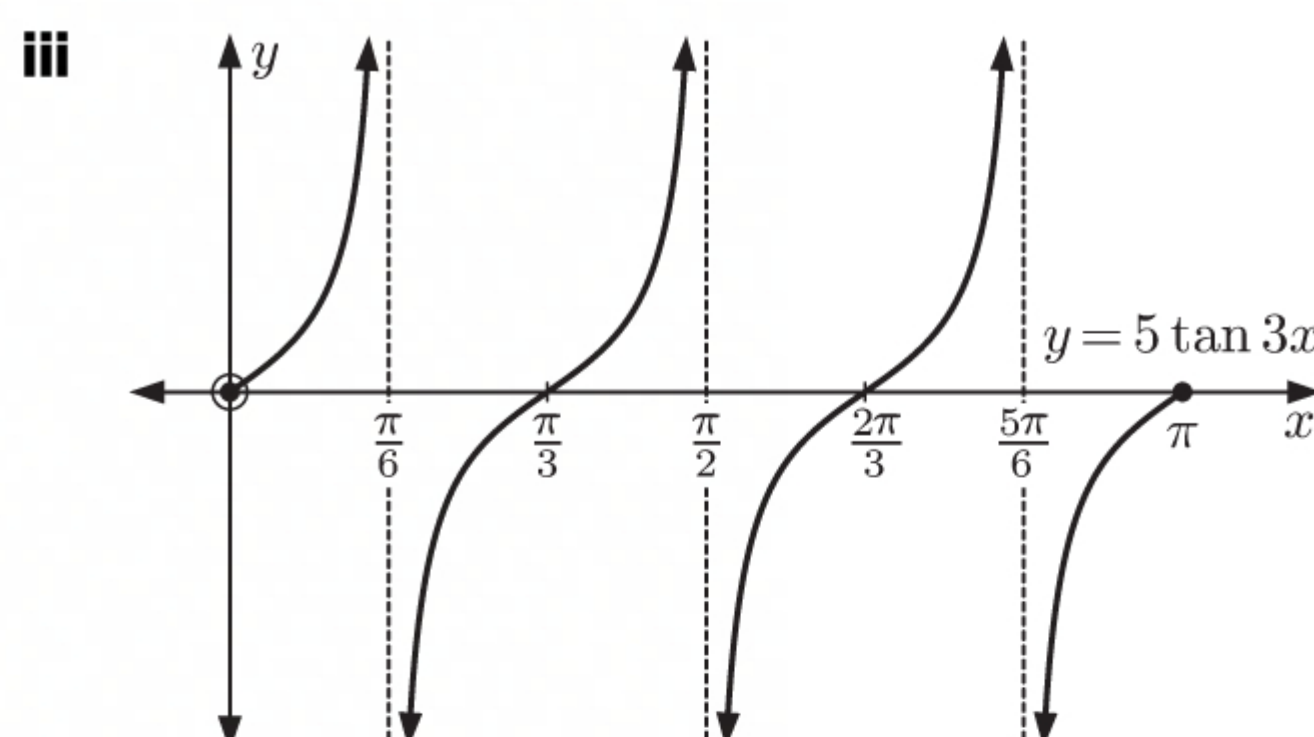
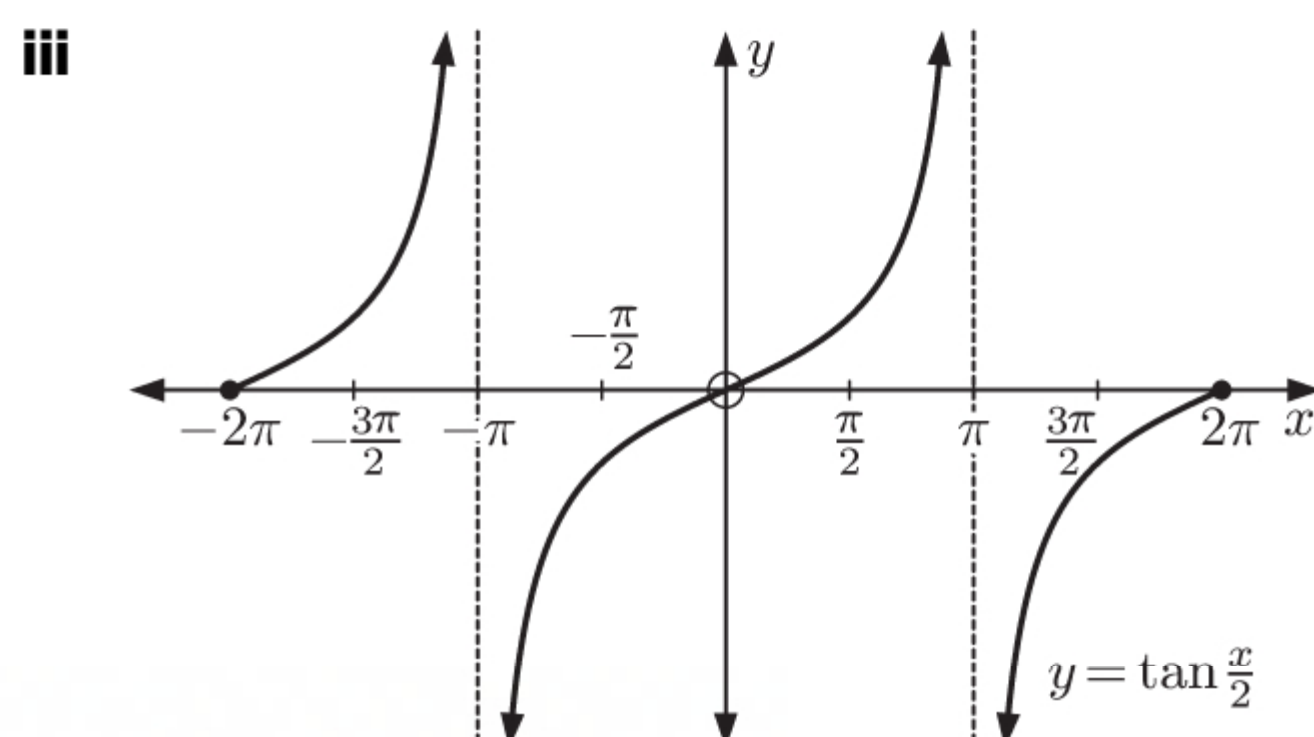
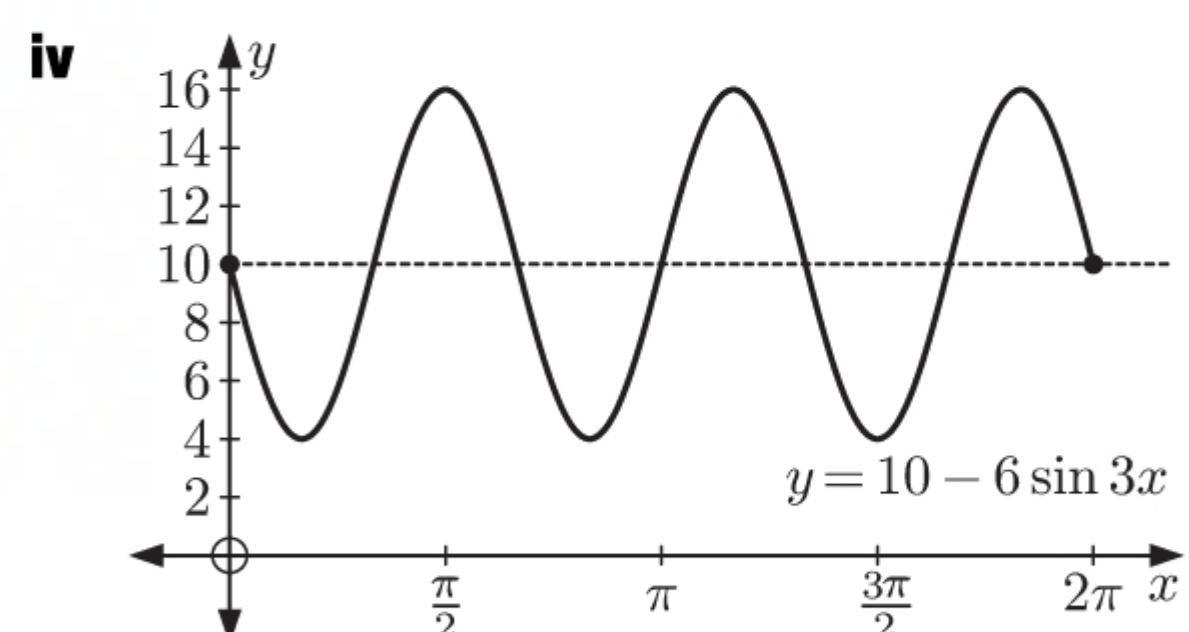
- ii** The principal axis is $y = -1$.



- ii** The principal axis is $y = 0$.



- ii** The principal axis is $y = 10$.

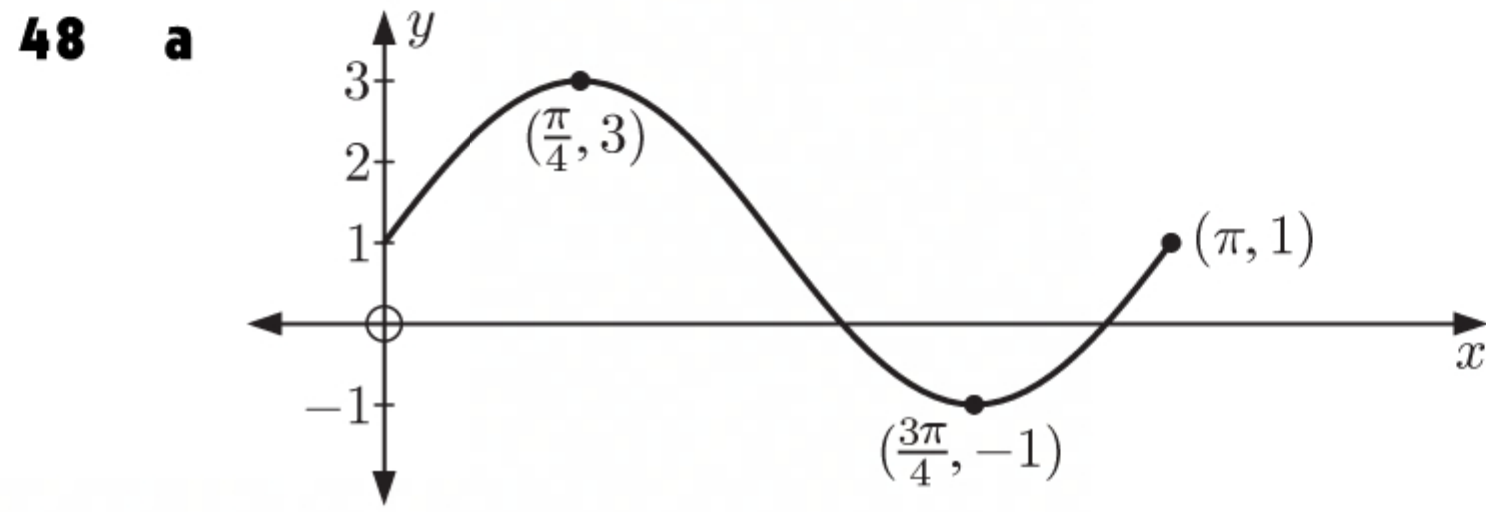


47 a $\sin x \xrightarrow[\text{scale factor 2}]{\text{vertical stretch}} 2 \sin x \xrightarrow[\text{scale factor 3}]{\text{horizontal stretch}} 2 \sin \frac{x}{3}$

A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 3 maps $y = \sin x$ onto $y = 2 \sin \frac{x}{3}$.

b $\sin x \xrightarrow[\text{translation } \begin{pmatrix} -\frac{\pi}{3} \\ -4 \end{pmatrix}]{\text{translation}} \sin\left(x + \frac{\pi}{3}\right) - 4$

A translation $\frac{\pi}{3}$ units left and 4 units downwards maps $y = \sin x$ onto $y = \sin\left(x + \frac{\pi}{3}\right) - 4$.



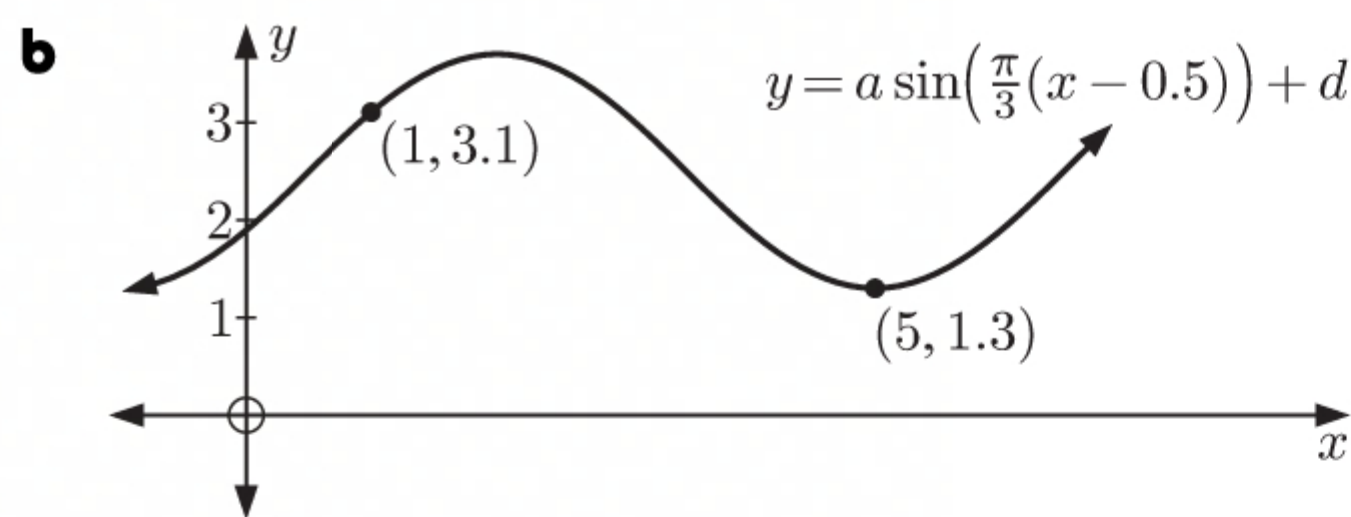
The amplitude is 2, so $a = 2$.

The period is π , so $\frac{2\pi}{b} = \pi$ and $\therefore b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 1$, so $d = 1$.

\therefore the equation of the function is $y = 2 \sin 2x + 1$.



When $x = 1$, $y = 3.1$

and when $x = 5$, $y = 1.3$

$$\therefore a \sin\left(\frac{\pi}{3}(1 - 0.5)\right) + d = 3.1$$

$$\therefore a \sin\left(\frac{\pi}{3}(5 - 0.5)\right) + d = 1.3$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 0.5\right) + d = 3.1$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 4.5\right) + d = 1.3$$

$$\therefore a \sin \frac{\pi}{6} + d = 3.1$$

$$\therefore a \sin \frac{3\pi}{2} + d = 1.3$$

$$\therefore \frac{1}{2}a + d = 3.1 \quad \dots (1)$$

$$\therefore -a + d = 1.3 \quad \dots (2)$$

$$-\frac{1}{2}a - d = -3.1 \quad \{(1) \times -1\}$$

$$-a + d = 1.3$$

Adding, $-\frac{3}{2}a = -1.8$

$$\therefore a = 1.2$$

Substituting $a = 1.2$ into (2) gives $-1.2 + d = 1.3$

$$\therefore d = 2.5$$

\therefore the equation of the sine function is $y = 1.2 \sin\left(\frac{\pi}{3}(x - 0.5)\right) + 2.5$.

49 a i The amplitude is 5, so $a = 5$.

ii The period is 4, so $\frac{2\pi}{b} = 4$ and $b = \frac{\pi}{2}$.

iii The principal axis is $y = \frac{7 + (-3)}{2}$ which is $y = 2$, so $d = 2$.

iv The function would start its first period at $x = 4 - 4 = 0$.

So, there is no horizontal translation and $c = 0$.

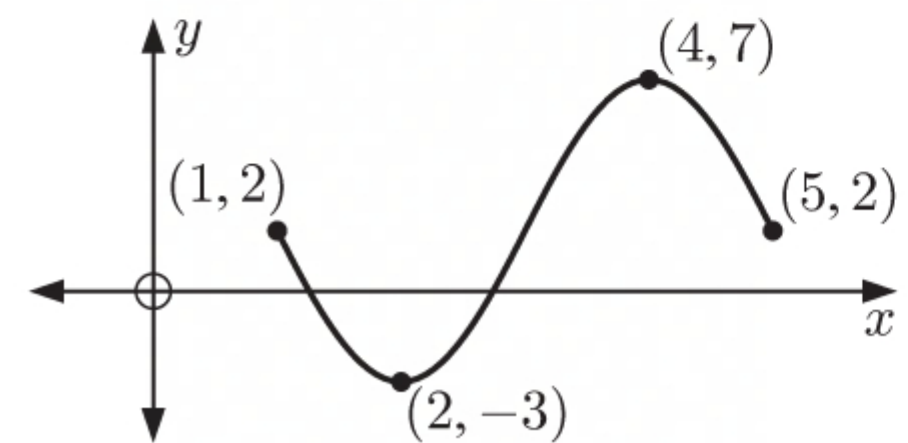
b From **a**, $y = 5 \cos \frac{\pi}{2}x + 2$

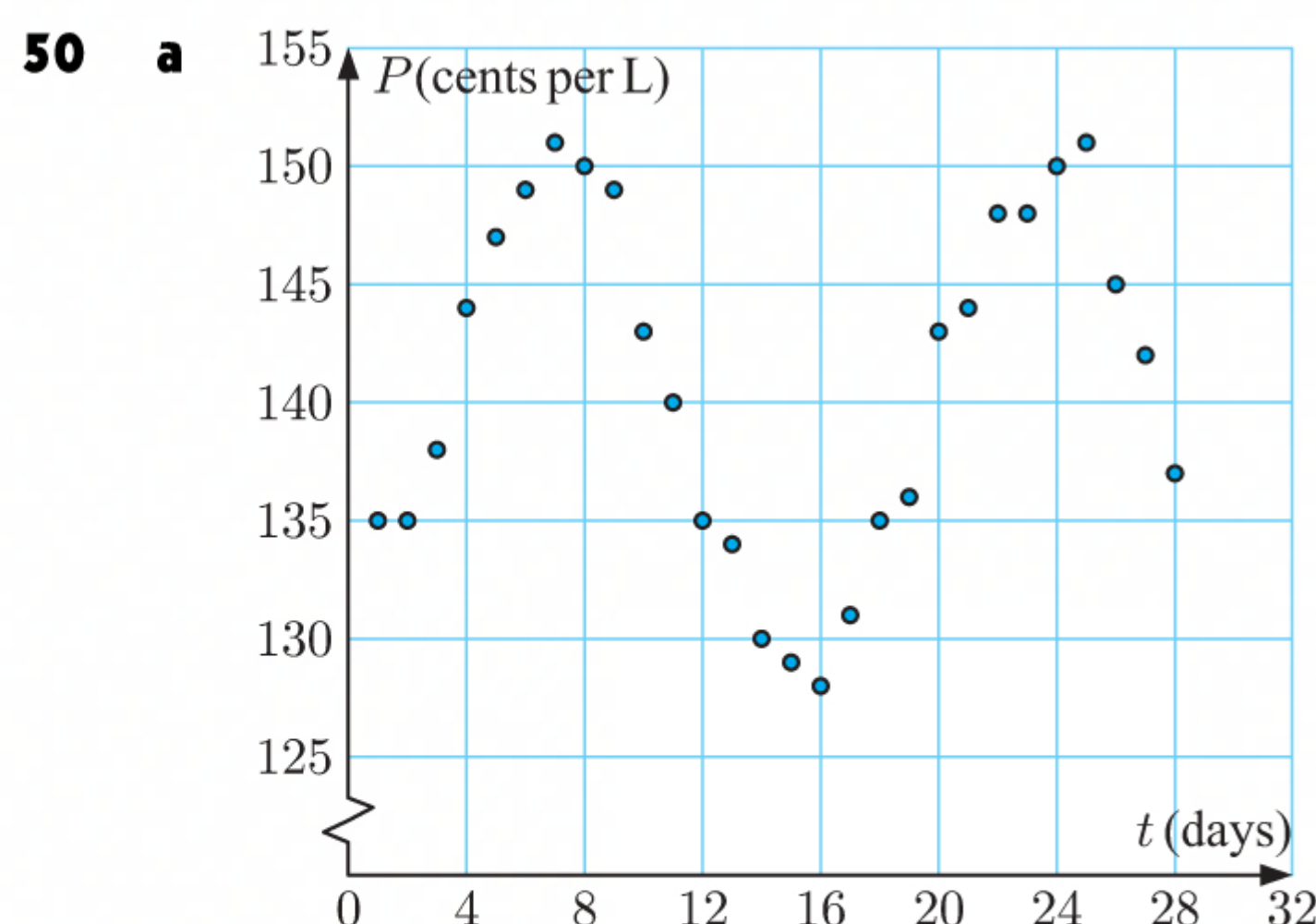
$$\therefore y = 5 \sin\left(\frac{\pi}{2} - \frac{\pi}{2}x\right) + 2 \quad \{\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)\}$$

$$\therefore y = 5 \sin\left(\frac{\pi}{2}(1 - x)\right) + 2$$

$$\therefore y = 5 \sin\left(-\frac{\pi}{2}(x - 1)\right) + 2$$

$$\therefore y = -5 \sin\left(\frac{\pi}{2}(x - 1)\right) + 2 \quad \{\sin(-\theta) = -\sin \theta\}$$





- b i** From the scatter diagram, the time between peaks (period) is about 16 days.

So, $\frac{2\pi}{b} \approx 16$ and $\therefore b \approx \frac{\pi}{8}$.

ii The amplitude $= \frac{\max - \min}{2} \approx \frac{151 - 128}{2} \approx 11.5$, so $a \approx 11.5$.

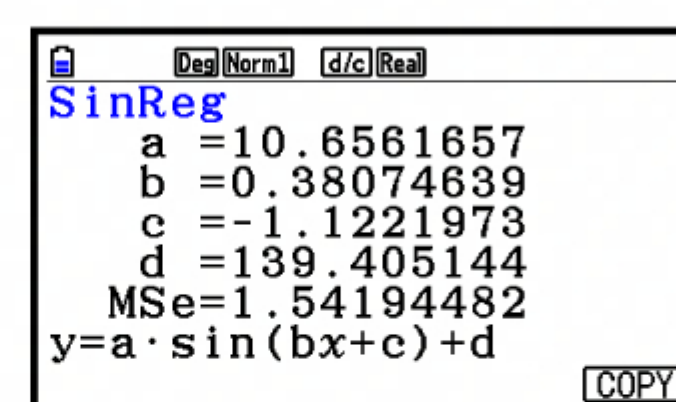
iii The principal axis is midway between the maximum and minimum, so $d \approx \frac{151 + 128}{2} \approx 139.5$.

iv The model is $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - c)\right) + 139.5$ for some constant c .

From the scatter diagram, the first period starts somewhere between $t = 3$ and $t = 4$. So we estimate $c \approx 3.5$.

c From **b**, our model is $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - 3.5)\right) + 139.5$
 $\approx 11.5 \sin(0.393t - 1.37) + 139.5$

Using technology, $P \approx 10.7 \sin(0.381t - 1.12) + 139.4$



51 $f(x) = 2 \tan(3(x - 1)) + 4$, $-1 \leq x \leq 1$

a The period of $y = f(x)$ is $\frac{\pi}{b} = \frac{\pi}{3}$.

b $y = f(x)$ is undefined when $3(x - 1) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$$\therefore x - 1 = \frac{\pi}{6} + \frac{k\pi}{3}, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3} + 1, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3} + 1, \quad k = -2, -1 \quad \{-1 \leq x \leq 1\}$$

$$\therefore x = \frac{\pi}{6} - \frac{2\pi}{3} + 1, \quad \frac{\pi}{6} - \frac{\pi}{3} + 1$$

$$\therefore x = 1 - \frac{\pi}{2}, \quad 1 - \frac{\pi}{6}$$

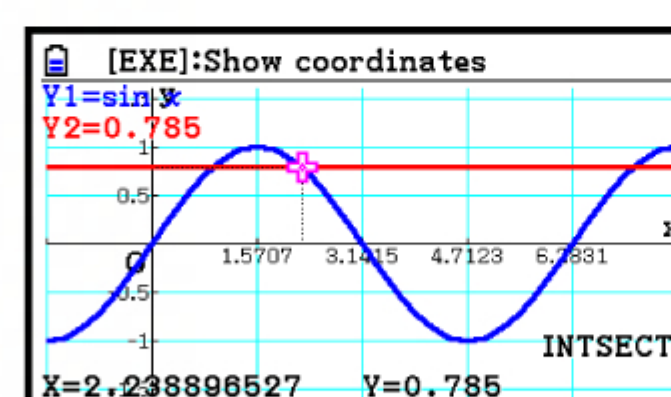
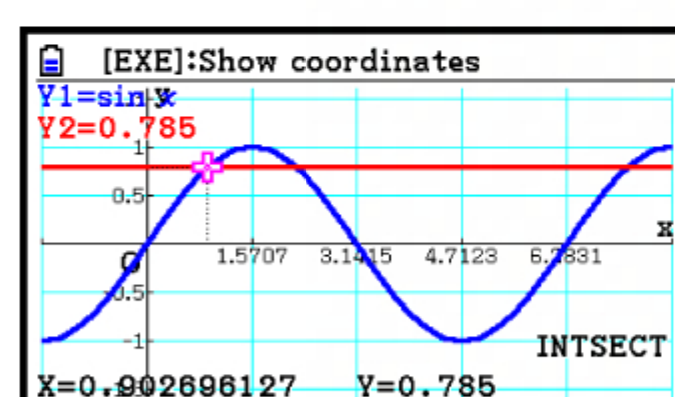
c $\tan x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 2} 2 \tan x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{3}} 2 \tan 3x \xrightarrow[\text{translation}]{\begin{pmatrix} 1 \\ 4 \end{pmatrix}} 2 \tan(3(x - 1)) + 4$

So, a vertical stretch with scale factor 2, then a horizontal stretch with scale factor $\frac{1}{3}$, then a horizontal translation 1 unit to the right, then a vertical translation 4 units upwards will map $y = \tan x$ onto $y = f(x)$.

d Using **b**, the domain of $y = f(x)$ is $\{x \mid -1 \leq x \leq 1, x \neq 1 - \frac{\pi}{2}, x \neq 1 - \frac{\pi}{6}\}$.

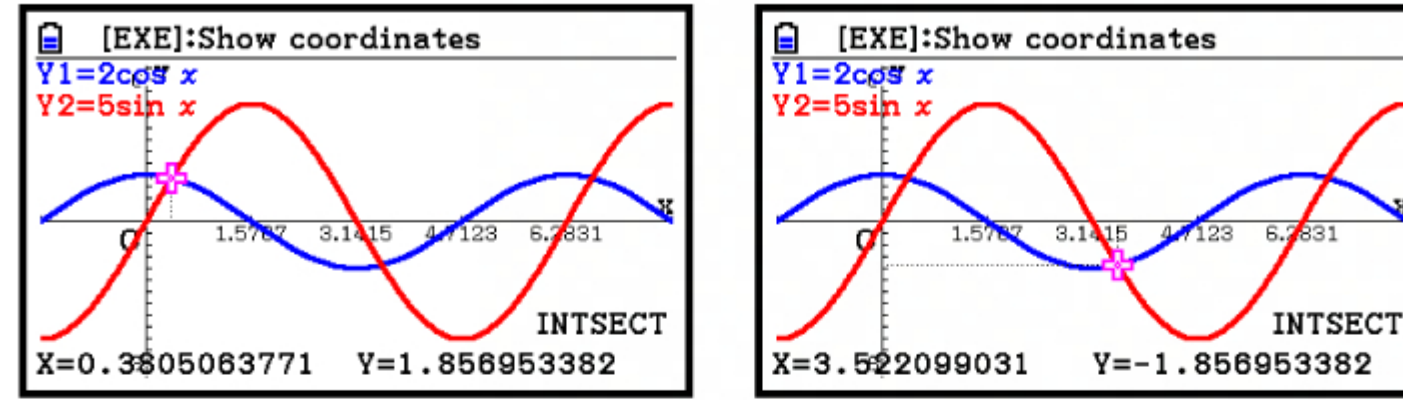
$2 \tan(3(x - 1)) + 4$ can take any real value for $1 - \frac{\pi}{2} < x < 1 - \frac{\pi}{6}$. So, the range of $y = f(x)$ is $\{y \mid y \in \mathbb{R}\}$.

52 a We graph the functions $Y_1 = \sin X$ and $Y_2 = 0.785$ on the same set of axes.



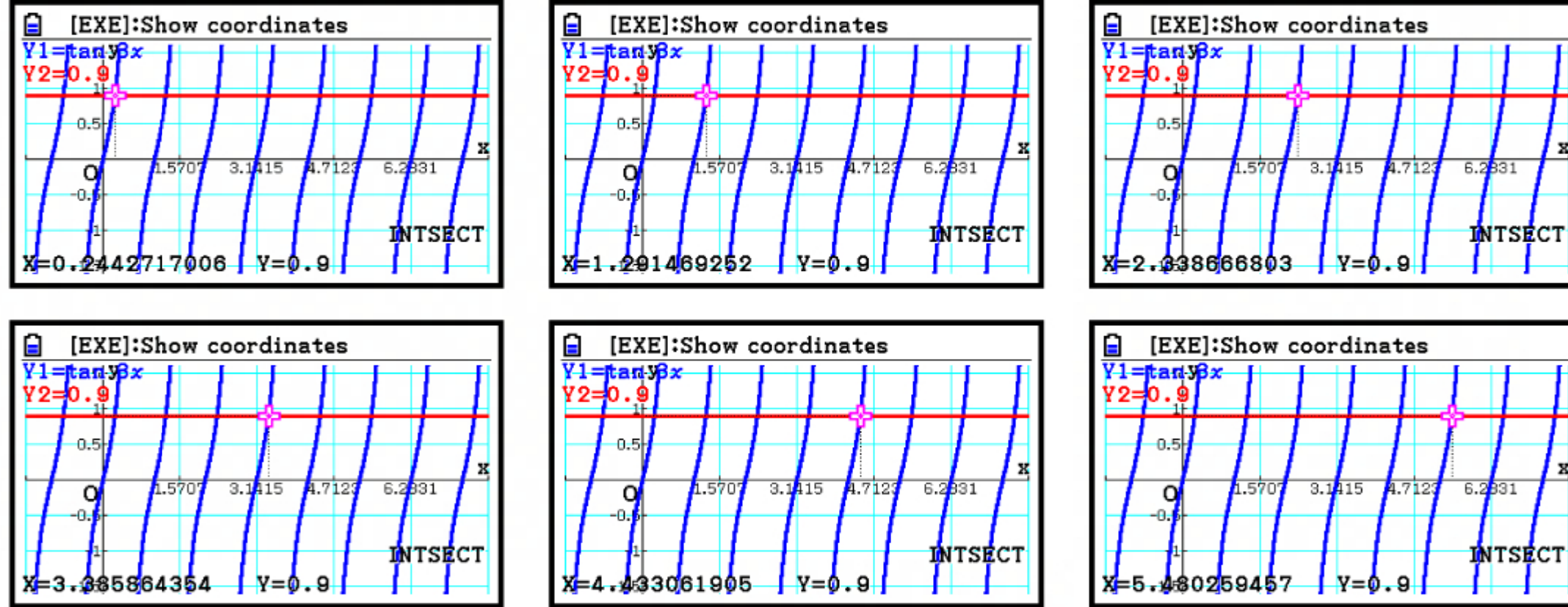
The solutions are $x \approx 0.903, 2.24$.

- b** We graph the functions $Y_1 = 2 \cos X$ and $Y_2 = 5 \sin X$ on the same set of axes.



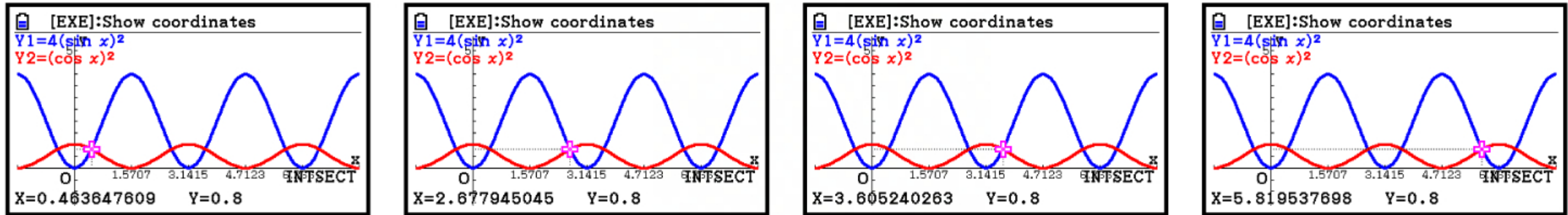
The solutions are $x \approx 0.381, 3.52$.

- c** We graph the functions $Y_1 = \tan 3X$ and $Y_2 = 0.9$ on the same set of axes.



The solutions are $x \approx 0.244, 1.29, 2.34, 3.39, 4.43, 5.48$.

- d** We graph the functions $Y_1 = 4 \sin^2 X$ and $Y_2 = \cos^2 X$ on the same set of axes.



The solutions are $x \approx 0.464, 2.68, 3.61, 5.82$.

53 a $\sqrt{3} \tan \frac{x}{2} = -1$

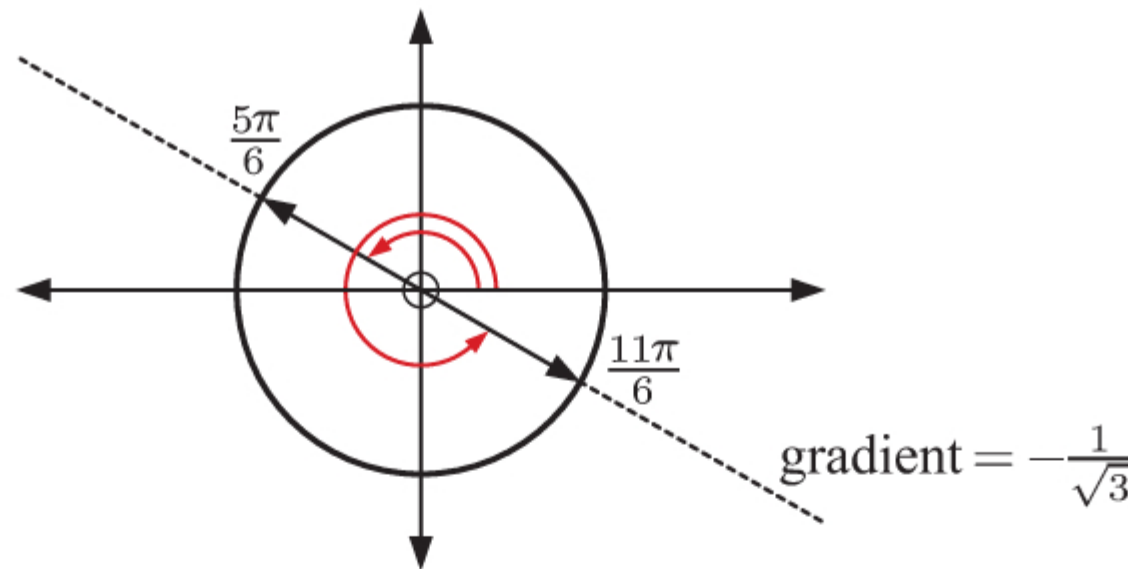
$$\therefore \tan \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

Since $-\pi \leq x \leq 3\pi$,

$$-\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{3\pi}{2}$$

So, $\frac{x}{2} = -\frac{\pi}{6}$ and $\frac{5\pi}{6}$

$$\therefore x = -\frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$



b $\sqrt{3} + 2 \sin 2x = 0$

$$\therefore 2 \sin 2x = -\sqrt{3}$$

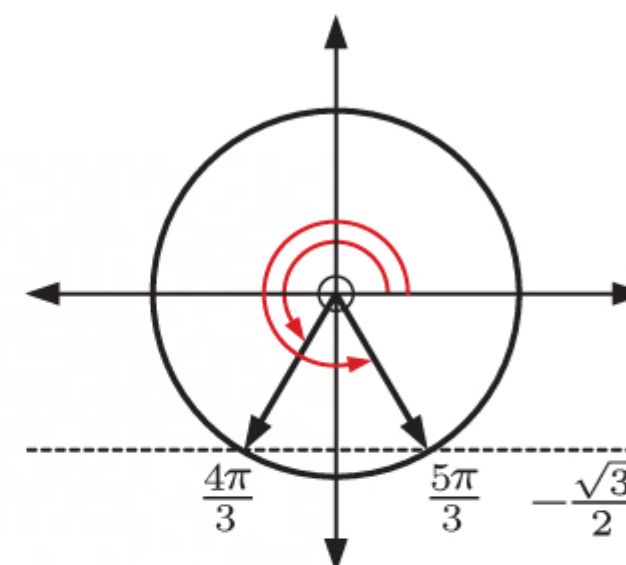
$$\therefore \sin 2x = -\frac{\sqrt{3}}{2}$$

Since $-\pi \leq x \leq 3\pi$,

$$-2\pi \leq 2x \leq 6\pi$$

So, $2x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$

$$\therefore x = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}, \frac{8\pi}{3}, \frac{17\pi}{6}$$



c $1 - \sqrt{2} \cos 3x = 0$

$$\therefore -\sqrt{2} \cos 3x = -1$$

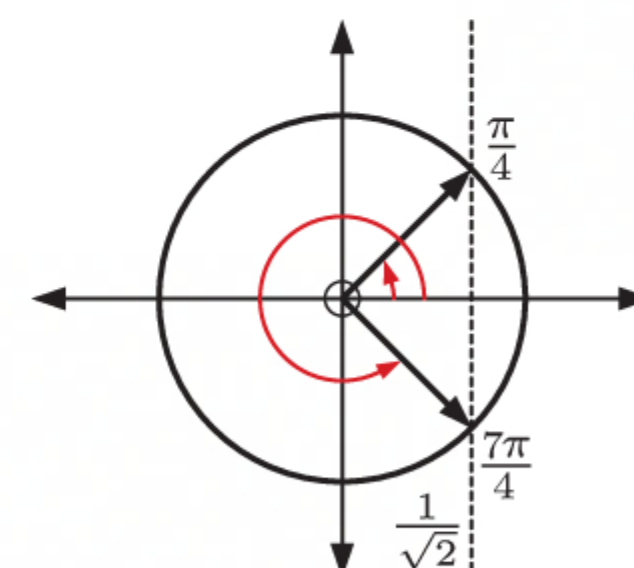
$$\therefore \cos 3x = \frac{1}{\sqrt{2}}$$

Since $-\pi \leq x \leq 3\pi$

$$-3\pi \leq 3x \leq 9\pi$$

So, $3x = -\frac{9\pi}{4}, -\frac{7\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}, \frac{25\pi}{4}, \frac{31\pi}{4}, \frac{33\pi}{4}$

$$\therefore x = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}, \frac{25\pi}{12}, \frac{31\pi}{12}, \frac{11\pi}{4}$$



$$\mathbf{d} \quad 10 \sin \frac{x}{3} = 5\sqrt{3}$$

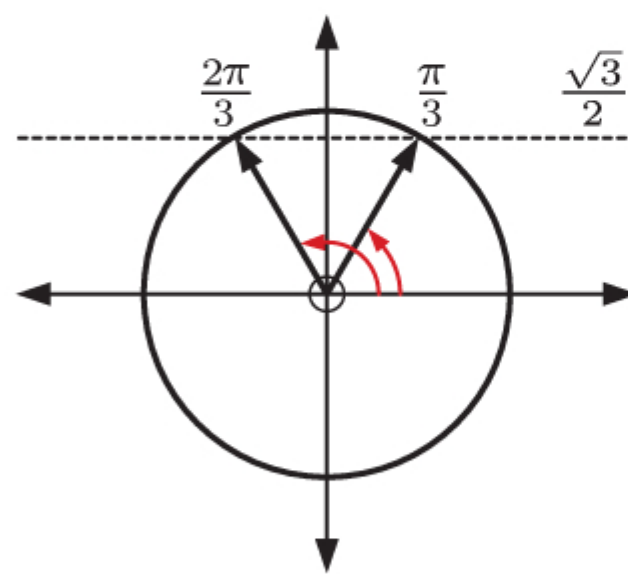
$$\therefore \sin \frac{x}{3} = \frac{\sqrt{3}}{2}$$

$$\text{Since } -\pi \leq x \leq 3\pi$$

$$\therefore -\frac{\pi}{3} \leq \frac{x}{3} \leq \pi$$

$$\text{So, } \frac{x}{3} = \frac{\pi}{3} \text{ and } \frac{2\pi}{3}$$

$$\therefore x = \pi \text{ and } 2\pi$$



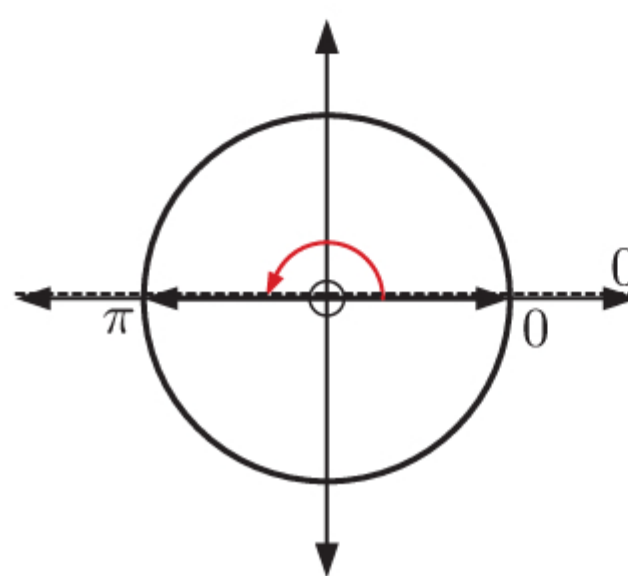
$$\mathbf{54} \quad \mathbf{a} \quad \sin\left(x - \frac{\pi}{3}\right) = 0$$

$$\text{Since } -2\pi \leq x \leq \pi$$

$$-\frac{7\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{2\pi}{3}$$

$$\text{So, } x - \frac{\pi}{3} = -2\pi, -\pi, 0$$

$$\therefore x = -\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}$$



$$\mathbf{b} \quad \cos\left(3x + \frac{\pi}{4}\right) = -\frac{1}{2}$$

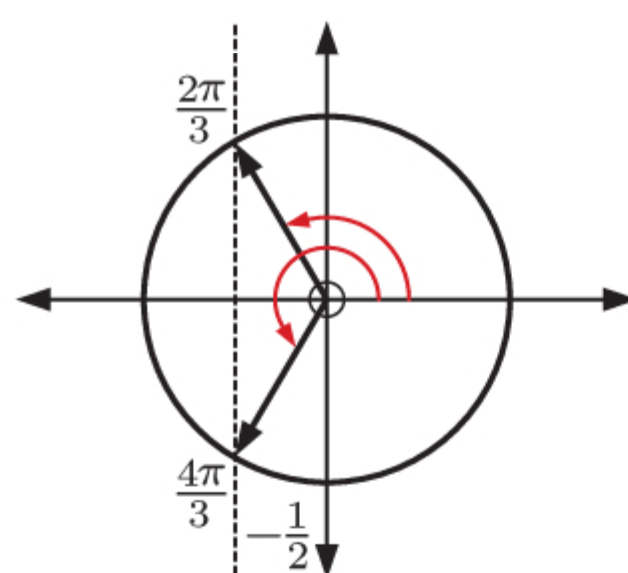
$$\text{Since } 0 \leq x \leq \pi$$

$$\therefore \frac{\pi}{4} \leq 3x + \frac{\pi}{4} \leq \frac{13\pi}{4}$$

$$\text{So, } 3x + \frac{\pi}{4} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

$$\therefore 3x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{29\pi}{12}$$

$$\therefore x = \frac{5\pi}{36}, \frac{13\pi}{36}, \frac{29\pi}{36}$$

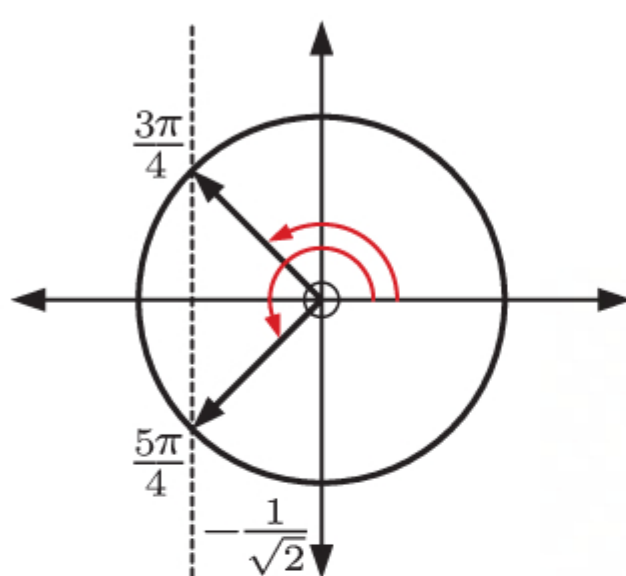


$$\mathbf{55} \quad \mathbf{a} \quad \sqrt{2} \cos x + 1 = 0$$

$$\therefore \sqrt{2} \cos x = -1$$

$$\therefore \cos x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

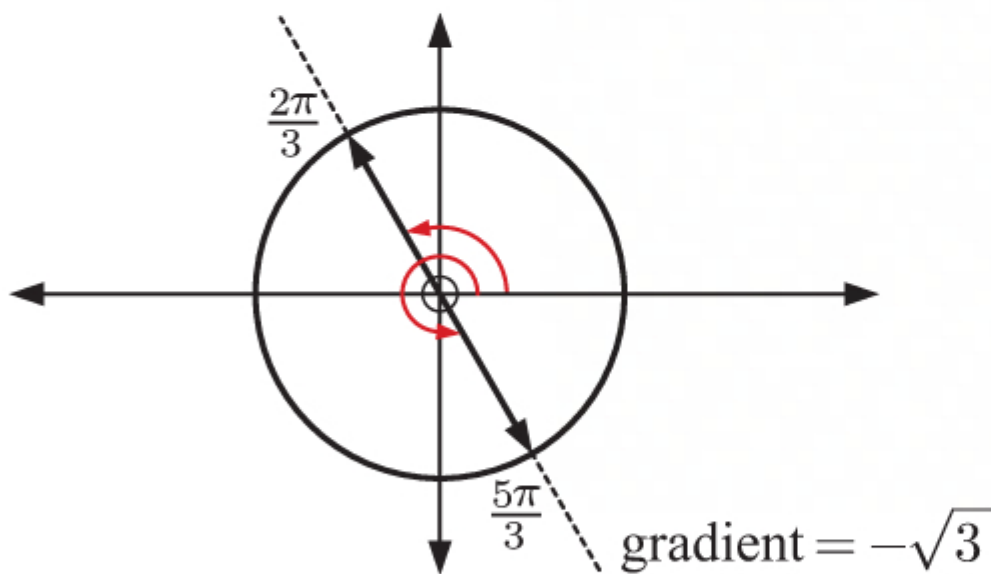


$$\mathbf{b} \quad \sin x = -\sqrt{3} \cos x$$

$$\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$$

$$\therefore \tan x = -\sqrt{3}$$

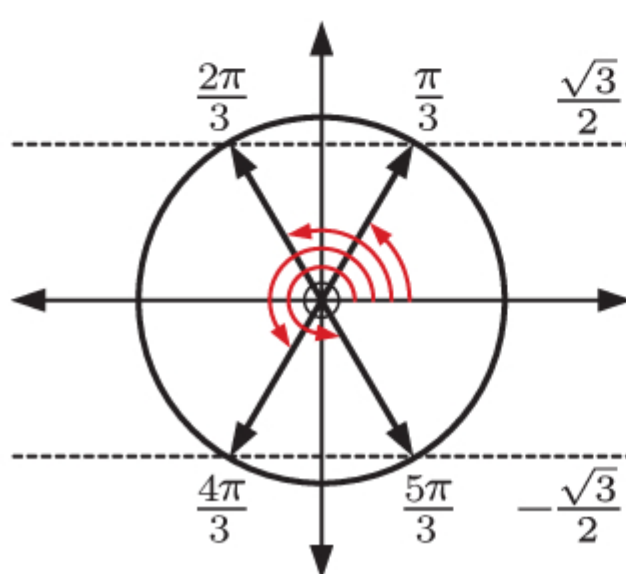
$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$



$$\mathbf{c} \quad \sin^2 x = \frac{3}{4}$$

$$\therefore \sin x = \pm \frac{\sqrt{3}}{2}$$

$$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$



$$\mathbf{d} \quad \tan^3 2x - 3 \tan 2x = 0$$

$$\therefore \tan 2x (\tan^2 2x - 3) = 0$$

$$\therefore \tan 2x = 0 \quad \text{or} \quad \tan^2 2x = 3$$

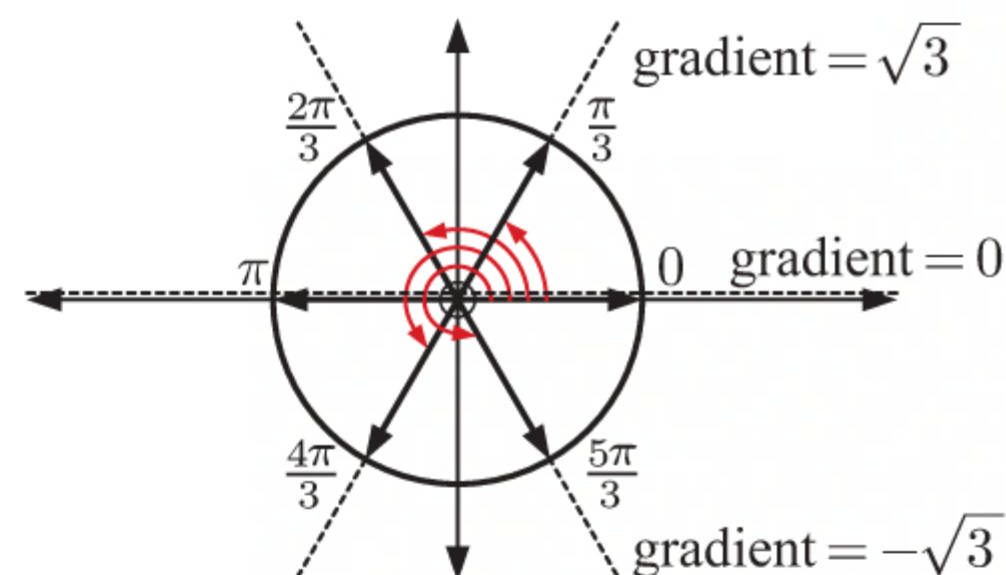
$$\therefore \tan 2x = \pm \sqrt{3}$$

$$\text{Since } 0 \leq x \leq 2\pi,$$

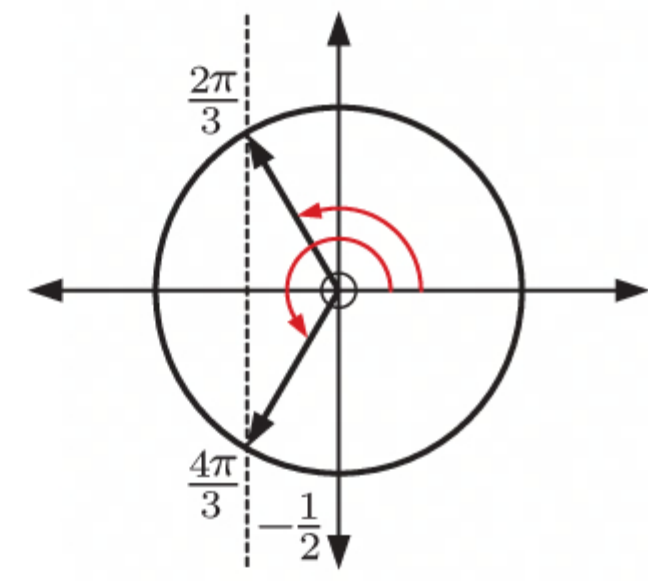
$$0 \leq 2x \leq 4\pi$$

$$\text{So, } 2x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, \frac{8\pi}{3}, 3\pi, \frac{10\pi}{3}, \frac{11\pi}{3}, 4\pi$$

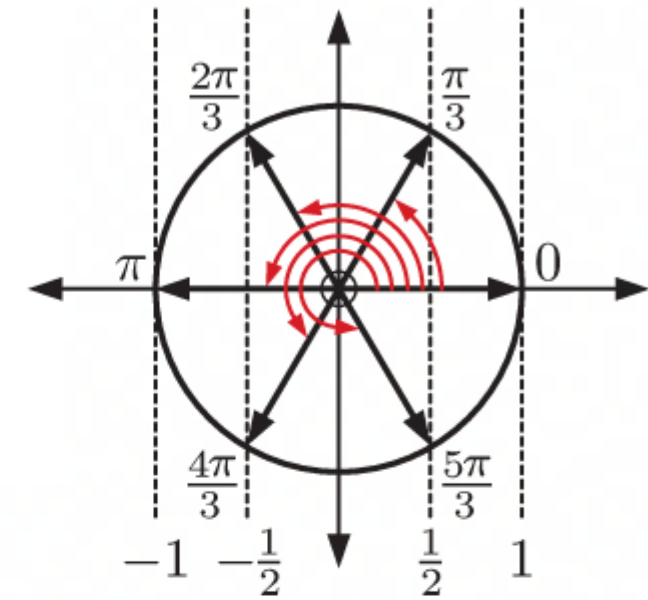
$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$$



$$\begin{aligned}
 \text{e} \quad & 4 \cos^2 x - 3 = 4 \cos x \\
 & \therefore 4 \cos^2 x - 4 \cos x - 3 = 0 \\
 & \therefore 4 \cos^2 x + 2 \cos x - 6 \cos x - 3 = 0 \\
 & \therefore 2 \cos x(2 \cos x + 1) - 3(2 \cos x + 1) = 0 \\
 & \therefore (2 \cos x + 1)(2 \cos x - 3) = 0 \\
 & \therefore \cos x = -\frac{1}{2} \quad \{-1 \leq \cos x \leq 1 \text{ for all } x\} \\
 & \therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}
 \end{aligned}$$



$$\begin{aligned}
 \text{f} \quad & 4 \cos^4 x + 1 = 5 \cos^2 x \\
 & \therefore 4 \cos^4 x - 5 \cos^2 x + 1 = 0 \\
 & \therefore 4 \cos^4 x - 4 \cos^2 x - \cos^2 x + 1 = 0 \\
 & \therefore 4 \cos^2 x(\cos^2 x - 1) - (\cos^2 x - 1) = 0 \\
 & \therefore (\cos^2 x - 1)(4 \cos^2 x - 1) = 0 \\
 & \therefore \cos^2 x = 1 \quad \text{or} \quad \cos^2 x = \frac{1}{4} \\
 & \therefore \cos x = \pm 1 \quad \text{or} \quad \cos x = \pm \frac{1}{2} \\
 & \therefore x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi
 \end{aligned}$$



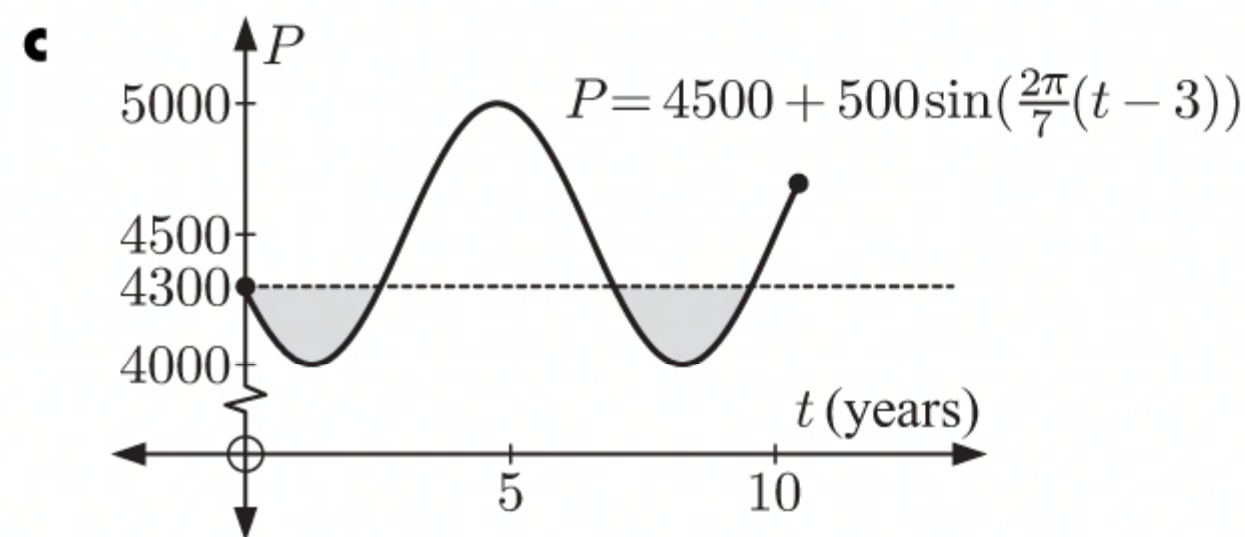
$$56 \quad P(t) = 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right), \quad 0 \leq t \leq 10$$

$$\begin{aligned}
 \text{a i} \quad & P(0) = 4500 + 500 \sin\left(\frac{2\pi}{7}(0-3)\right) \\
 & = 4500 + 500 \sin\left(-\frac{6\pi}{7}\right) \\
 & \approx 4283.06 \\
 & \approx 4280 \text{ butterflies}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & P(3) = 4500 + 500 \sin\left(\frac{2\pi}{7}(3-3)\right) \\
 & = 4500 + 500 \sin 0 \\
 & = 4500 \text{ butterflies}
 \end{aligned}$$

$$\begin{aligned}
 \text{b i} \quad & \text{We need to solve } P(t) = 4200, \text{ so} \\
 & 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4200 \\
 & \text{Using technology, } t \approx 0.217, 2.28, 7.22, 9.28. \\
 & \text{So, the population is 4200 after about 0.217 years,} \\
 & 2.28 \text{ years, 7.22 years, and 9.28 years.}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & \text{We need to solve } P(t) = 4900, \text{ so} \\
 & 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4900 \\
 & \text{Using technology, } t \approx 4.03, 5.47. \\
 & \text{So, the population is 4900 after about 4.03 years and} \\
 & 5.47 \text{ years.}
 \end{aligned}$$



$$\begin{aligned}
 & \text{We need to solve } P(t) = 4300, \text{ so} \\
 & 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4300 \\
 & \text{Using technology, } t \approx 2.54, 6.96, 9.54. \\
 & \text{So, the population drops below 4300 between 0 years and} \\
 & \approx 2.54 \text{ years, and between } \approx 6.96 \text{ years and } \approx 9.54 \text{ years.}
 \end{aligned}$$

$$57 \quad \text{a} \quad 2 \sin^2 \theta + 3 \sin^2 \theta = 5 \sin^2 \theta$$

$$\begin{aligned}
 \text{b} \quad & \cos x \tan x - 2 \sin x \\
 & = \cos x \times \frac{\sin x}{\cos x} - 2 \sin x \\
 & = \sin x - 2 \sin x \\
 & = -\sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{-\cos\left(\frac{\pi}{2} - \theta\right) \sin(-\theta)}{\cos(-\theta) \sin(\pi - \theta)} \\
 & = \frac{-\sin \theta (-\sin \theta)}{\cos \theta \sin \theta} \\
 & = \frac{\sin \theta}{\cos \theta} \\
 & = \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 58 \quad \text{a} \quad & -3 \sin^2 \theta - 3 \cos^2 \theta \\
 & = -3(\sin^2 \theta + \cos^2 \theta) \\
 & = -3(1) \\
 & = -3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sin \theta \cos^2 \theta + \sin^3 \theta \\
 & = \sin \theta (\cos^2 \theta + \sin^2 \theta) \\
 & = \sin \theta (1) \\
 & = \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{\sin^2 \theta - 1}{\cos \theta} \\
 & = \frac{-\cos^2 \theta}{\cos \theta} \\
 & = -\cos \theta
 \end{aligned}$$

$$\begin{aligned}
 59 \quad \text{a} \quad & (\sin^2 x + 3)^2 \\
 & = (\sin^2 x)^2 + 6 \sin^2 x + 9 \\
 & = \sin^4 x + 6 \sin^2 x + 9
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (\tan \alpha + 1)^2 \\
 & = \tan^2 \alpha + 2 \tan \alpha + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (\sin x - 1)(\sin x + 1) \\
 & = \sin^2 x - 1 \\
 & = -\cos^2 x
 \end{aligned}$$

60 a $2 \sin^2 x + 3 \cos x = 3, \quad 0 \leq x \leq 2\pi$

$$\therefore 2(1 - \cos^2 x) + 3 \cos x = 3$$

$$\therefore 2 - 2 \cos^2 x + 3 \cos x = 3$$

$$\therefore 2 \cos^2 x - 3 \cos x + 1 = 0$$

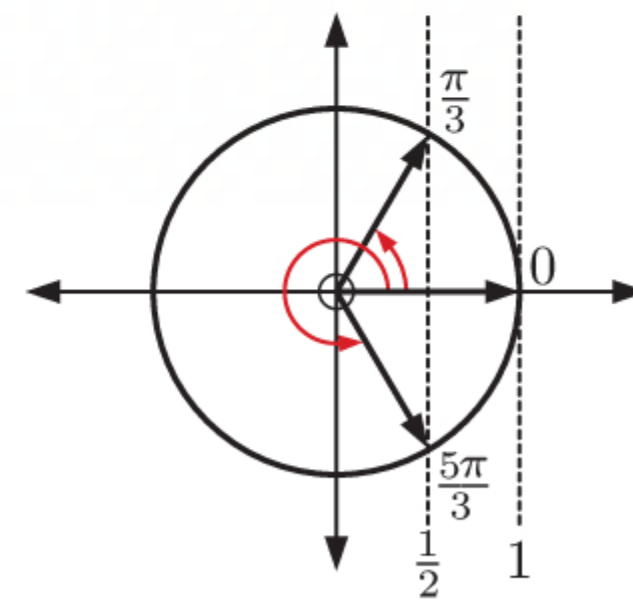
$$\therefore 2 \cos^2 x - 2 \cos x - \cos x + 1 = 0$$

$$\therefore 2 \cos x(\cos x - 1) - (\cos x - 1) = 0$$

$$\therefore (\cos x - 1)(2 \cos x - 1) = 0$$

$$\therefore \cos x = 1 \text{ or } \cos x = \frac{1}{2}$$

$$\therefore x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$$



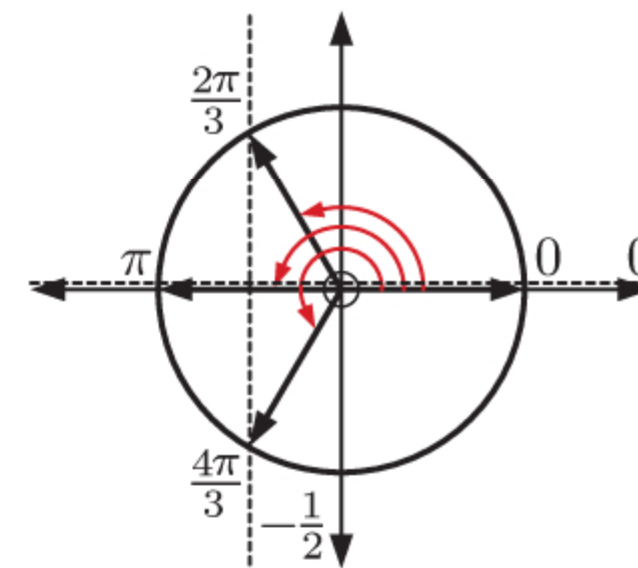
b $\sin 2x + \sin x = 0$

$$\therefore 2 \sin x \cos x + \sin x = 0 \quad \{\text{double angle formula}\}$$

$$\therefore \sin x(2 \cos x + 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = -\frac{1}{2}$$

$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$$



61 a $1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$

b $2 \cos^2 \alpha - 7 \cos \alpha \sin \alpha - 4 \sin^2 \alpha = 2 \cos^2 \alpha + \cos \alpha \sin \alpha - 8 \cos \alpha \sin \alpha - 4 \sin^2 \alpha$
 $= \cos \alpha(2 \cos \alpha + \sin \alpha) - 4 \sin \alpha(2 \cos \alpha + \sin \alpha)$
 $= (2 \cos \alpha + \sin \alpha)(\cos \alpha - 4 \sin \alpha)$

c $2 \cos^2 \theta - 3 \sin \theta = 2(1 - \sin^2 \theta) - 3 \sin \theta$
 $= 2 - 2 \sin^2 \theta - 3 \sin \theta$
 $= -2 \sin^2 \theta - 4 \sin \theta + \sin \theta + 2$
 $= -2 \sin \theta(\sin \theta + 2) + (\sin \theta + 2)$
 $= (\sin \theta + 2)(1 - 2 \sin \theta)$

62 $\sin^2 x - \cos^2 x = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \cos^2 x - \sin^2 x = 0$$

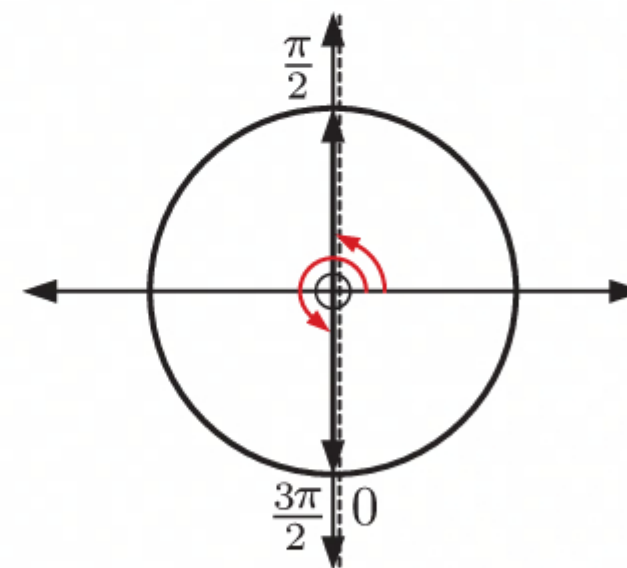
$$\therefore \cos 2x = 0$$

Since $0 \leq x \leq 2\pi,$

$$0 \leq 2x \leq 4\pi$$

So, $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



63 α is obtuse and $\sin \alpha = \frac{2}{3} \therefore \cos \alpha$ is negative.

a $\sin^2 \alpha + \cos^2 \alpha = 1$

$$\therefore \left(\frac{2}{3}\right)^2 + \cos^2 \alpha = 1$$

$$\therefore \frac{4}{9} + \cos^2 \alpha = 1$$

$$\therefore \cos^2 \alpha = \frac{5}{9}$$

$$\therefore \cos \alpha = -\frac{\sqrt{5}}{3} \quad \{\cos \alpha < 0\}$$

b $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
 $= \frac{5}{9} - \frac{4}{9} \quad \{\text{using a}\}$
 $= \frac{1}{9}$

64 α is acute and $\cos 2\alpha = \frac{5}{13} \therefore \cos \alpha$ and $\sin \alpha$ are positive.

a $\cos 2\alpha = 1 - 2 \sin^2 \alpha$

$$\therefore \frac{5}{13} = 1 - 2 \sin^2 \alpha$$

$$\therefore -2 \sin^2 \alpha = -\frac{8}{13}$$

$$\therefore \sin^2 \alpha = \frac{4}{13}$$

$$\therefore \sin \alpha = \frac{2}{\sqrt{13}} \quad \{\sin \alpha > 0\}$$

b $\sin^2 \alpha + \cos^2 \alpha = 1$

$$\therefore \frac{4}{13} + \cos^2 \alpha = 1 \quad \{\text{using a}\}$$

$$\therefore \cos^2 \alpha = \frac{9}{13}$$

$$\therefore \cos \alpha = \frac{3}{\sqrt{13}} \quad \{\cos \alpha > 0\}$$

$$\begin{aligned}
 \text{c } \tan \alpha &= \frac{\sin \alpha}{\cos \alpha} \\
 &= \frac{\frac{2}{\sqrt{13}}}{\frac{3}{\sqrt{13}}} \quad \{\text{using a and b}\} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$65 \quad \cos 2x = \frac{5}{8}$$

$$\text{Now } \cos 2x = 1 - 2\sin^2 x$$

$$\therefore \frac{5}{8} = 1 - 2\sin^2 x$$

$$\therefore -2\sin^2 x = -\frac{3}{8}$$

$$\therefore \sin^2 x = \frac{3}{16}$$

$$\therefore \sin x = \pm \frac{\sqrt{3}}{4}$$

$$66 \quad \text{a} \quad \sin 2x = \sin x, \quad -\pi \leq x \leq \pi$$

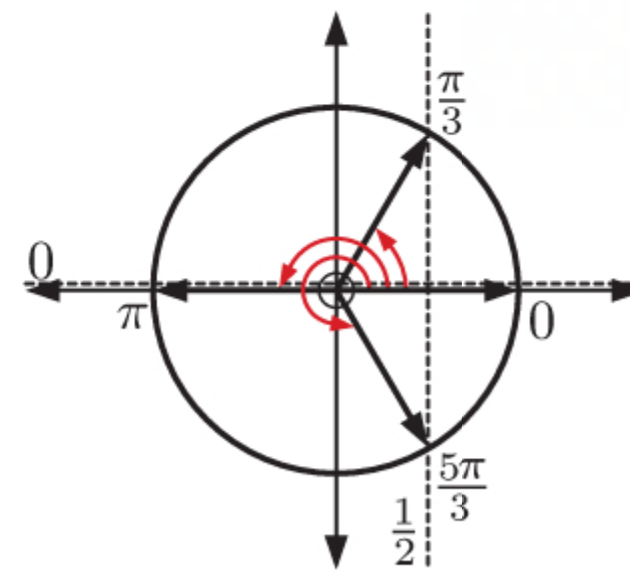
$$\therefore 2\sin x \cos x = \sin x$$

$$\therefore 2\sin x \cos x - \sin x = 0$$

$$\therefore \sin x(2\cos x - 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\therefore x = -\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$$



$$\text{b} \quad -3\cos 2x - 14\sin x + 11 = 0, \quad -\pi \leq x \leq \pi$$

$$\therefore -3(1 - 2\sin^2 x) - 14\sin x + 11 = 0$$

$$\therefore -3 + 6\sin^2 x - 14\sin x + 11 = 0$$

$$\therefore 6\sin^2 x - 14\sin x + 8 = 0$$

$$\therefore 3\sin^2 x - 7\sin x + 4 = 0$$

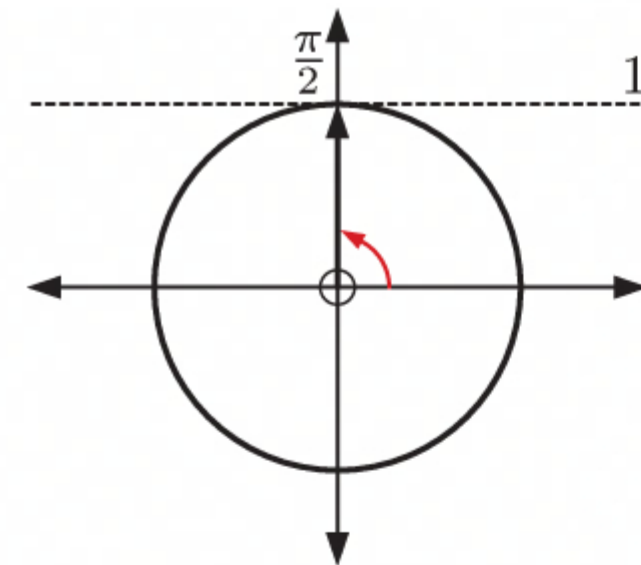
$$\therefore 3\sin^2 x - 3\sin x - 4\sin x + 4 = 0$$

$$\therefore 3\sin x(\sin x - 1) - 4(\sin x - 1) = 0$$

$$\therefore (\sin x - 1)(3\sin x - 4) = 0$$

$$\therefore \sin x = 1 \quad \{-1 \leq \sin x \leq 1 \text{ for all } x\}$$

$$\therefore x = \frac{\pi}{2}$$



$$\text{c} \quad \sin x + \cos x = 1, \quad -\pi \leq x \leq \pi$$

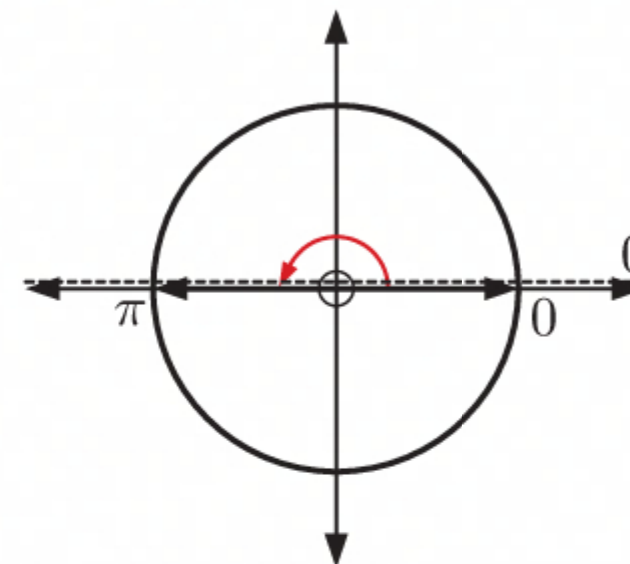
$$\therefore (\sin x + \cos x)^2 = 1^2$$

$$\therefore \sin^2 x + 2\sin x \cos x + \cos^2 x = 1$$

$$\therefore 2\sin x \cos x + 1 = 1$$

$$\therefore 2\sin x \cos x = 0$$

$$\therefore \sin 2x = 0$$



Since $-\pi \leq x \leq \pi$,

$$-2\pi \leq 2x \leq 2\pi$$

$$\text{So, } 2x = -2\pi, -\pi, 0, \pi, 2\pi$$

$$\therefore x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

Since we squared both sides of the equation, we must check that all the solutions satisfy the original equation.

$$\text{Check: If } x = -\pi, \quad \sin(-\pi) + \cos(-\pi) = 0 + (-1) = -1 \quad \times$$

$$\text{If } x = -\frac{\pi}{2}, \quad \sin(-\frac{\pi}{2}) + \cos(-\frac{\pi}{2}) = -1 + 0 = -1 \quad \times$$

$$\text{If } x = 0, \quad \sin 0 + \cos 0 = 0 + 1 = 1 \quad \checkmark$$

$$\text{If } x = \frac{\pi}{2}, \quad \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1 + 0 = 1 \quad \checkmark$$

$$\text{If } x = \pi, \quad \sin \pi + \cos \pi = 0 + (-1) = -1 \quad \times$$

So the solutions are $x = 0, \frac{\pi}{2}$.

TOPIC 4 SKILL BUILDER QUESTIONS

- 1 a** The sample size of only 10 students from a total of 500 is far too small, so this approach may produce a coverage error.
- b** The tape measure only allows Gerard to *estimate* the exact height of each student. With only 10 students being measured, any errors will significantly impact the results, so this approach may produce a measurement error.

2

	Boys	Girls
Year 8	135	140
Year 9	130	145
Year 10	125	130

- a i** A survey of 50 students can be completed in a reasonable time frame. It is also large enough that the results can be representative of the whole student body.
- ii** Surveying all students is time-consuming and often impractical. A non-response error may be produced if students are absent.

b Total number of students = $135 + 140 + 130 + 145 + 125 + 130 = 805$

i For the survey, $\frac{135}{805} \times 50 \approx 8$ Year 8 boys will be selected.

ii For the survey, $\frac{140 + 145 + 130}{805} \times 50 \approx 26$ girls will be selected.

c A stratified sample is better than a random sample in this case as a stratified sample will fairly represent each year level and gender. A random sample cannot guarantee such a fair representation.

- 3 a** This is a convenience sample because it is more convenient for Marie to sample the first 10 people to visit her office than to sample 10 random people from the whole building for example.

b The preferences of the first 10 people to visit Marie's office are likely to come from people who work with her. This may not be representative of the preferences of all people in the building, and so the sample may be biased.

c Marie could use a stratified sample where the subgroups may correspond to departments, floor number, and so on. In this way, a fair representation of preferences is more likely to be obtained.

- 4 a** The ticket inspector selects passengers at regular intervals, so the sampling method used is systematic sampling.

b The first passenger to be checked is the 8th passenger.

So, the next 6 passengers to be checked are the 28th, 48th, 68th, 88th, 108th, and 128th passengers.

c 5000 passengers left the terminal, and every 20th passenger was checked.

$$\therefore \frac{5000}{20} = 250 \text{ passengers were checked.}$$

- 5 a** The number of houses on a particular street can be counted, so it is a discrete variable.

b The number of hours spent travelling on an airplane can be measured and can take any value between certain limits, so it is a continuous variable.

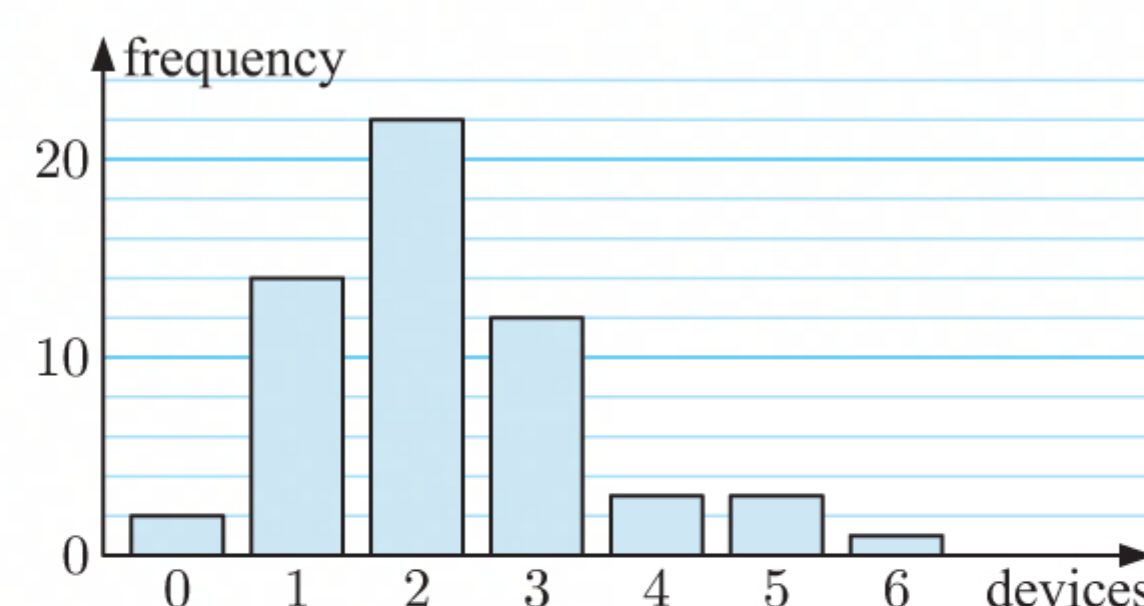
c The brand of laptop someone uses is a categorical variable.

- 6 a** $2 + 14 + 22 + 12 + 3 + 3 + 1 = 57$ people were surveyed.

b The mode of the data is 2 devices.

c $\frac{14 + 22}{57} \times 100\% \approx 63.2\%$ of people browsed the internet using 1 or 2 devices.

d The data is positively skewed with no outliers.



- 7 a** 26 is the data value which occurs most often, so the mode is 26 customers.

b As $n = 9$, $\frac{n+1}{2} = 5$

The ordered data set is: ~~14~~ ~~16~~ ~~18~~ ~~23~~ **24** ~~25~~ ~~26~~ ~~26~~ ~~34~~

↑
5th value

\therefore median = 24 customers.

c mean = $\frac{14 + 23 + \dots + 16 + 25}{9}$ ← sum of all the data values
← 9 data values
 $= \frac{206}{9}$
 ≈ 22.9 customers

8 a i mean number of visitors for exhibit A

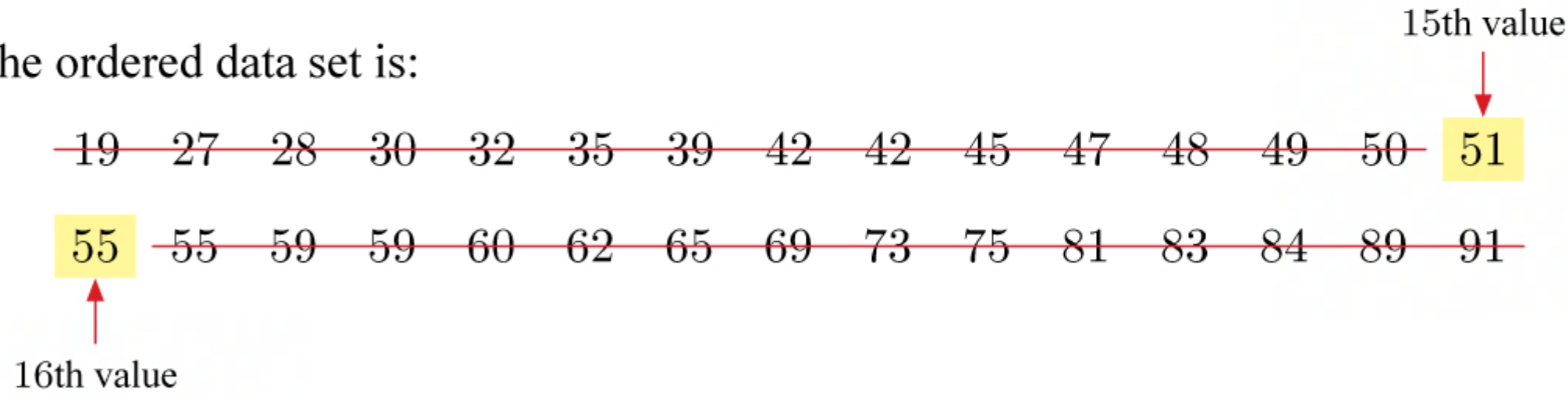
$$\begin{aligned}
 &= \frac{42 + 49 + 55 + 48 + \dots + 32}{30} \\
 &= \frac{1644}{30} \\
 &= 54.8 \text{ visitors}
 \end{aligned}$$

mean number of visitors for exhibit B

$$\begin{aligned}
 &= \frac{59 + 51 + 60 + 44 + \dots + 46}{30} \\
 &= \frac{1711}{30} \\
 &\approx 57.0 \text{ visitors}
 \end{aligned}$$

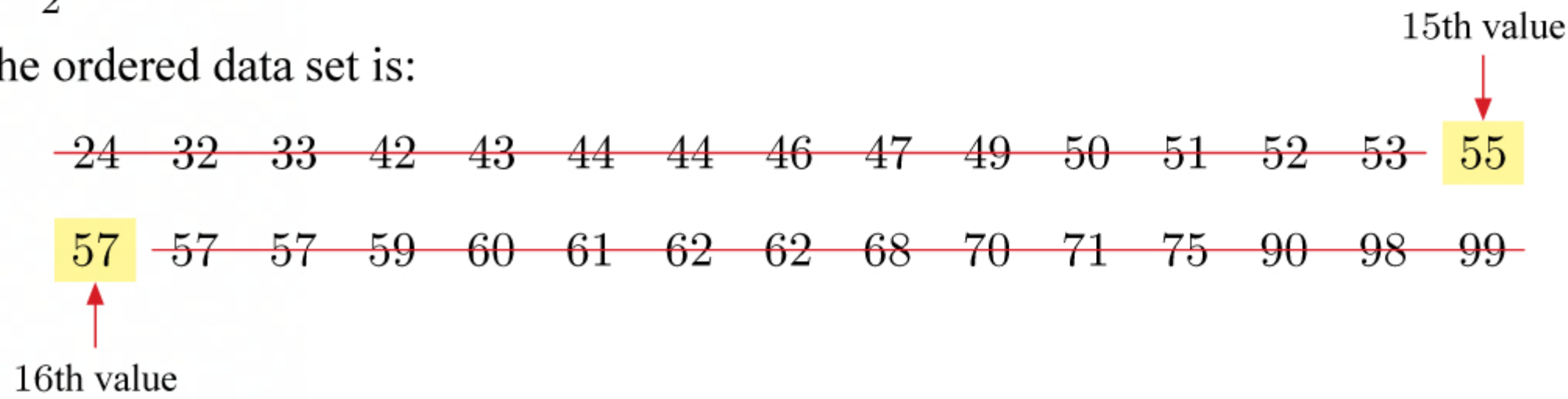
ii As $n = 30$, and $\frac{n+1}{2} = 15.5$, for both data sets, the median is the average of the 15th and 16th ordered data values.

For exhibit A, the ordered data set is:



$$\therefore \text{median} = \frac{51 + 55}{2} = 53 \text{ visitors}$$

For exhibit B, the ordered data set is:



$$\therefore \text{median} = \frac{55 + 57}{2} = 56 \text{ visitors}$$

b Exhibit B was more popular as the mean and median are both higher for exhibit B than for exhibit A.

9 a

$$\begin{aligned}
 \frac{9 + 10 + a + 13 + b + 16 + 21}{7} &= 14 \\
 \therefore \frac{69 + a + b}{7} &= 14 \\
 \therefore 69 + a + b &= 98 \\
 \therefore a + b &= 29
 \end{aligned}$$

Now, a and b are integers such that $10 \leq a \leq 13$ and $13 \leq b \leq 16$.

\therefore the only possible solution is $a = 13$ and $b = 16$.

b Since $n = 6$, $\frac{n+1}{2} = 3.5$

So the median is the average of the 3rd and 4th ordered data values.

The ordered data set is: $1 \quad 5 \quad 9 \quad 11 \quad 16 \quad p$
two middle data values

$$\therefore \text{median} = \frac{9 + 11}{2} = 10$$

$$\text{Now, } \frac{1 + 5 + 9 + 11 + 16 + p}{6} = 10$$

$$\therefore \frac{42 + p}{6} = 10$$

$$\therefore 42 + p = 60$$

$$\therefore p = 18$$

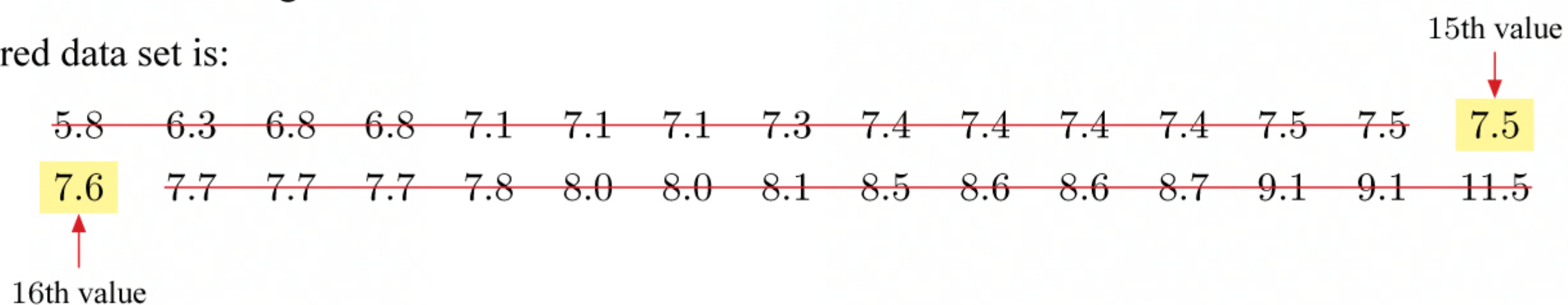
10 a

$$\begin{aligned}
 \text{mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{30} \\
 &= \frac{233.1}{30} \\
 &= 7.77 \text{ hours}
 \end{aligned}$$

$$\text{As } n = 30, \frac{n+1}{2} = 15.5$$

So the median is the average of the 15th and 16th data values.

The ordered data set is:



$$\therefore \text{median} = \frac{7.5 + 7.6}{2} = 7.55 \text{ hours}$$

b The outlier is 11.5 hours.

$$\begin{aligned} \text{c i mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{29} \\ &= \frac{221.6}{29} \\ &\approx 7.64 \text{ hours} \end{aligned}$$

As $n = 29$ with the outlier removed, $\frac{n+1}{2} = 15$.

The ordered data set is:

15th value
↓

5.8	6.3	6.8	6.8	7.1	7.1	7.1	7.3	7.4	7.4	7.4	7.4	7.5	7.5	7.5
7.6	7.7	7.7	7.7	7.8	8.0	8.0	8.1	8.5	8.6	8.6	8.7	9.1	9.1	

\therefore median = 7.5 hours

- ii The measure of centre which is most affected by extreme values is the mean. So, the mean is most affected if the outlier is removed.

11 a

Number of cars	Frequency	Cumulative frequency
0	78	78
1	117	195
2	69	264
3	18	282
4	2	284
Total	284	

$$\begin{aligned} \text{b i mean} &= \frac{0 \times 78 + 1 \times 117 + 2 \times 69 + 3 \times 18 + 4 \times 2}{284} \\ &= \frac{317}{284} \\ &\approx 1.12 \text{ cars} \end{aligned}$$

- ii There are 284 data values, so $n = 284$. $\frac{n+1}{2} = 142.5$, so the median is the average of the 142nd and 143rd ordered data values.

From the cumulative frequency column, the 79th to 195th ordered data values are 1 car.

\therefore the 142nd and 143rd data values are 1 car.

$$\therefore \text{median} = \frac{1+1}{2} = 1 \text{ car}$$

- iii Looking down the frequency column, the highest frequency is 117. This corresponds to 1 car, so the mode is 1 car.

12

Score	7	9	a	13	16
Frequency	1	2	1	2	1

$$\begin{aligned} \text{a} \quad \bar{x} &= \frac{\sum xf}{\sum f} \\ \therefore 11 &= \frac{7 \times 1 + 9 \times 2 + a \times 1 + 13 \times 2 + 16 \times 1}{1 + 2 + 1 + 2 + 1} \\ \therefore 11 &= \frac{a + 67}{7} \\ \therefore a + 67 &= 77 \\ \therefore a &= 10 \end{aligned}$$

- b** Let k be the number of goals that Kai will need to score in the next game.

Since she averaged 11 goals in her first 7 games, her average after the next game = $\frac{7 \times 11 + k}{8} = \frac{k + 77}{8}$.

For her overall average to improve to 12, we require $\frac{k + 77}{8} = 12$

$$\therefore k + 77 = 96$$

$$\therefore k = 19$$

So, Kai will need to score 19 goals in the next game to improve her overall average to 12.

13

Weekly rent (€ <i>r</i>)	Frequency (<i>f</i>)	Midpoint (<i>x</i>)	Product (<i>x f</i>)
$80 \leq r < 100$	3	90	270
$100 \leq r < 120$	15	110	1650
$120 \leq r < 140$	26	130	3380
$140 \leq r < 160$	30	150	4500
$160 \leq r < 180$	14	170	2380
$180 \leq r < 200$	1	190	190
<i>Total</i>	$\sum f = 89$		$\sum x f = 12\,370$

a $\bar{x} = \frac{\sum x f}{\sum f}$
 $= \frac{12\,370}{89}$
 ≈ 139

b $P(r \geq 140) = \frac{30 + 14 + 1}{89}$
 ≈ 0.506

\therefore the mean weekly rent was about €139.

14 The ordered data sets are:

Cailan: 79 80 81 83 84 85 86 87 90 92 (10 data values)

Miles: 82 82 83 84 84 85 87 88 90 91 (10 data values)

a *Cailan*:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum} \\ &= 92 - 79 \\ &= 13\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

lower half
upper half

79 80 **81** 83 84
85 86 **87** 90 92

$$Q_1 = \text{median of lower half} = 81$$

$$Q_3 = \text{median of upper half} = 87$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 87 - 81 \\ &= 6\end{aligned}$$

Miles:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum} \\ &= 91 - 82 \\ &= 9\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

lower half
upper half

82 82 **83** 84 84
85 87 **88** 90 91

$$Q_1 = \text{median of lower half} = 83$$

$$Q_3 = \text{median of upper half} = 88$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 88 - 83 \\ &= 5\end{aligned}$$

b i Miles' data has the lower range.

ii Miles' data has the lower interquartile range.

c The interquartile range is more appropriate than the range for determining who is generally the more consistent golfer as it is less affected by outliers.

- 15 a** The ordered data set is:

4 9 10 12 12 14 14 15 16 16 16 17 18 18 18 20 23 26 31 {19 data values}

$Q_1 = 12$
 $\text{median} = 16$
 $Q_3 = 18$

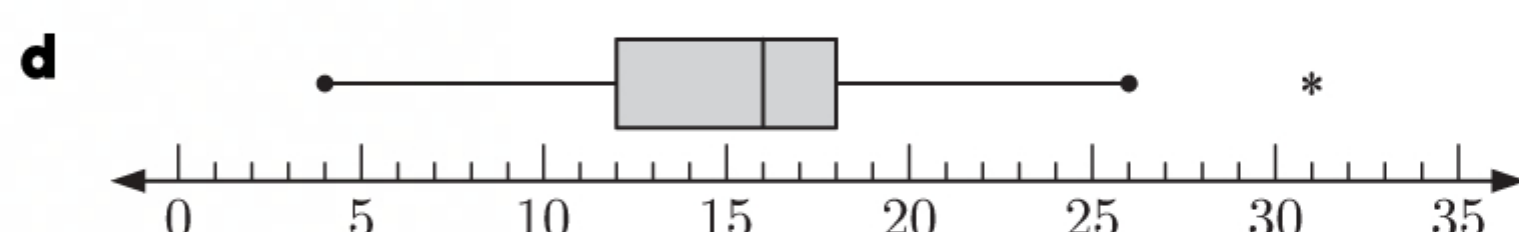
So the five-number summary is:

$$\begin{cases} \text{minimum} = 4 & Q_1 = 12 \\ \text{median} = 16 & Q_3 = 18 \\ \text{maximum} = 31 \end{cases}$$

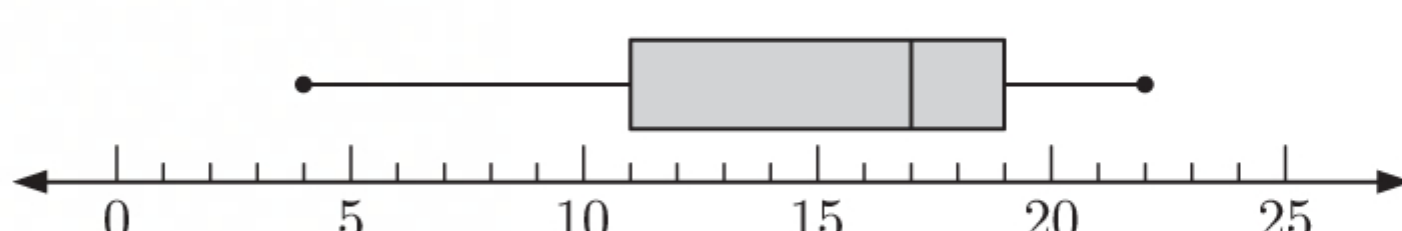
b $\text{IQR} = Q_3 - Q_1$
 $= 18 - 12$
 $= 6$

c upper boundary lower boundary
 $= \text{upper quartile} + 1.5 \times \text{IQR}$ $= \text{lower quartile} - 1.5 \times \text{IQR}$
 $= 18 + 1.5 \times 6$ $= 12 - 1.5 \times 6$
 $= 27$ $= 3$

31 is above the upper boundary, so it is an outlier.



16



a minimum value = 4 cm

b maximum value = 22 cm

c median = 17 cm

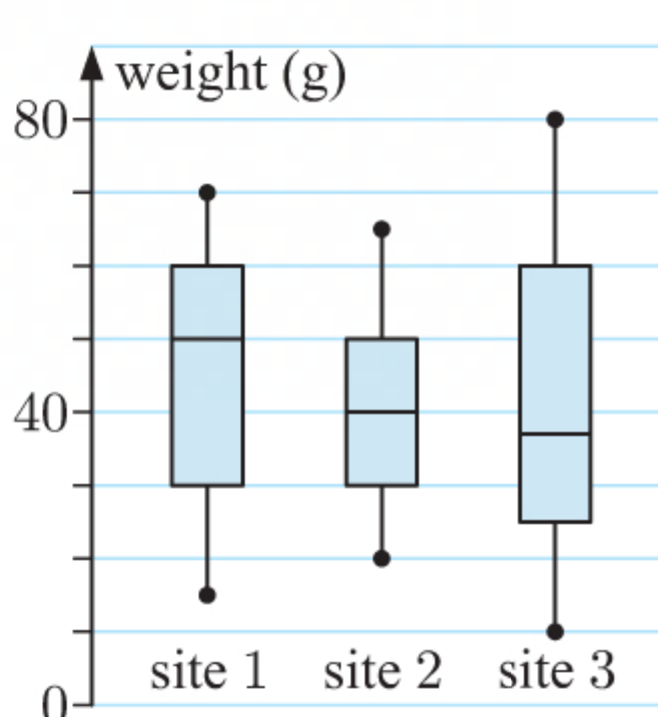
d upper quartile = 19 cm

e lower quartile = 11 cm

f range = maximum - minimum
 $= 22 - 4$
 $= 18 \text{ cm}$

g $\text{IQR} = Q_3 - Q_1$
 $= 19 - 11$
 $= 8 \text{ cm}$

17



- a** The five-number summary for site 1 is:

$$\begin{cases} \text{minimum} = 15 & Q_1 = 30 \\ \text{median} = 50 & Q_3 = 60 \\ \text{maximum} = 70 \end{cases}$$

- b** Site 3 has the greatest range of weights.

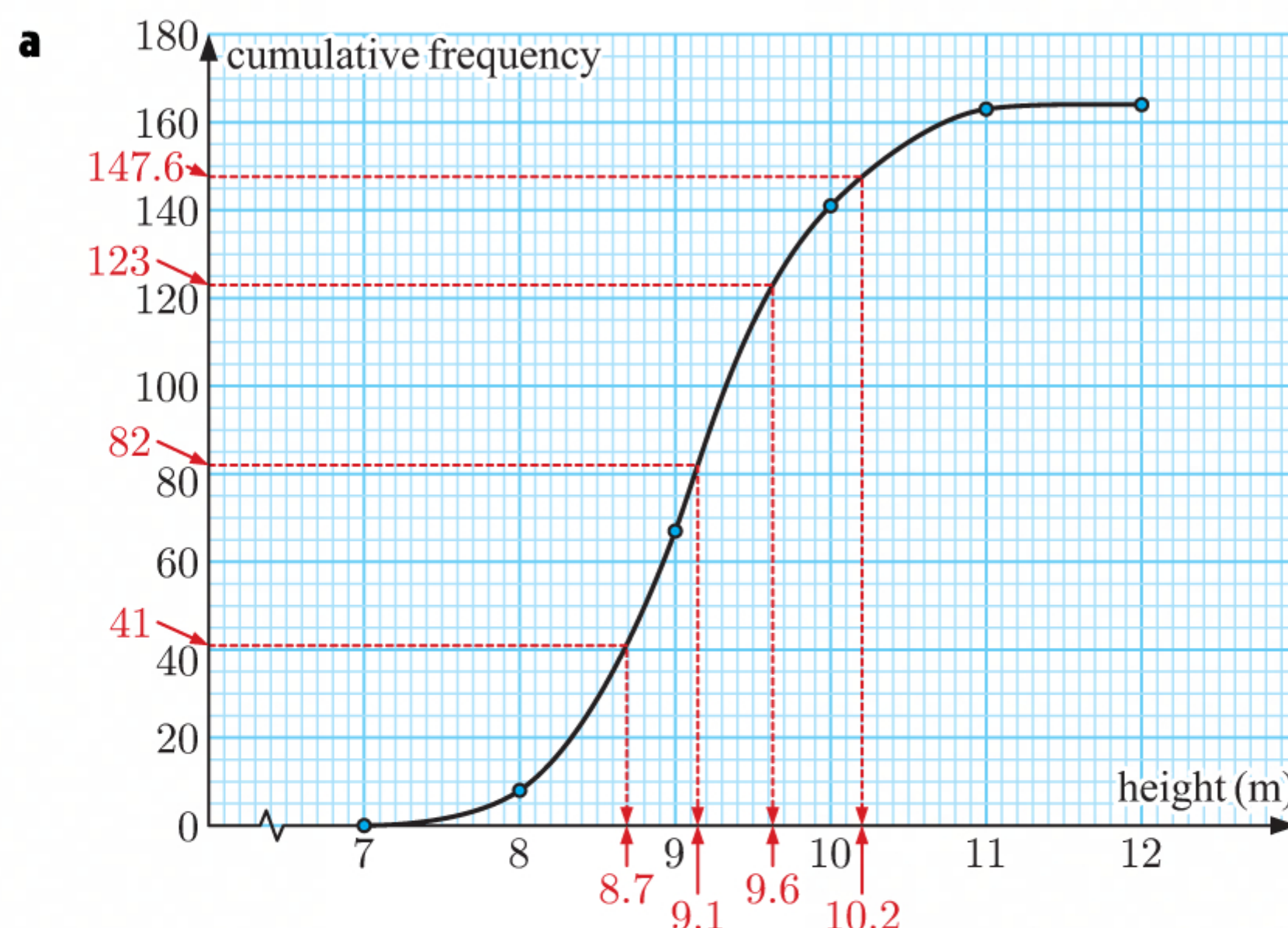
- c** The weights of fungi have the least variation at site 2.

- d** Site 1 has the highest median weight of fungi.

- e** Site 1 has the highest proportion of weights above 40 grams.

18

Height (h m)	Frequency	Cumulative frequency
$7 \leq h < 8$	8	8
$8 \leq h < 9$	59	67
$9 \leq h < 10$	74	141
$10 \leq h < 11$	22	163
$11 \leq h < 12$	1	164



- b** The median is the 50th percentile. As 50% of 164 is 82, we start with the cumulative frequency 82 and find the corresponding height.

From the graph, the median ≈ 9.1 m.

- c** Q_1 is the 25th percentile. As 25% of 164 is 41, we start with the cumulative frequency 41 and find the corresponding height.

From the graph, $Q_1 \approx 8.7$ m

Q_3 is the 75th percentile. As 75% of 164 is 123, we start with the cumulative frequency 123 and find the corresponding height.

From the graph, $Q_3 \approx 9.6$ m

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &\approx 9.6 - 8.7 \\ &\approx 0.9 \text{ m}\end{aligned}$$

- d** As 90% of 164 is 147.6, we start with the cumulative frequency 147.6 and find the corresponding height.

The 90th percentile ≈ 10.2 m which means that 90% of trees are shorter than about 10.2 m.

19 Anthony: $1\frac{1}{2}, 2, 2\frac{1}{2}, 4, 4\frac{1}{2}, 3, 3\frac{1}{2}, 5, 6, 6$

Katherine: $3, 3\frac{1}{2}, 4, 3, 3, 3\frac{1}{2}, 4, 4, 4\frac{1}{2}, 4$

- a** Using technology:

Anthony:

	1-Variable
\bar{x}	=3.8
Σx	=38
Σx^2	=167
σx	=1.50332963
sx	=1.58464857
n	=10

The mean $\mu = 3.8$ hours and the standard deviation $\sigma \approx 1.50$ hours.

Katherine:

	1-Variable
\bar{x}	=3.65
Σx	=36.5
Σx^2	=135.75
σx	=0.50249378
sx	=0.52967495
n	=10

The mean $\mu = 3.65$ hours and the standard deviation $\sigma \approx 0.502$ hours.

- b** Anthony's mean is higher than Katherine's, so Anthony generally practised for longer.
- c** Katherine's standard deviation is lower than Anthony's, so there is less deviation from the mean for her data set. Katherine therefore practised more consistently than Anthony.

20

Mark	3	4	5	6	7	8	9	10
Frequency	1	3	5	8	4	2	0	1

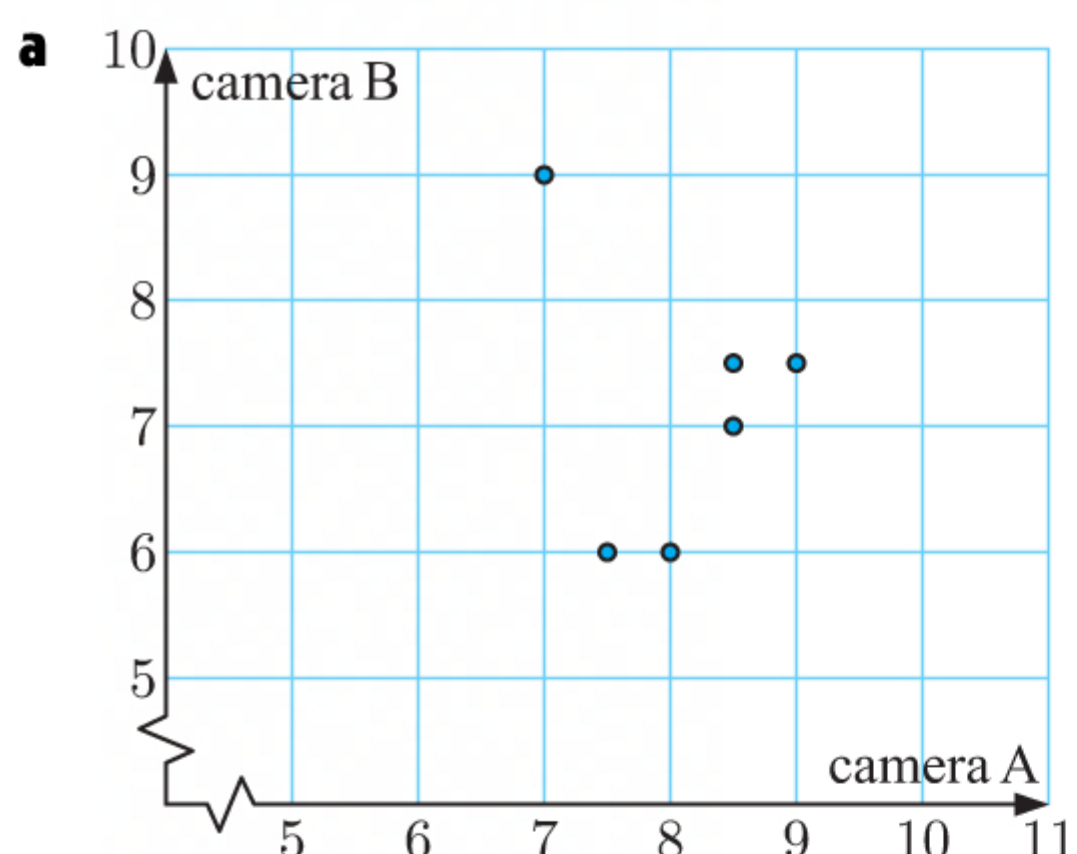
Using technology:

	1-Variable
\bar{x}	=5.91666666
Σx	=142
Σx^2	=894
σx	=1.49768839
sx	=1.52989532
n	=24

The mean test score $\mu \approx 5.92$, and the population standard deviation $\sigma \approx 1.50$.

21

Camera A	8.5	8	9	7	8.5	7.5
Camera B	7	6	7.5	9	7.5	6



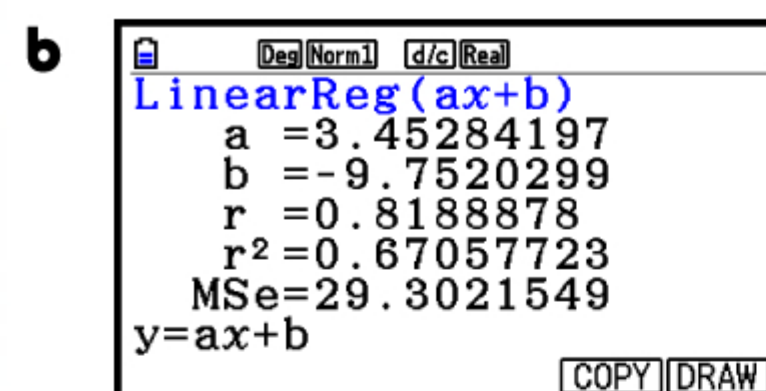
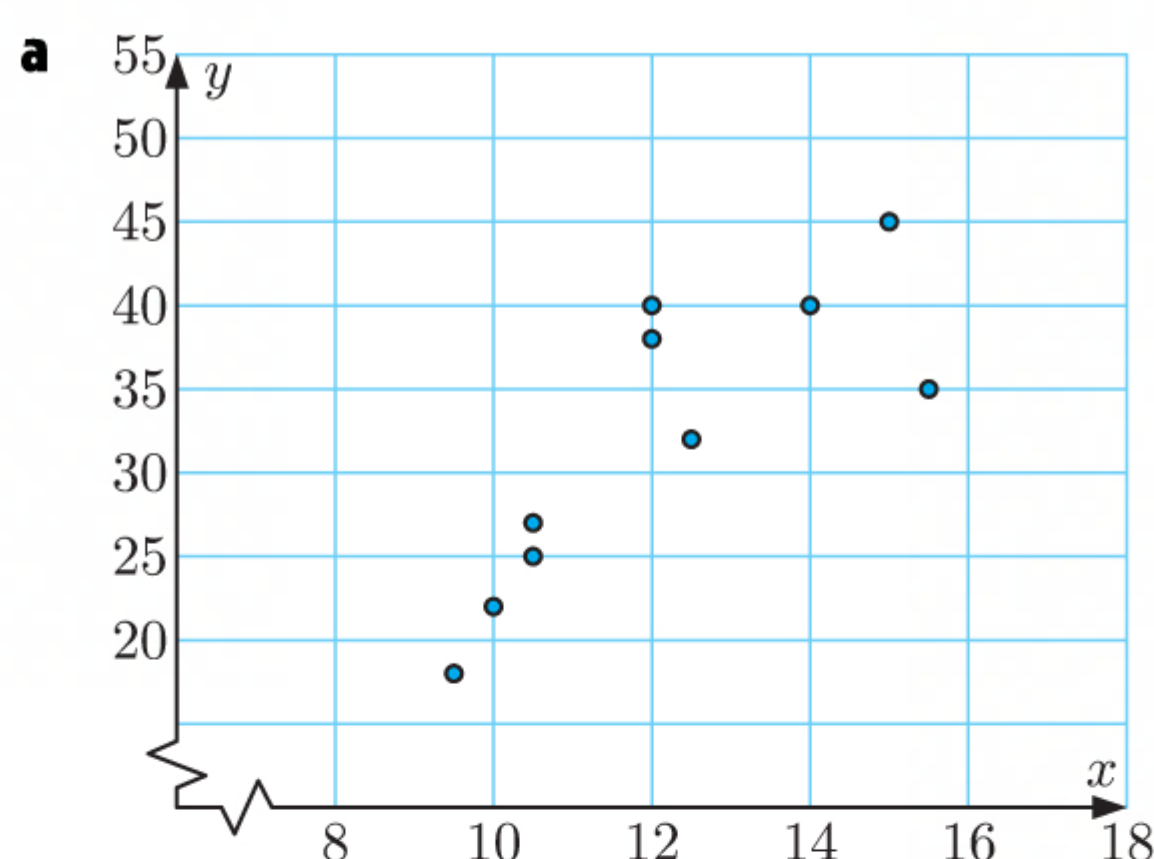
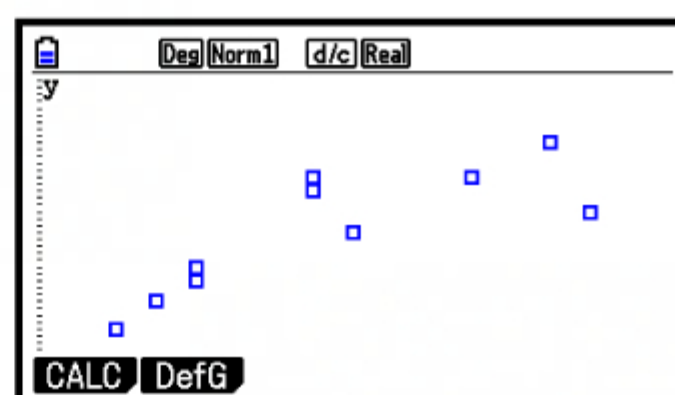
- b** The point (7, 9) appears to be an outlier. This corresponds to the review scoring camera A a 7, and camera B a 9.

- c**
- i** With the outlier removed, there appears to be a strong, positive, linear correlation between camera A's scores and camera B's scores.
 - ii** No, an increase in camera A's scores is not likely to cause an increase in camera B's scores. It is more likely that both scores are related to the preferences of each reviewer.

22

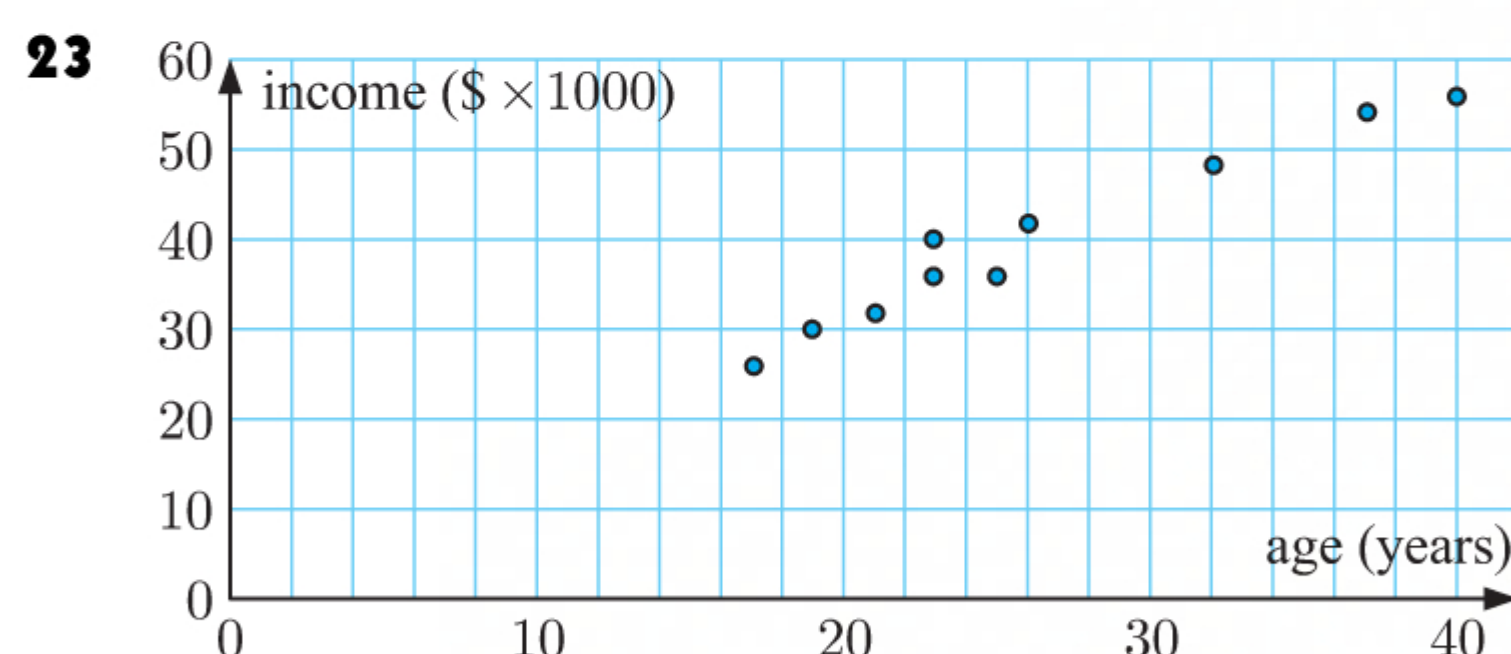
<i>Language</i> (x)	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
<i>Mathematics</i> (y)	32	45	27	38	18	25	35	22	40	40

	List 1	List 2	List 3	List 4
SUB				
1	12.5	32		
2	15	45		
3	10.5	27		
4	12	38		
				38

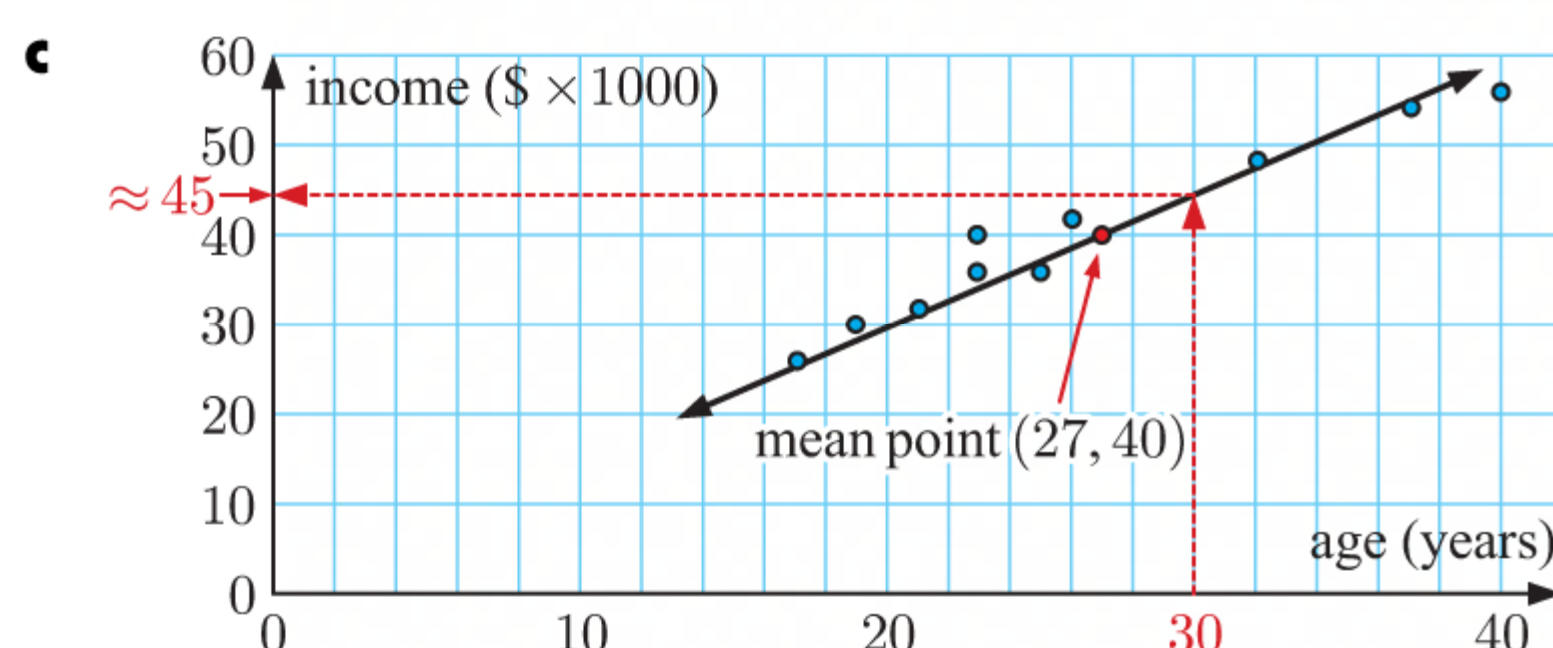


So, $r \approx 0.819$.

- c** The data suggests that there is a moderate, positive, linear correlation between the students' language scores and their mathematics scores. So, "Those who do well in languages also do well in mathematics." is a moderately reasonable statement.



- a** There is a strong, positive correlation between the age of an individual and their annual income.
- b** No, the relationship is more likely dependent on the amount of professional experience or qualifications an individual has.



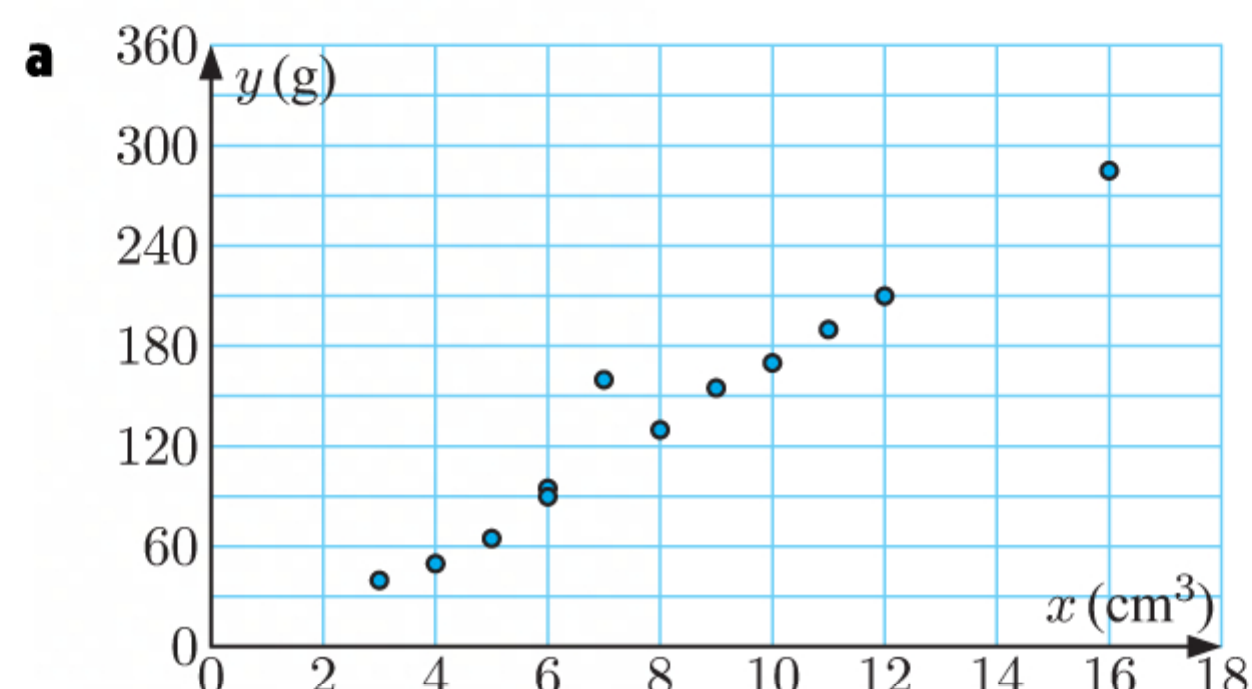
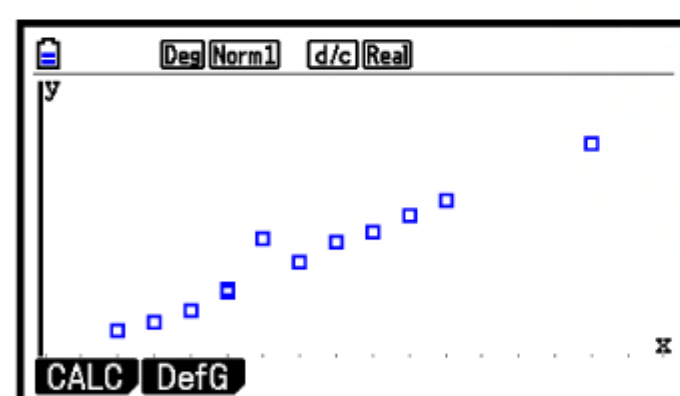
- d** When $x = 30$, $y \approx 45$.

The annual income of someone who is 30 years old is approximately \$45 000. This is an interpolation, so the estimate is reliable.

24

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ($x \text{ cm}^3$)	3	6	4	7	16	8	5	12	9	6	10	11
Mass ($y \text{ g}$)	40	95	50	160	285	130	65	210	155	90	170	190

	List 1	List 2	List 3	List 4
SUB				
1	3	40		
2	6	95		
3	4	50		
4	7	160		



b

LinearReg(ax+b)
a = 19.1171662
b = -17.86376
r = 0.98019806
r² = 0.96078823
MSe = 228.081743
y = ax + b

c There appears to be a strong, positive correlation between the *volume* of a sample of silver and its *mass*.

d The data point (7, 160) which corresponds to sample D appears to be an outlier. We therefore agree with the jeweller that there is a fake sample.

So, $r \approx 0.980$.

e i

LinearReg(ax+b)
a = 19.4604316
b = -24.676258
r = 0.9987246
r² = 0.99745083
MSe = 16.3069544
y = ax + b

ii When $x = 7$, $y \approx 19.5(7) - 24.7$
 ≈ 112

So, a sample of silver with volume 7 cm^3 would weigh approximately 112 g.

Using technology, the regression line is
 $y \approx 19.5x - 24.7$.

25

Study time ($x \text{ h}$)	7	6	3	16	15	11	18	32	20
Result ($y \%$)	56	42	25	80	65	60	85	96	90

a

LinearReg(ax+b)
a = 2.43264433
b = 31.9579472
r = 0.91326692
r² = 0.83405646
MSe = 104.88158
y = ax + b

Using technology, the least squares regression line is $y \approx 2.43x + 32.0$.

b From **a**, $r \approx 0.913$.

So, there is a strong, positive correlation between the number of hours that a student studies and their examination result.

c Yes, this is a causal relationship as spending more time studying for the examination is likely to cause a better result.

d When $y = 70$, $70 \approx 2.43x + 32.0$
 $\therefore 38 \approx 2.43x$
 $\therefore x \approx 15.6$

So, Tony studied for approximately 15.6 hours.

e The y -intercept of the line of best fit ≈ 32.0 . This indicates that if a student did not spend any time studying, they would obtain a result of 32% on average.

The gradient of the line of best fit ≈ 2.43 . This indicates that for every additional hour of study, the result obtained increases by an average of 2.43%.

26

Time (t days)	0	1	2	3	4	5	6	7	8	9
Height (h mm)	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3

a

	Des	Norm1	d/c	Real
LinearReg(ax+b)				
a	=0.48787878			
b	=4.91454545			
r	=0.99265002			
r ²	=0.98535406			
MSe	=0.03648484			
y=ax+b				
COPY DRAW				

So, $r \approx 0.993$.

- b** r is very close to 1 which indicates a very strong correlation between the variables.
The sign of r is positive which indicates that the variables are positively correlated.
An increase in one variable results in an increase in the other.

c $h \approx 0.4879t + 4.9145$

i When $t = 14$, $h \approx 0.4879(14) + 4.9145$
 ≈ 11.7

\therefore after 14 days, the grass is about 11.7 mm high.

ii When $h = 20$, $20 \approx 0.4879t + 4.9145$
 $\therefore 15.0855 \approx 0.4879t$
 $\therefore t \approx 30.9$

\therefore the grass reaches a height of 20 mm after about 30.9 days.

27

Distance from shore (x km)	3.7	1.3	4.3	2.8	0.9
Fish caught (y)	5	4	9	5	2

- a** We should use the regression line of x against y , since the number of fish caught can be more precisely measured than the distance from the shore.

b

	List 1	List 2	List 3	List 4
SUB				
1	3.7		5	
2	1.3		4	
3	4.3		9	
4	2.8		5	
				5

GRAPH CALC TEST INTR DIST

	Des	Norm1	d/c	Real
LinearReg(ax+b)				
a	=0.5076923			
b	=0.06153846			
r	=0.8766556			
r ²	=0.76852505			
MSe	=0.67282051			
y=ax+b				
COPY DRAW				

The regression line of x against y is $x \approx 0.508y + 0.0615$.

When $x = 7$, $7 \approx 0.508y + 0.0615$
 $\therefore 6.9385 \approx 0.508y$
 $\therefore y \approx 13.7$

If the distance from the shore is 7 km, we expect about 14 fish to be caught.

- c** The estimate in **b** is an extrapolation, so it may not be reliable.

28

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

a $P(\text{player in the 2017 tournament played in division 1}) \approx \frac{4}{61}$ ← number of division 1 players in 2017 tournament
← total number of players in 2017 tournament
 ≈ 0.0656

- b** There were $13 + 12 + 14 = 39$ division 3 players in total,
and $61 + 51 + 57 = 169$ players in total.

$\therefore P(\text{player in any of the past tournaments played in division 3}) \approx \frac{39}{169}$
 ≈ 0.231

- c** In the 2019 tournament, 8 players played in division 2 and 14 players played in division 4. So, $57 - 8 - 14 = 35$ players in the 2019 tournament did *not* play in division 2 or 4.

$\therefore P(\text{player in the 2019 tournament did not play in division 2 or 4}) \approx \frac{35}{57}$
 ≈ 0.614

29 a

	< 40	40 - 59	≥ 60	Total
Male	56	127	419	602
Female	75	113	230	418
Total	131	240	649	1020

b i 602 out of the 1020 patients were male.

$$\therefore P(\text{male}) \approx \frac{602}{1020} \approx 0.590$$

ii 75 out of the 1020 patients were female and younger than 40.

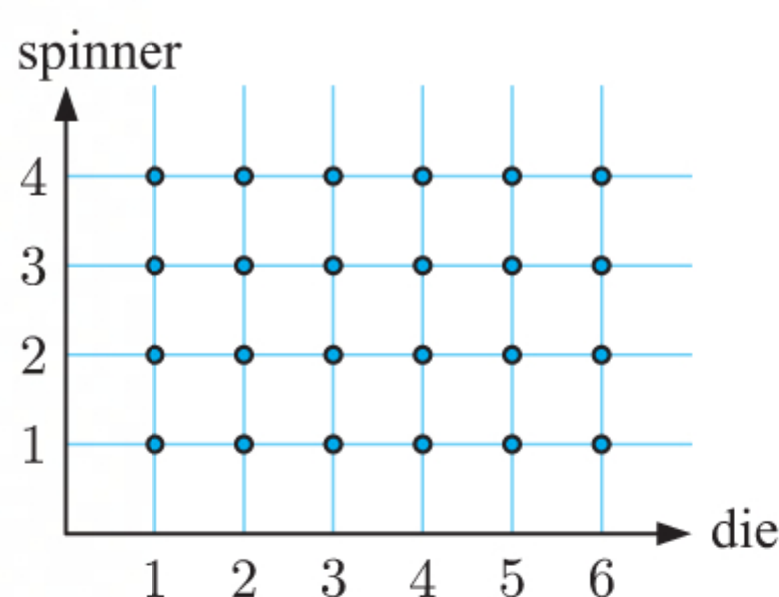
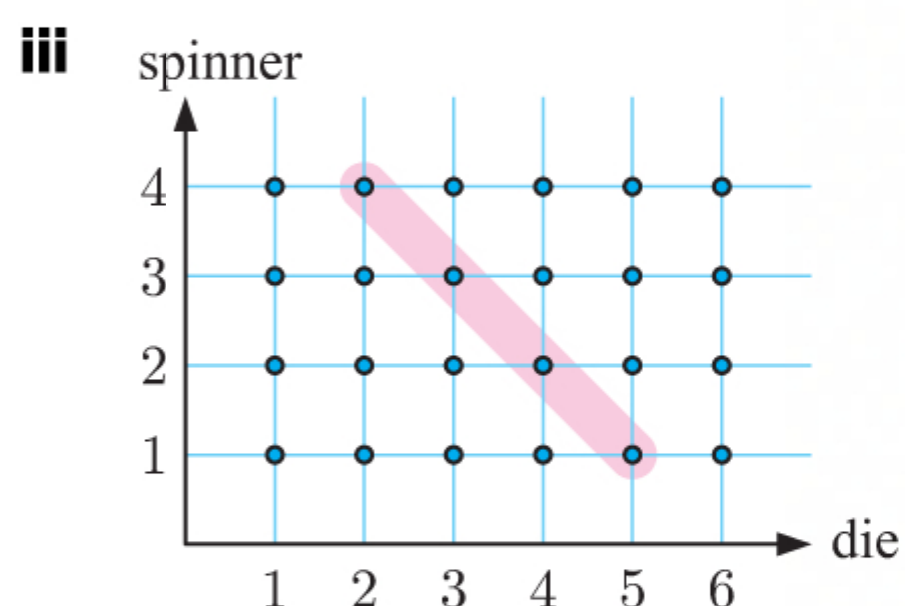
$$\therefore P(\text{female and younger than 40}) \approx \frac{75}{1020} \approx 0.0735$$

iii 230 out of the 418 female patients were 60 or older.

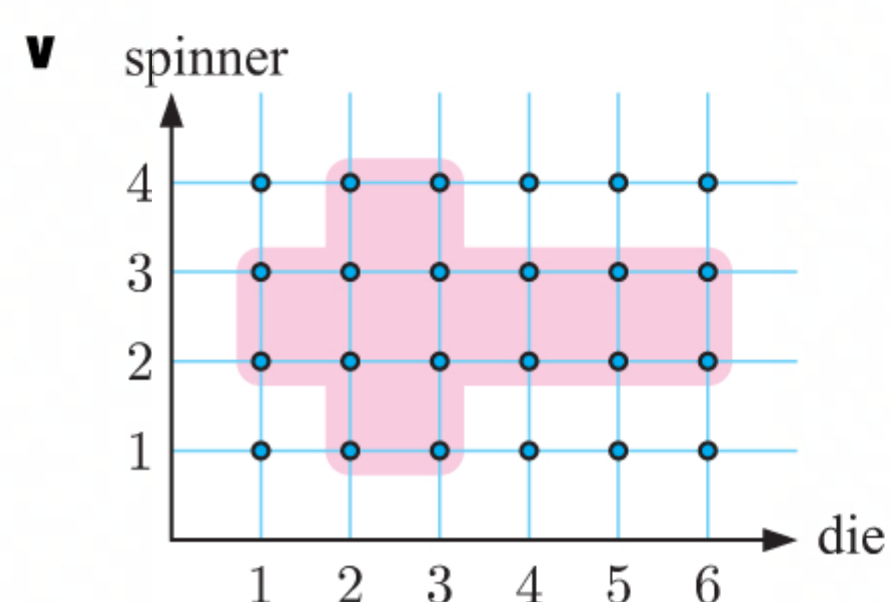
$$\therefore P(60 \text{ or older, given they were female}) \approx \frac{230}{418} \approx 0.550$$

iv $127 + 419 = 546$ out of the $240 + 649 = 889$ patients who were 40 or older were male.

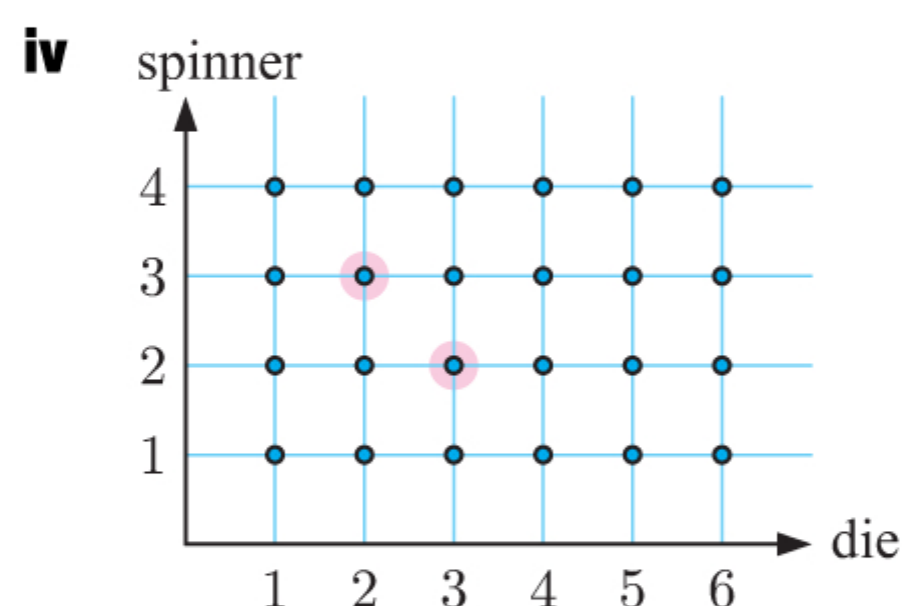
$$\therefore P(\text{male, given they were 40 or older}) \approx \frac{546}{889} \approx 0.614$$

30 a**b i** $P(\text{two 1s}) = \frac{1}{24}$ 

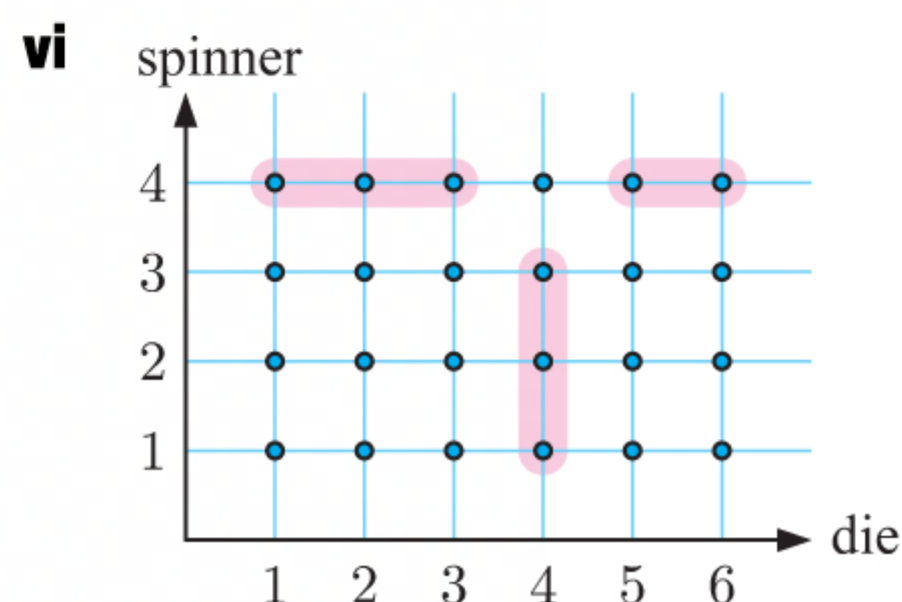
$$P(\text{a sum of 6}) = \frac{4}{24} \\ = \frac{1}{6}$$



$$P(\text{a 2 or a 3 (or both)}) = \frac{16}{24} \\ = \frac{2}{3}$$

ii $P(\text{two 5s}) = 0$ {the spinner does not have a 5}

$$P(\text{a 2 and a 3}) = \frac{2}{24} \\ = \frac{1}{12}$$



$$P(\text{exactly one 4}) = \frac{8}{24} \\ = \frac{1}{3}$$

31 a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore 0.78 = 0.37 + 0.41 - P(A \cap B)$$

$$\therefore P(A \cap B) = 0$$

b Since $P(A \cap B) = 0$, A and B are mutually exclusive events.**32** $P(A \cup B) = 1 - P((A \cup B)')$

$$= 1 - \frac{1}{12}$$

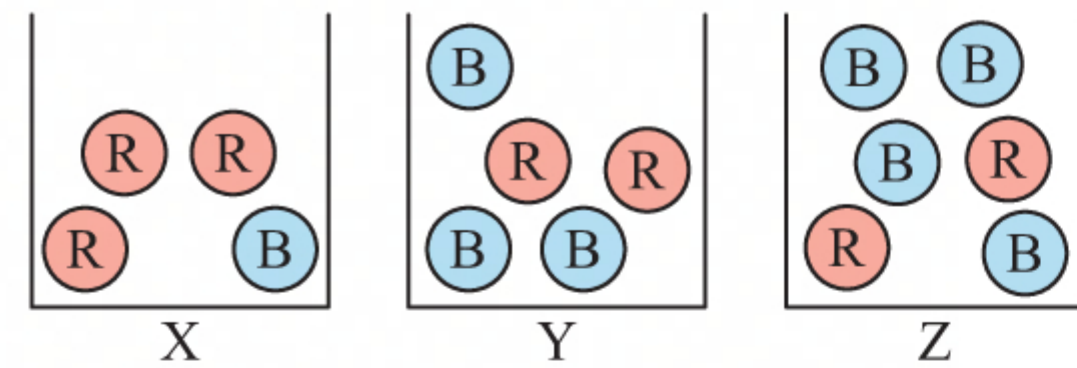
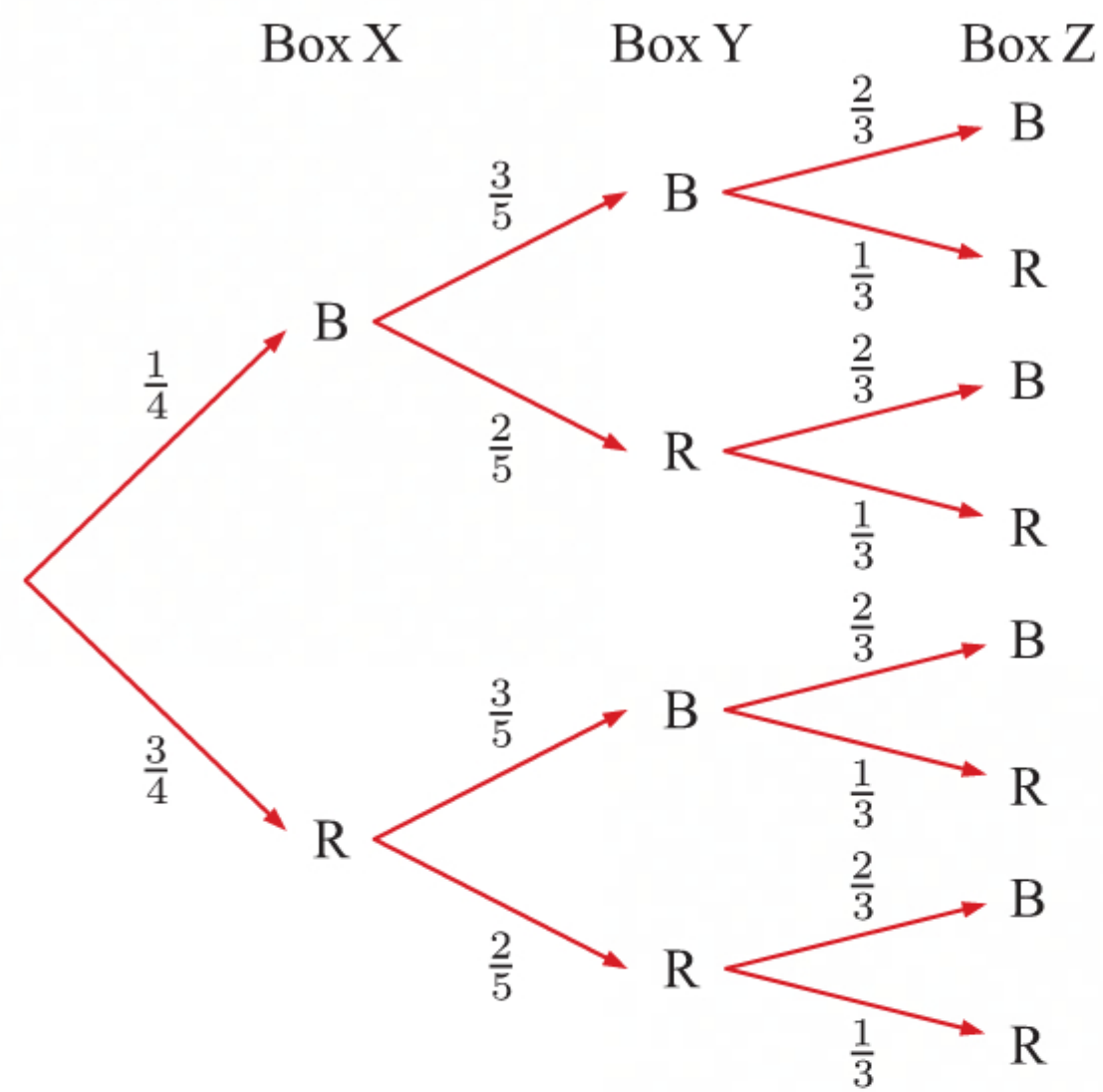
$$= \frac{11}{12}$$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \frac{11}{12} = \frac{23}{50} + \frac{5}{7} - P(A \cap B)$$

$$\therefore P(A \cap B) = \frac{541}{2100}$$

33 a


 b i $P(\text{exactly 2 red balls are drawn})$

$$\begin{aligned}
 &= P(RRB) + P(RBR) + P(BRR) \\
 &= \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{1}{3} \\
 &= \frac{1}{5} + \frac{3}{20} + \frac{1}{30} \\
 &= \frac{23}{60}
 \end{aligned}$$

 ii $P(\text{blue balls are drawn from boxes X and Z})$

$$\begin{aligned}
 &= P(BBB) + P(BRB) \\
 &= \frac{1}{4} \times \frac{3}{5} \times \frac{2}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{2}{3} \\
 &= \frac{1}{10} + \frac{1}{15} \\
 &= \frac{1}{6}
 \end{aligned}$$

 iii $P(\text{at most one blue ball is drawn})$

$$\begin{aligned}
 &= P(\text{no blue balls are drawn}) + P(\text{exactly one blue ball is drawn}) \\
 &= P(RRR) + [P(BRR) + P(RBR) + P(RRB)] \\
 &= \frac{3}{4} \times \frac{2}{5} \times \frac{1}{3} + \left[\frac{1}{4} \times \frac{2}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3} \right] \\
 &= \frac{1}{10} + \frac{1}{30} + \frac{3}{20} + \frac{1}{5} \\
 &= \frac{29}{60}
 \end{aligned}$$

c If an extra red ball is added to box Y, the probabilities in b i and b iii will be affected.

34 a

	outcome	probability
	T and E	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
	T and E'	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
	T' and E	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
	T' and E'	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
	total	$\frac{6}{6} = 1$

 b i $P(T \cap E') = \frac{1}{3}$

$$\begin{aligned}
 \text{ii } P(T \cup E') &= P(T) + P(E') - P(T \cap E') \\
 &= \frac{1}{2} + \frac{2}{3} - \frac{1}{3} \quad \{\text{from a}\} \\
 &= \frac{5}{6}
 \end{aligned}$$

 35 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore 0.63 = P(A) + 0.36 - P(A)P(B) \quad \{A \text{ and } B \text{ are independent}\}$$

$$\therefore 0.27 = P(A) - 0.36 \times P(A)$$

$$\therefore 0.27 = 0.64 \times P(A)$$

$$\therefore P(A) = \frac{0.27}{0.64} \approx 0.422$$

 36 a $P(2 \text{ white truffles}) = P(\text{first is white} \cap \text{second is white})$

$$= P(\text{first is white}) \times P(\text{second is white given first is white})$$

$$= \frac{2}{12} \times \frac{1}{11}$$

$$= \frac{2}{132}$$

$$= \frac{1}{66}$$

b $P(2 \text{ white truffles}) = \frac{1}{66}$ {from **a**}

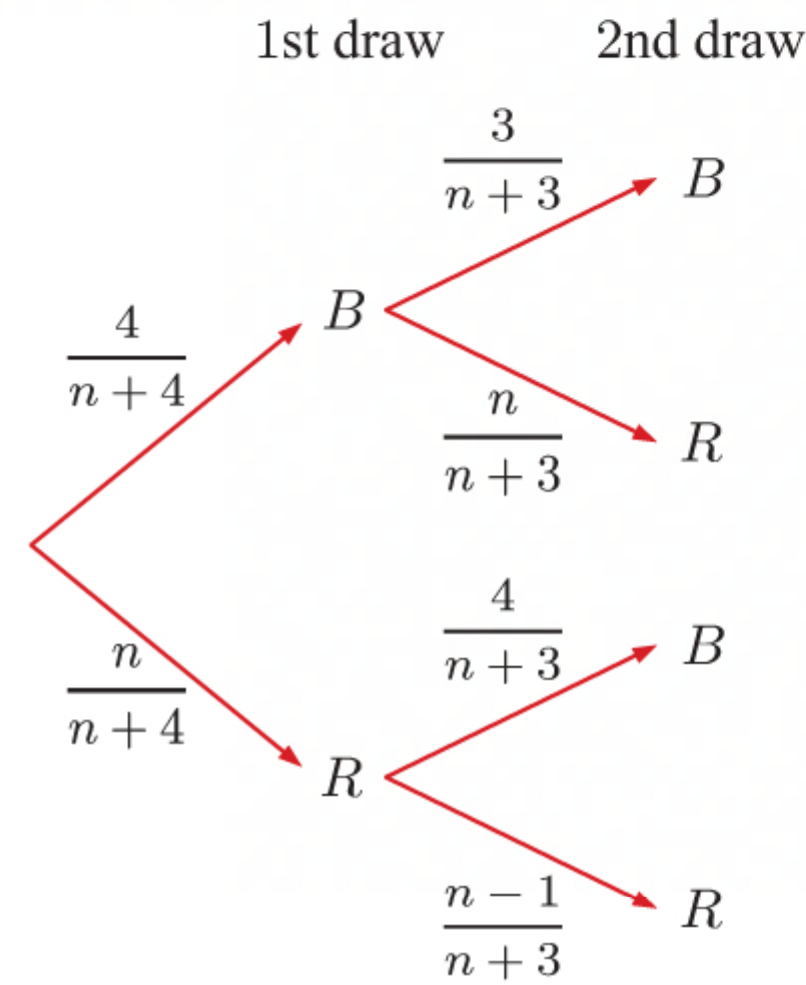
$$\begin{aligned} P(2 \text{ dark brown truffles}) &= P(\text{first is dark brown} \cap \text{second is dark brown}) \\ &= P(\text{first is dark brown}) \times P(\text{second is dark brown given first is dark brown}) \\ &= \frac{6}{12} \times \frac{5}{11} \\ &= \frac{30}{132} \\ &= \frac{5}{22} \end{aligned}$$

$$\begin{aligned} P(2 \text{ light brown truffles}) &= P(\text{first is light brown} \cap \text{second is light brown}) \\ &= P(\text{first is light brown}) \times P(\text{second is light brown given first is light brown}) \\ &= \frac{4}{12} \times \frac{3}{11} \\ &= \frac{12}{132} \\ &= \frac{1}{11} \end{aligned}$$

$$\begin{aligned} P(\text{different coloured truffles}) &= 1 - P(\text{same coloured truffles}) \\ &= 1 - \left(\frac{1}{66} + \frac{5}{22} + \frac{1}{11} \right) \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

- 37** Let B be the event that a blue ball is drawn, and
 R be the event that a red ball is drawn.

$$\begin{aligned} \text{Now } P(\text{both red}) &= \frac{1}{3} \\ \therefore P(R \cap R) &= \frac{1}{3} \\ \therefore \frac{n}{n+4} \times \frac{n-1}{n+3} &= \frac{1}{3} \\ \therefore \frac{n(n-1)}{(n+4)(n+3)} &= \frac{1}{3} \\ \therefore 3n(n-1) &= (n+4)(n+3) \\ \therefore 3n^2 - 3n &= n^2 + 7n + 12 \\ \therefore 2n^2 - 10n - 12 &= 0 \\ \therefore n^2 - 5n - 6 &= 0 \\ \therefore (n-6)(n+1) &= 0 \\ \therefore n &= 6 \quad \{n \geq 0\} \end{aligned}$$



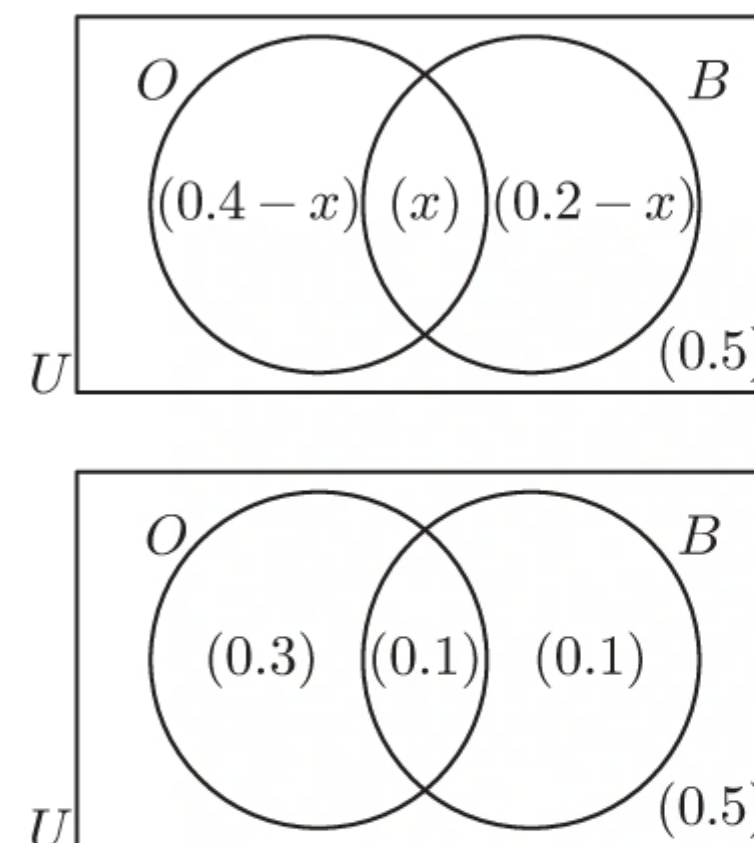
- 38 a** Let O represent a student who owns an orange highlighter, and
 B represent a student who owns a blue highlighter.

Let the proportion of students in $O \cap B$ be x .

\therefore the proportion in $O \cap B'$ is $0.4 - x$ and
the proportion in $O' \cap B$ is $0.2 - x$.

The proportion in $O' \cap B'$ is 0.5 .

$$\begin{aligned} \therefore (0.4 - x) + x + (0.2 - x) &= 1 - 0.5 \\ \therefore 0.6 - x &= 0.5 \\ \therefore x &= 0.1 \end{aligned}$$



b i $P(B | O) = \frac{P(B \cap O)}{P(O)}$

$$\begin{aligned} &= \frac{0.1}{0.4} \\ &= \frac{1}{4} \end{aligned}$$

ii $P(O | B') = \frac{P(O \cap B')}{P(B')}$

$$\begin{aligned} &= \frac{0.3}{0.8} \\ &= \frac{3}{8} \end{aligned}$$

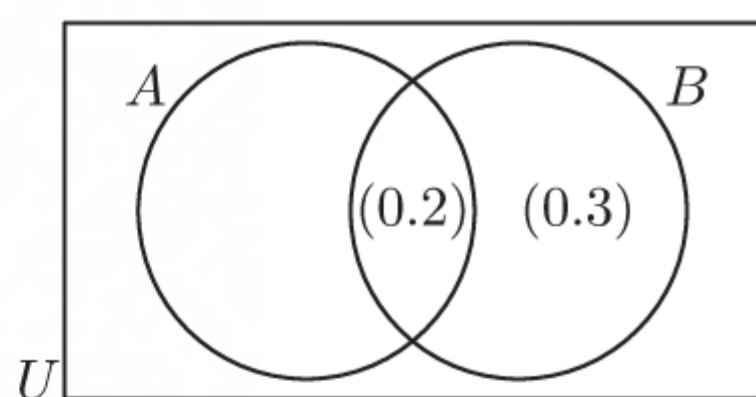
$$\begin{aligned}
 \text{39 a i } P(X \cap Y) &= P(X) + P(Y) - P(X \cup Y) \\
 &= \frac{3}{7} + \frac{2}{9} - \frac{3}{5} \\
 &= \frac{16}{315}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } P(X | Y) &= \frac{P(X \cap Y)}{P(Y)} \\
 &= \frac{\frac{16}{315}}{\frac{2}{9}} \quad \{\text{using i}\} \\
 &= \frac{8}{35}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } P(Y | X) &= \frac{P(Y \cap X)}{P(X)} \\
 &= \frac{\frac{16}{315}}{\frac{3}{7}} \quad \{\text{using i}\} \\
 &= \frac{16}{135}
 \end{aligned}$$

b X and Y are not independent as $P(X | Y) \neq P(X)$ and $P(Y | X) \neq P(Y)$.

40



$$\begin{aligned}
 P(B) &= P(B \cap A) + P(B \cap A') \\
 &= 0.2 + 0.3 \\
 &= 0.5
 \end{aligned}$$

$$\begin{aligned}
 \text{and } P(A' \cap B) &= P(A') \times P(B) \quad \{A' \text{ and } B \text{ are independent}\} \\
 \therefore 0.3 &= 0.5 \times P(A') \\
 \therefore P(A') &= 0.6
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) \\
 &= 0.6 + 0.5 - 0.3 \\
 &= 0.8
 \end{aligned}$$

41 Let H represent a head, and T represent a tail.

$$\begin{aligned}
 \text{a } P(2 \text{ heads and } 1 \text{ tail}) &= P(\text{HHT, HTH, or THH}) \\
 &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\
 &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
 &= \frac{3}{8}
 \end{aligned}$$

b $n = 400$ times

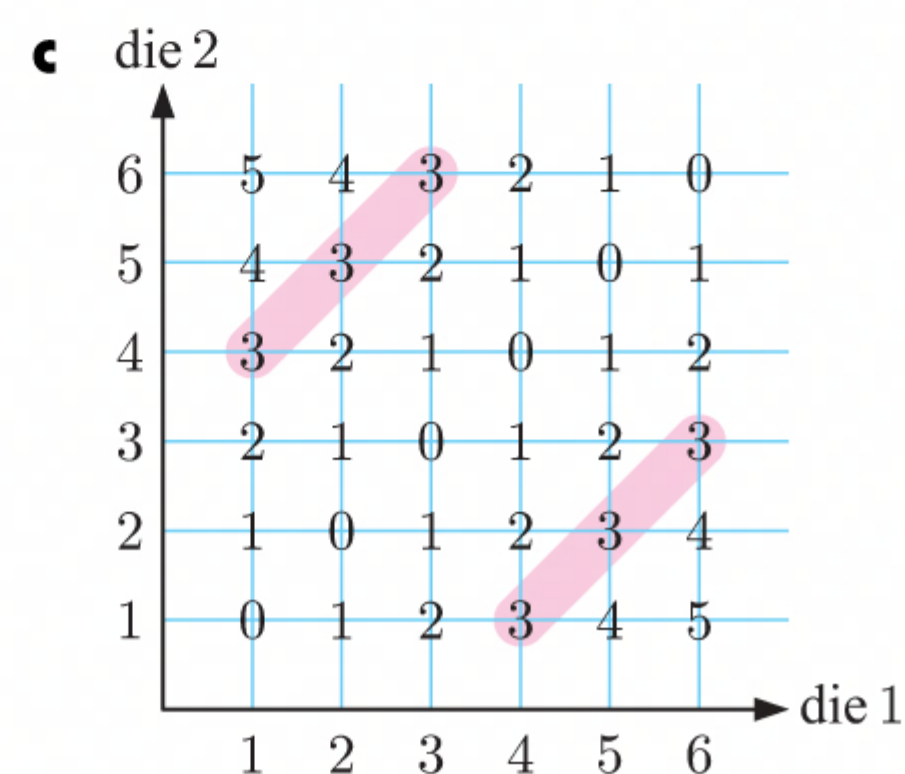
$$p = P(\text{exactly 1 tail}) = \frac{3}{8} \quad \{\text{from a}\}$$

You would expect to see exactly one tail

$$np = 400 \times \frac{3}{8} = 150 \text{ times.}$$

42 a X is the difference of a number from one die and a number from the other die. So X is a discrete random variable because X has a set of distinct possible values.

b $X = 0, 1, 2, 3, 4, 5$



$$\begin{aligned}
 P(X = 3) &= \frac{6}{36} \\
 &= \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{43 a } P(x) &= k(x+3), \quad x = 0, 1, 2, 3, 4 \\
 \therefore P(0) &= 3k, \quad P(1) = 4k, \quad P(2) = 5k, \\
 P(3) &= 6k, \quad P(4) = 7k
 \end{aligned}$$

Since $P(x)$ is a probability mass function,

$$\begin{aligned}
 \sum_{i=1}^n P(x_i) &= 1 \\
 \therefore 3k + 4k + 5k + 6k + 7k &= 1 \\
 \therefore 25k &= 1 \\
 \therefore k &= \frac{1}{25}
 \end{aligned}$$

$$\text{b } P(x) = \frac{k^{x-3}}{x-1}, \quad x = 3, 4, 5$$

$$\begin{aligned}
 \therefore P(3) &= \frac{k^0}{3-1}, \quad P(4) = \frac{k^1}{4-1}, \quad P(5) = \frac{k^2}{5-1} \\
 &= \frac{1}{2} \quad \quad \quad = \frac{k}{3} \quad \quad \quad = \frac{k^2}{4}
 \end{aligned}$$

Since $P(x)$ is a probability mass function,

$$\begin{aligned}
 \sum_{i=1}^n P(x_i) &= 1 \\
 \therefore \frac{1}{2} + \frac{k}{3} + \frac{k^2}{4} &= 1 \\
 \therefore \frac{k^2}{4} + \frac{k}{3} - \frac{1}{2} &= 0 \\
 \therefore 3k^2 + 4k - 6 &= 0 \\
 \therefore k &= \frac{-4 \pm \sqrt{16 + 72}}{2 \times 3} \\
 &= \frac{-4 \pm \sqrt{88}}{6} \\
 &= \frac{-2 + \sqrt{22}}{3} \quad \{k > 0\}
 \end{aligned}$$

44 a $P(x) = \frac{a}{(x-3)^2}, \quad x = 0, 1, 2$

$$\therefore P(0) = \frac{a}{9}, \quad P(1) = \frac{a}{4}, \quad P(2) = a$$

Since $P(x)$ is a probability mass function, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore \frac{a}{9} + \frac{a}{4} + a = 1$$

$$\therefore \frac{49}{36}a = 1$$

$$\therefore a = \frac{36}{49}$$

b $P(X = 2) = P(2)$
 $= a$
 $= \frac{36}{49} \quad \{\text{from a}\}$

c Since $P(X = 2) = \frac{36}{49}$ is the greatest probability, the mode of the distribution is 2.

$$p_1 = \frac{a}{9} = \frac{\frac{36}{49}}{9} = \frac{4}{49} \approx 0.0816$$

$$p_1 + p_2 = \frac{a}{9} + \frac{a}{4} = \frac{4}{49} + \frac{\frac{36}{49}}{4} = \frac{4}{49} + \frac{9}{49} = \frac{13}{49} \approx 0.265$$

Since $p_1 + p_2 + p_3 = 1 \geq 0.5$, the median is 2.

45 a $X = 2, 3, 4$

b Let B represent a blue ticket, and R represent a red ticket.

The possible selections that can be made are:

BR	BBR	RRRB
RB	RRB	↓
($X = 2$)	($X = 3$)	($X = 4$)

So, $P(X = 2) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{3}{5}$

$$P(X = 3) = \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} + \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{3}{10}$$

$$P(X = 4) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{1}{10}$$

\therefore the probability distribution of X is

x	2	3	4
$P(X = x)$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

c It is most likely that 2 tickets are drawn, so the mode is 2.

d $E(X) = \sum_{i=1}^n x_i p_i$
 $= 2\left(\frac{3}{5}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{1}{10}\right)$
 $= \frac{5}{2} = 2.5$

46

x	0	1	2	3
$P(x)$	0.05	k	0.5	0.3

a Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.05 + k + 0.5 + 0.3 = 1$$

$$\therefore k = 0.15$$

b Since $P(X = 2) = P(2) = 0.5$ is the greatest probability, the mode of the distribution is 2.

c $E(X) = \sum_{i=1}^n x_i p_i$
 $= 0(0.05) + 1(0.15) + 2(0.5) + 3(0.3)$
 $= 2.05$ safe hits

47

x	0	1	2	3
$P(X = x)$	0.3	0.2	m	n

Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.3 + 0.2 + m + n = 1$$

$$\therefore n = 0.5 - m \quad \dots (*)$$

Now if $E(X) = 1.55$, then $1.55 = \sum_{i=1}^n x_i p_i$

$$\therefore 1.55 = 0(0.3) + 1(0.2) + 2m + 3n$$

$$\therefore 1.35 = 2m + 3(0.5 - m) \quad \{\text{using } (*)\}$$

$$\therefore 1.35 = 2m + 1.5 - 3m$$

$$\therefore -0.15 = -m$$

$$\therefore m = 0.15$$

Substituting into (*), $n = 0.5 - 0.15 = 0.35$.

$$48 \quad a \quad P(x) = \frac{x^2 + kx}{50}, \quad x = 1, 2, 3, 4$$

$$\therefore P(1) = \frac{k+1}{50}, \quad P(2) = \frac{2k+4}{50}, \quad P(3) = \frac{3k+9}{50}, \quad P(4) = \frac{4k+16}{50}$$

$$\text{Since } P(x) \text{ is a probability mass function, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore \frac{k+1}{50} + \frac{2k+4}{50} + \frac{3k+9}{50} + \frac{4k+16}{50} = 1$$

$$\therefore \frac{10k+30}{50} = 1$$

$$\therefore 10k+30 = 50$$

$$\therefore 10k = 20$$

$$\therefore k = 2$$

$$b \quad \mu = E(X)$$

$$= \sum_{i=1}^n x_i p_i$$

$$= 1\left(\frac{3}{50}\right) + 2\left(\frac{8}{50}\right) + 3\left(\frac{15}{50}\right) + 4\left(\frac{24}{50}\right)$$

$$= 3.2$$

$$c \quad P(X \geq 2) = 1 - P(X = 1)$$

$$= 1 - P(1)$$

$$= 1 - \frac{3}{50}$$

$$= \frac{47}{50}$$

49 a Let X be the return from each game.

$$\begin{aligned} E(X) &= 40\left(\frac{1}{12}\right) + 20\left(\frac{3}{12}\right) + 5\left(\frac{8}{12}\right) \\ &= \frac{40 + 60 + 40}{12} \\ &= \frac{140}{12} \\ &\approx \$11.67 \end{aligned}$$

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{8}{12}$

b The expected return per game is \$11.67. It costs \$15 to play.

So, the expected gain $\approx \$11.67 - \$15 \approx -\$3.33$

It is not advisable to play this game many times as the player can expect to lose \$3.33 on average per game.

c Let k be the number of extra red tickets added to the bag.

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12+k}$	$\frac{k+3}{12+k}$	$\frac{8}{12+k}$

For the game to be fair, the expected return must equal the cost of each game.

$$\therefore E(X) = 40\left(\frac{1}{12+k}\right) + 20\left(\frac{k+3}{12+k}\right) + 5\left(\frac{8}{12+k}\right) = 15 \quad \{\text{the cost of the game is \$15}\}$$

$$\therefore \frac{40}{12+k} + \frac{20k+60}{12+k} + \frac{40}{12+k} = 15$$

$$\therefore \frac{20k+140}{12+k} = 15$$

$$\therefore 20k+140 = 15(12+k)$$

$$\therefore 20k+140 = 180+15k$$

$$\therefore 5k = 40$$

$$\therefore k = 8$$

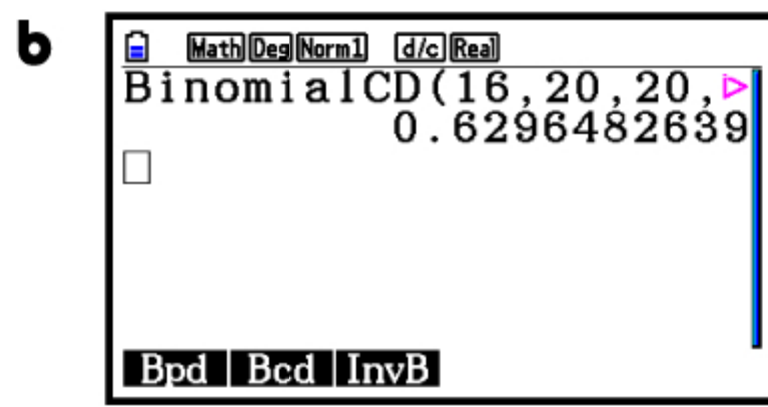
So, 8 extra red tickets should be added to the bag to make the game fair.

50 Let X be the number of residents who oppose the construction.

$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 80\% = 0.8$

$$\therefore X \sim B(20, 0.8)$$

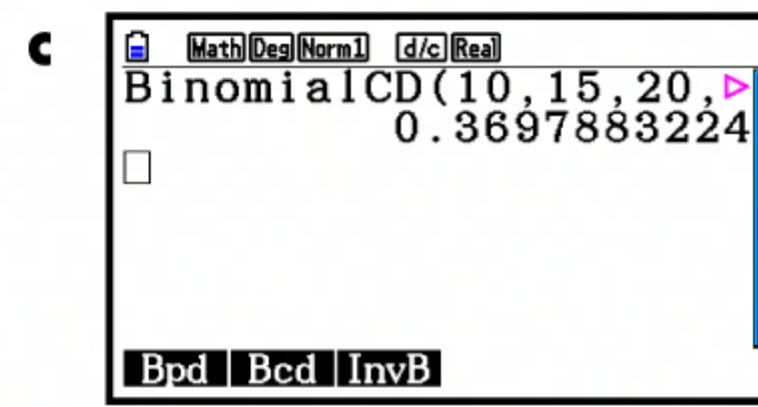
$$\begin{aligned} a \quad P(X = 16) &= \binom{20}{16} (0.8)^{16} (0.2)^4 \\ &\approx 0.218 \end{aligned}$$



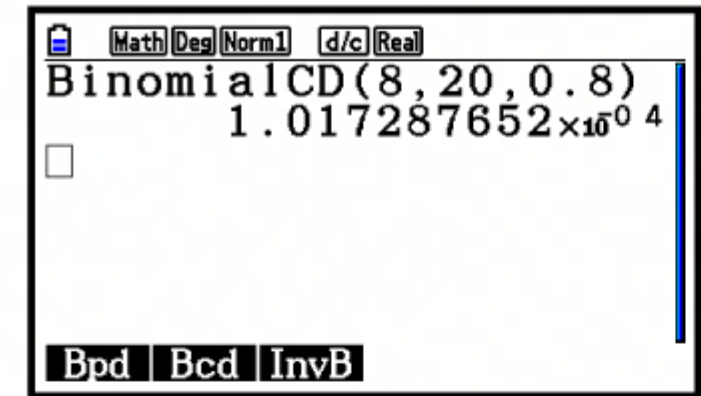
$$P(X \geq 16) \approx 0.630$$

- d** If more than 8 residents support the construction, then 8 or fewer residents oppose the construction.

$$P(X \leq 8) \approx 0.000\,102$$



$$P(10 \leq X \leq 15) \approx 0.370$$



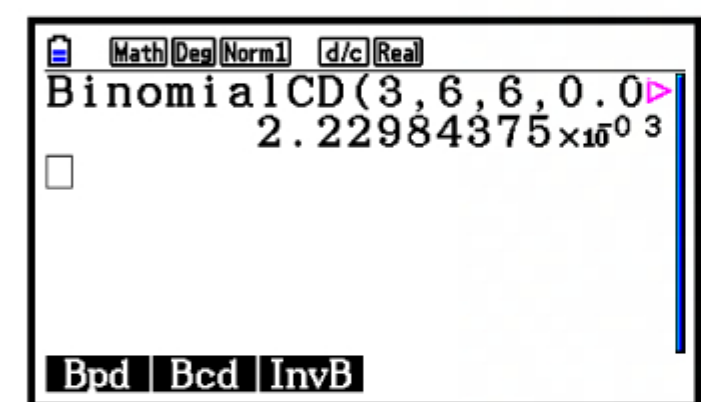
- 51 a** Let X be the number of defective items.

$n = 6$, so $X = 0, 1, 2, 3, 4, 5$, or 6 , and $p = 5\% = 0.05$

$$\therefore X \sim B(6, 0.05)$$

Using technology, $P(X > 2) = P(X \geq 3)$
 $\approx 0.002\,23$

\therefore the manufacturer will have to pay a refund on about $0.002\,23 \times 100\% \approx 0.223\%$ of boxes.



- b** Let Y be the number of boxes refunded.

$n = 10$, so $Y = 0, 1, 2, 3, \dots$, or 10 , and $p \approx 0.002\,23$ {from **a**}

$$\therefore Y \sim B(10, 0.002\,23)$$

$$P(Y = 1) \approx \binom{10}{1} (0.002\,23)^1 (1 - 0.002\,23)^9$$

$$\approx 0.0219$$

So, the probability that Patrick will get a refund for exactly 1 box is about 0.0219.

- 52** Let X be the number of germinations in one row.

$n = 10$, so $X = 0, 1, 2, 3, \dots$, or 10 , and $p = \frac{1}{2}$

$$\therefore X \sim B(10, \frac{1}{2})$$

$$P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10)$$

$$= \binom{10}{8} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + \binom{10}{9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + \binom{10}{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= \frac{7}{128}$$

So, the probability that at least 8 seeds germinate in one row is $\frac{7}{128}$.

Let Y be the number of rows with at least 8 seeds germinating.

$n = 10$, so $Y = 0, 1, 2, 3, \dots$, or 10 , and $p = \frac{7}{128}$

$$\therefore Y \sim B(10, \frac{7}{128})$$

$$P(Y \geq 1) = 1 - P(Y = 0)$$

$$= 1 - \binom{10}{0} \left(\frac{7}{128}\right)^0 \left(\frac{121}{128}\right)^{10}$$

$$\approx 0.430$$

So, the probability that the row with the maximum number of germinations contains at least 8 seedlings is about 0.430.

53

Score	1	2	3	4
Probability	$\frac{1}{12}$	k	$\frac{1}{4}$	$\frac{1}{3}$

- a** Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore \frac{1}{12} + k + \frac{1}{4} + \frac{1}{3} = 1$$

$$\therefore k = \frac{1}{3}$$

b Let X be the number of 2s rolled.

$n = 2400$, so $X = 0, 1, 2, 3, \dots$, or 2400 , and $p = \frac{1}{3}$ {from **a**}

$\therefore X \sim B(2400, \frac{1}{3})$

$$\begin{aligned} \text{So, } \mu &= np & \text{and } \sigma &= \sqrt{np(1-p)} \\ &= 2400 \times \frac{1}{3} & &= \sqrt{2400 \times \frac{1}{3} \times \frac{2}{3}} \\ &= 800 & &= \sqrt{\frac{1600}{3}} \\ & & &= \frac{40}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ & & &= \frac{40\sqrt{3}}{3} \end{aligned}$$

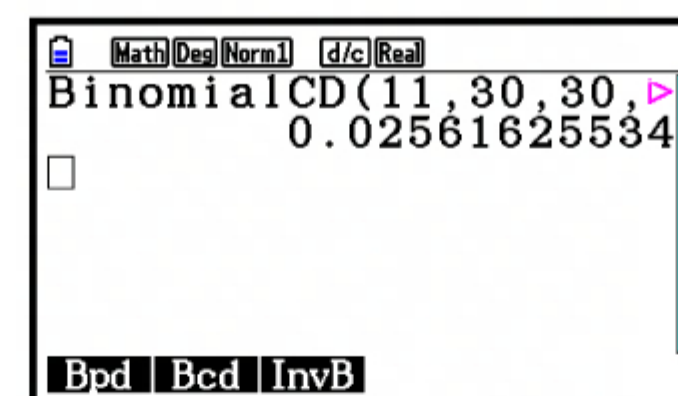
54 $Y \sim B(30, \frac{1}{5})$

$$\begin{aligned} \text{a } \mu &= np & \text{and } \sigma &= \sqrt{np(1-p)} \\ &= 30 \times \frac{1}{5} & &= \sqrt{30 \times \frac{1}{5} \times \frac{4}{5}} \\ &= 6 & &= \sqrt{\frac{24}{5}} \\ & & &\approx 2.19 \end{aligned}$$

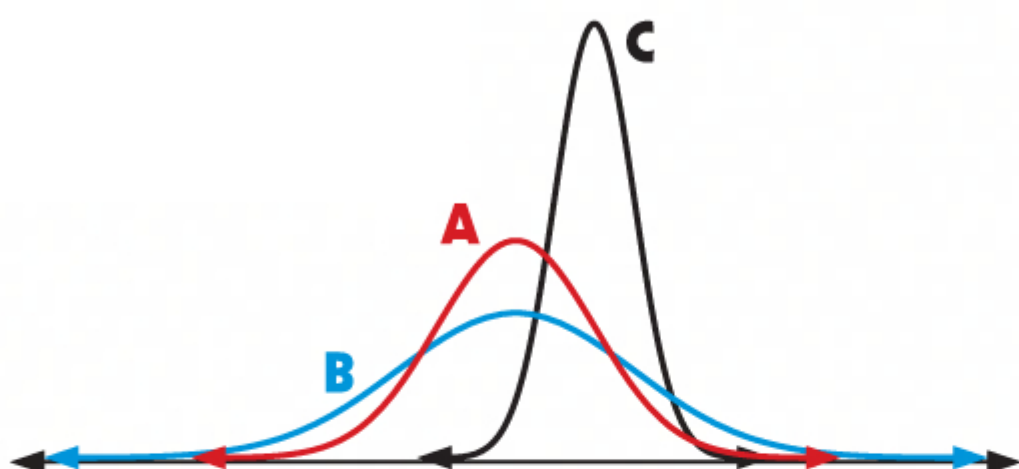
$$\begin{aligned} \text{b } P(Y = 20) &= \binom{30}{20} \left(\frac{1}{5}\right)^{20} \left(\frac{4}{5}\right)^{10} \\ &\approx 3.38 \times 10^{-8} \end{aligned}$$

$$\begin{aligned} \text{c } P(Y \geq \mu + 2\sigma) &= P\left(Y \geq 6 + 2\sqrt{\frac{24}{5}}\right) \quad \{\text{from a}\} \\ &= P(Y \geq 10.38) \\ &= P(Y \geq 11) \\ &\approx 0.0256 \end{aligned}$$

{using technology}



55



A and **B** both have $\mu = 2$, **C** has $\mu = 4$.

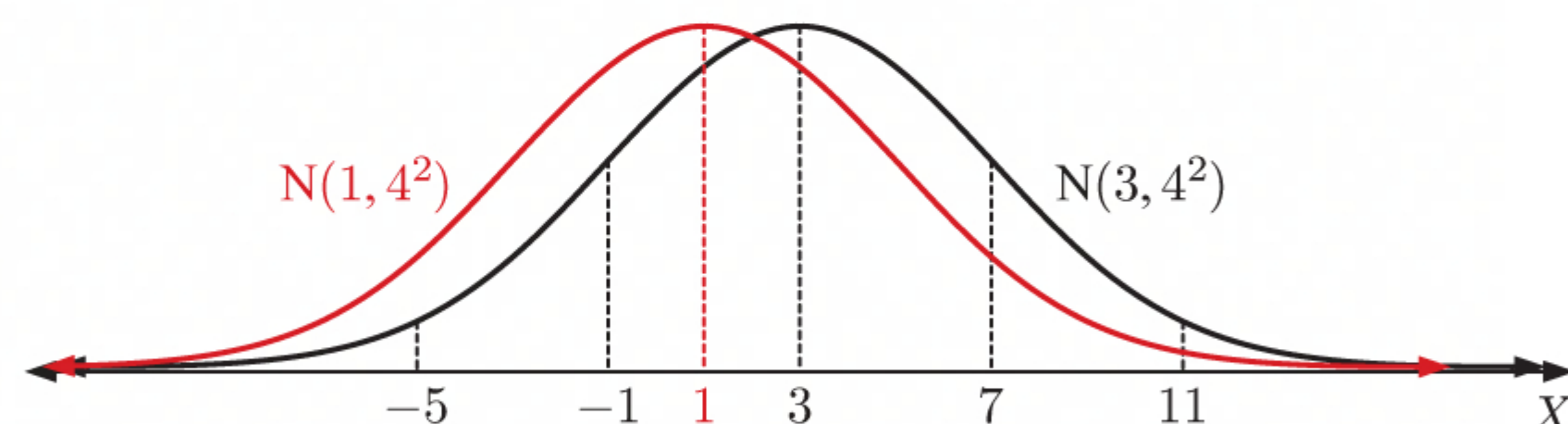
B has a greater spread, and hence a larger standard deviation than **A**.

a $\mu = 4$, $\sigma = 1$ corresponds to **C**

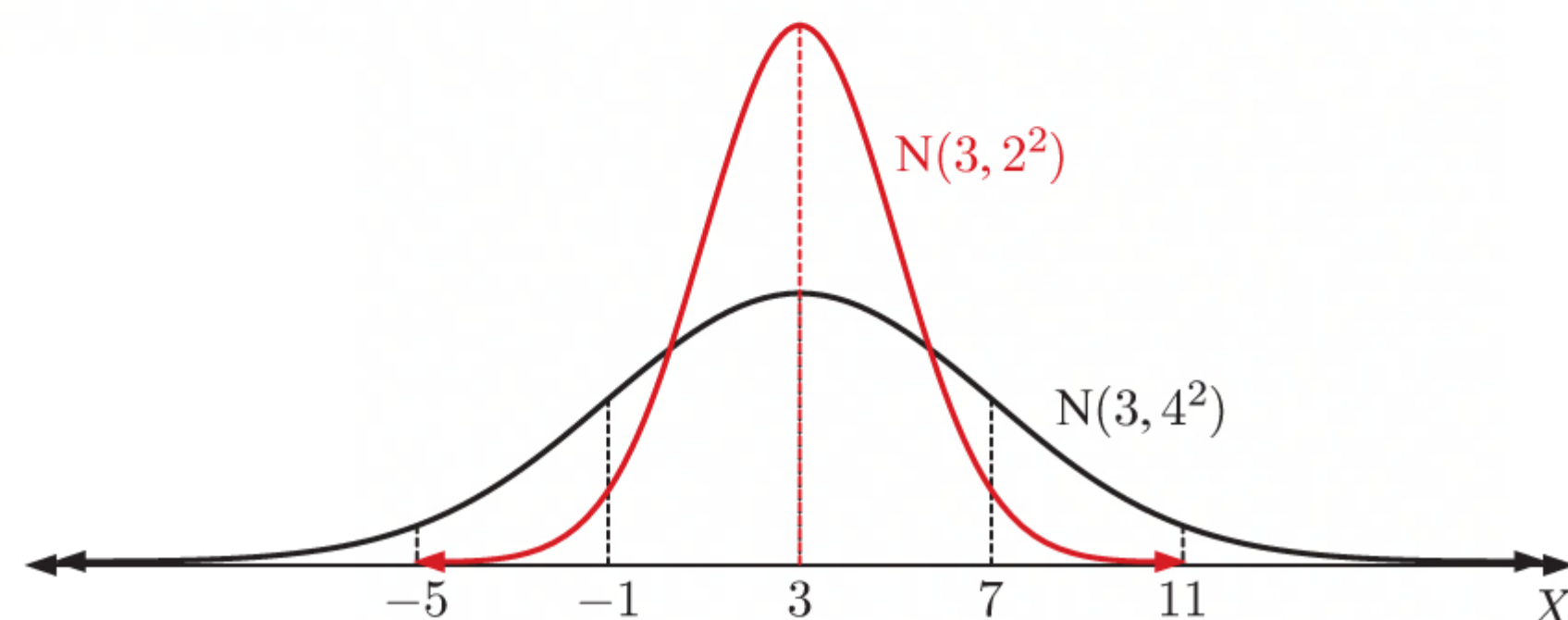
b $\mu = 2$, $\sigma = 2$ corresponds to **A**

c $\mu = 2$, $\sigma = 3$ corresponds to **B**

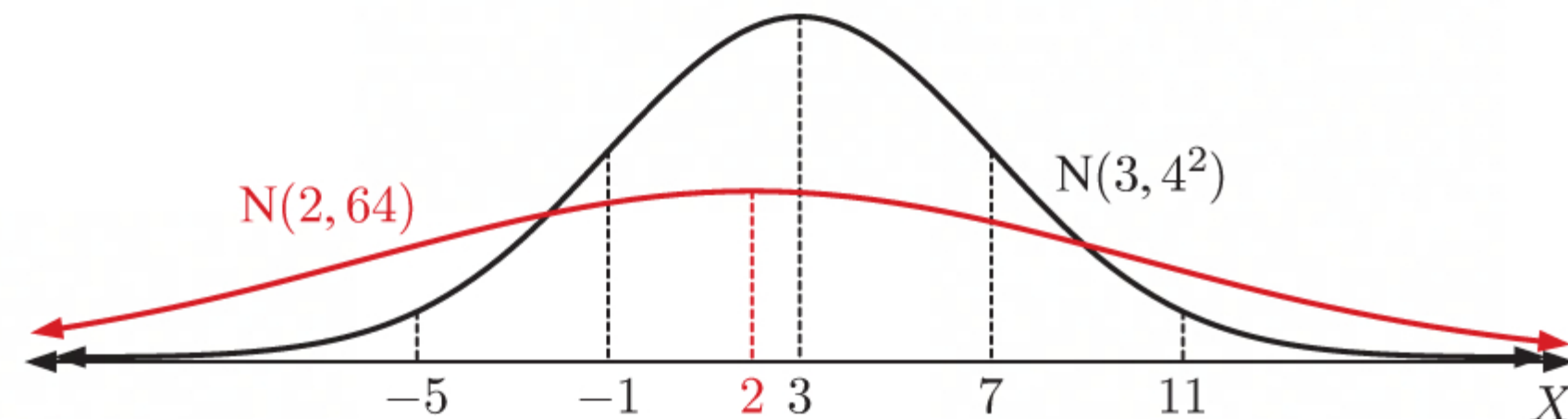
56 **a**



b



c



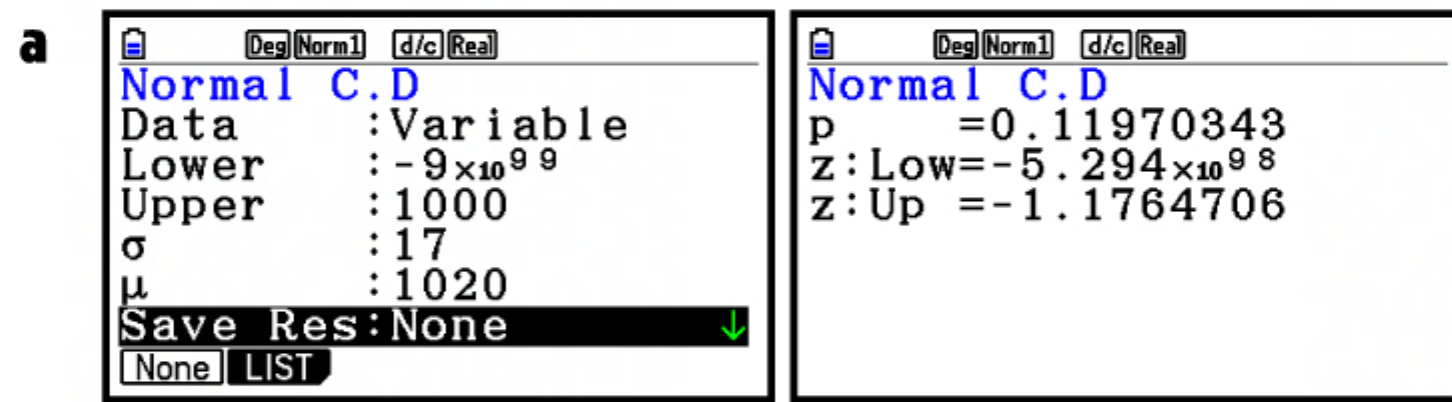
57 **a** Approximately 68% of the population lies between 25 and 35.

b Approximately 95% of the population lies between 20 and 40.

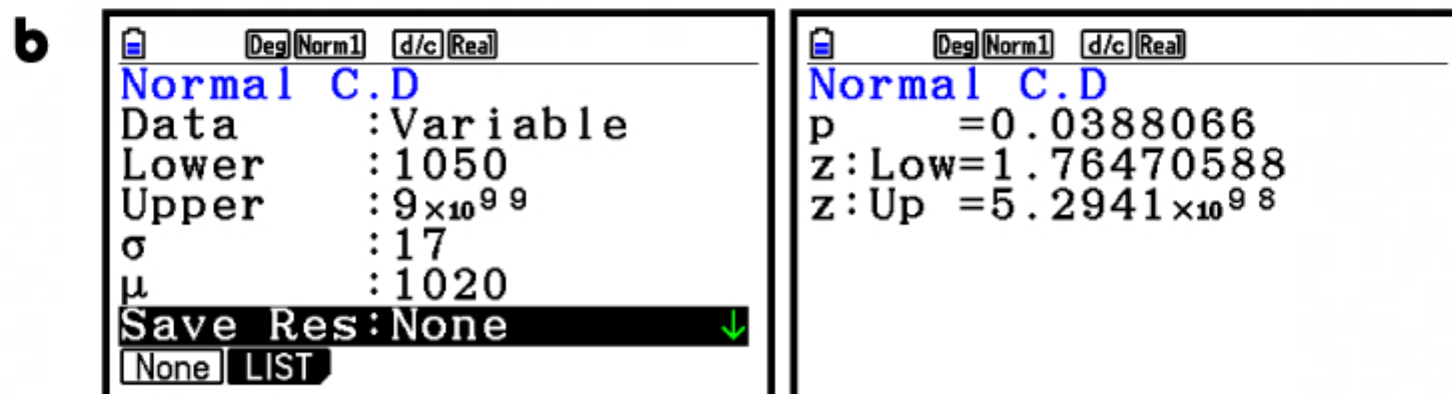
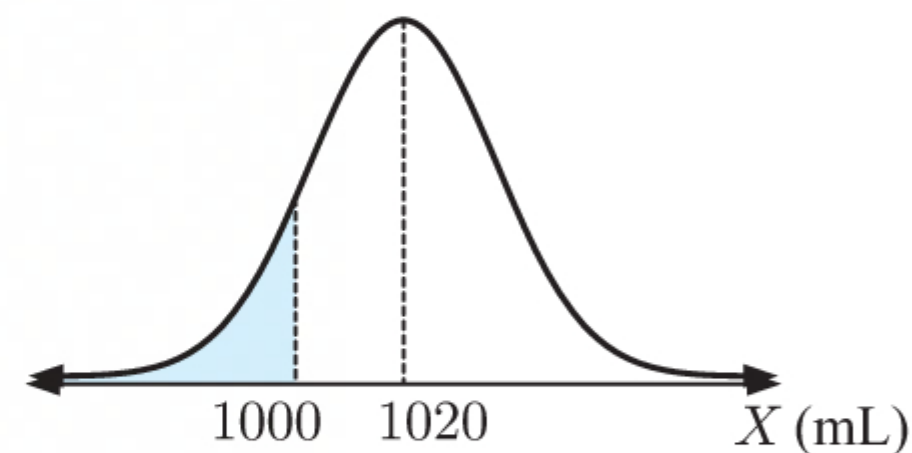
c Approximately 99.7% of the population lies between 15 and 45.

58 Let the capacity of a randomly selected container be X mL.

So, $X \sim N(1020, 17^2)$.

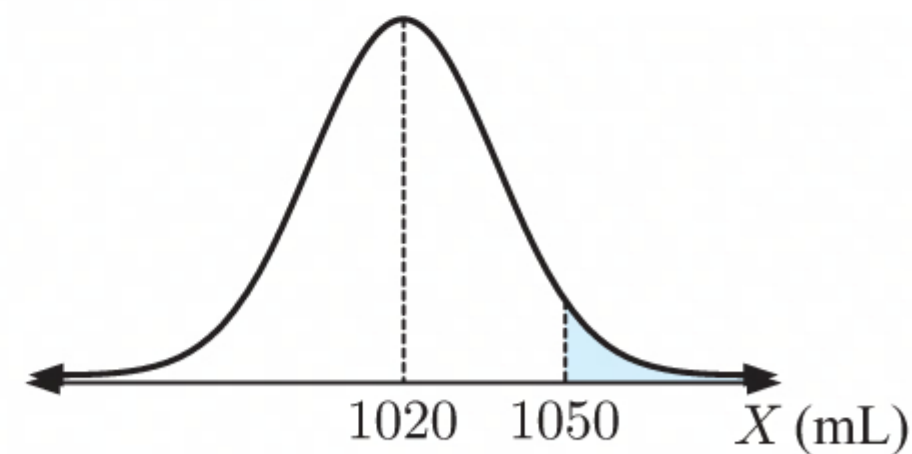


$$P(X \leq 1000) \approx 0.120$$



$$P(X \geq 1050) \approx 0.0388$$

\therefore about 3.88% of containers overflow.

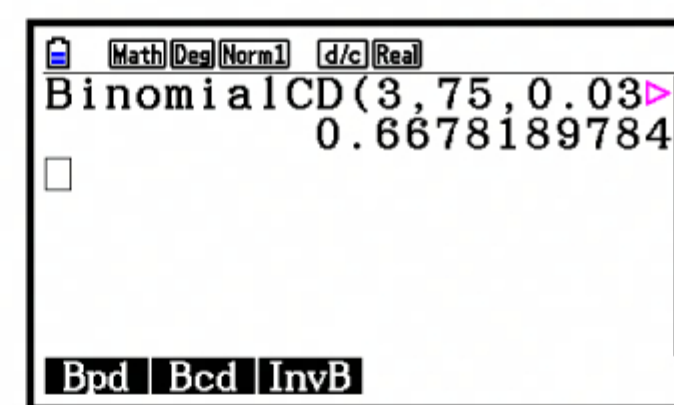


c Let Y be the number of containers which overflow.

$n = 75$, so $Y = 0, 1, 2, 3, \dots$, or 75 and $p \approx 0.0388$ {from **b**}

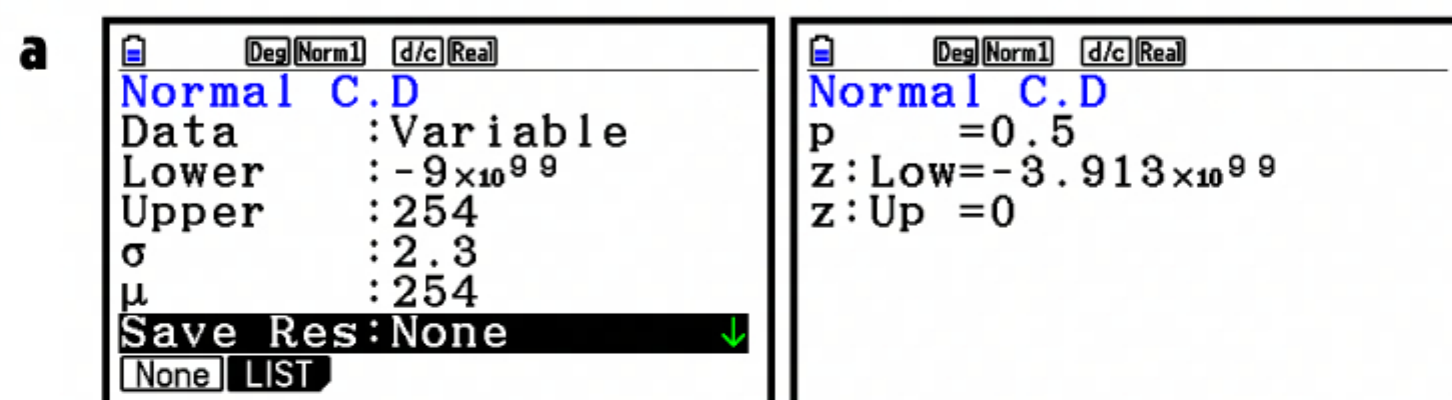
$\therefore Y \sim B(75, 0.0388)$

Using technology, $P(Y \leq 3) \approx 0.668$

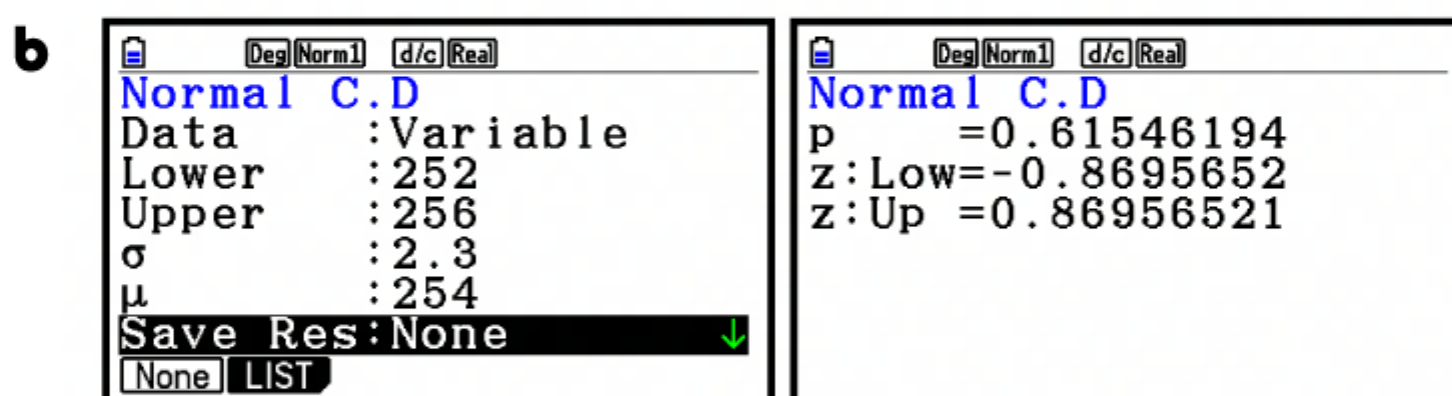
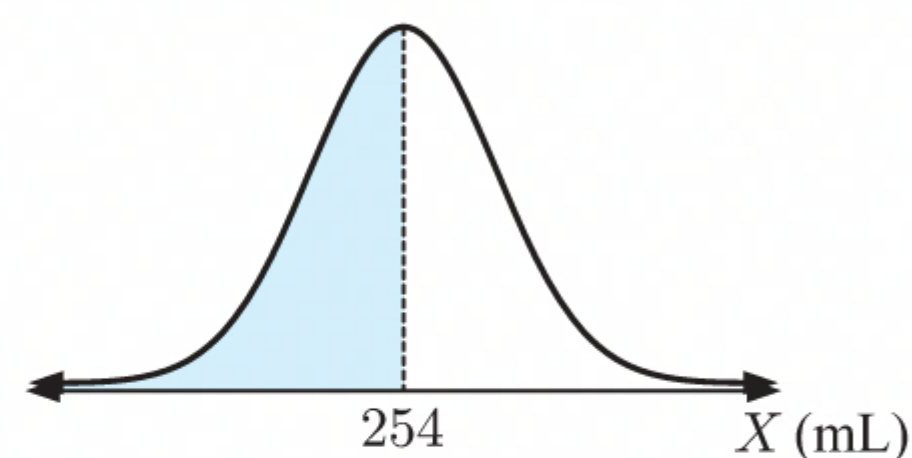


59 Let the volume of a randomly selected drink be X mL.

So, $X \sim N(254, (2.3)^2)$.

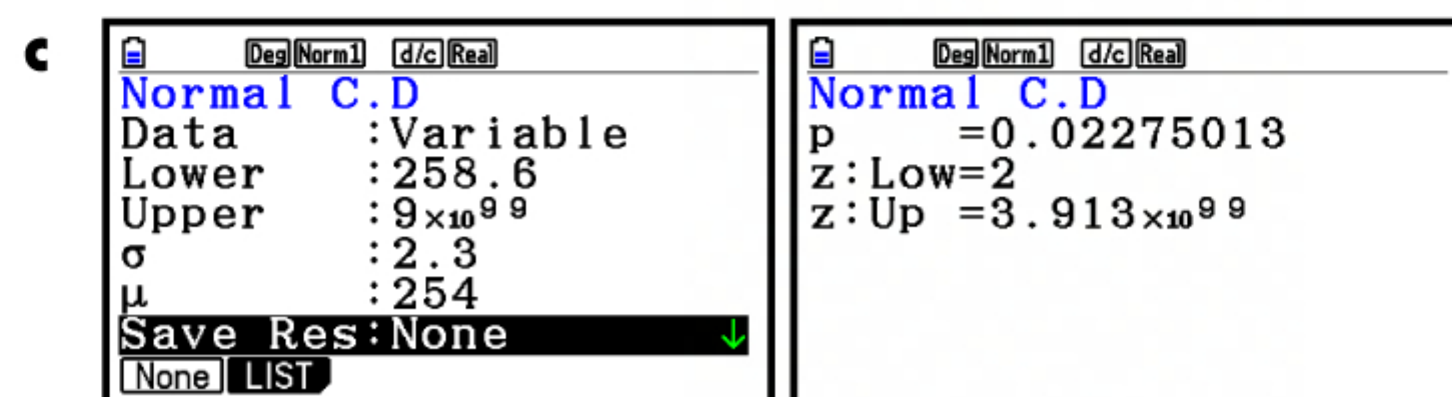
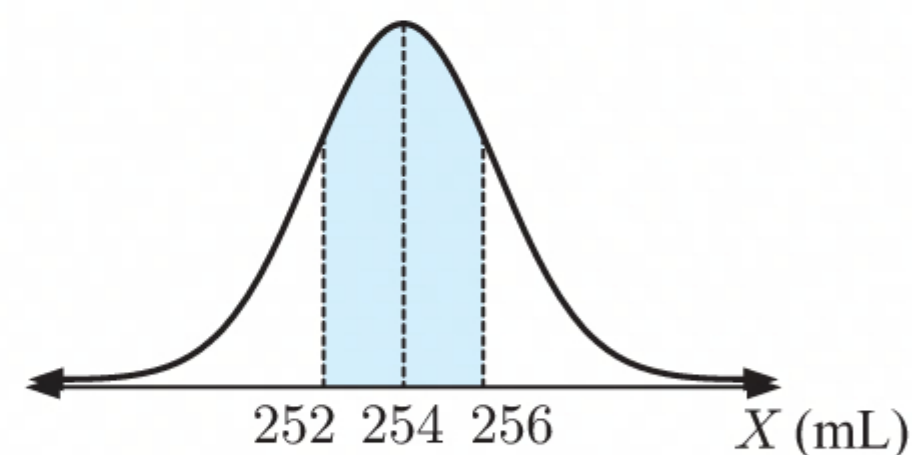


$$P(X < 254) = 0.5$$

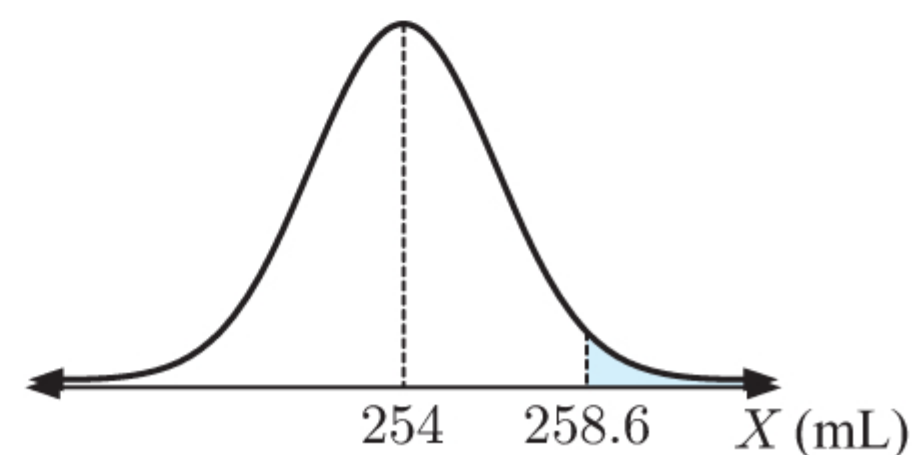


$$P(252 \leq X \leq 256) \approx 0.615$$

\therefore about 61.5% of drinks dispensed by the machine have volume between 252 mL and 256 mL.

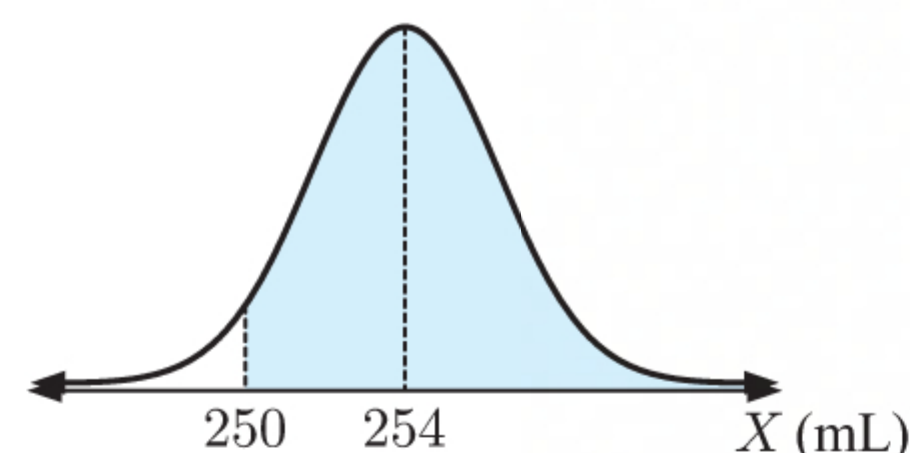


$$P(X \geq 254 + 2 \times 2.3) = P(X \geq 258.6) \approx 0.0228$$



\therefore we expect about $0.0228 \times 80 \approx 2$ drinks to have volume at least two standard deviations above the mean.

d i	<pre> Normal C.D Data :Variable Lower :250 Upper :9×10⁹⁹ σ :2.3 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.95899408 z:Low=-1.7391304 z:Up =3.913×10⁹⁹ </pre>
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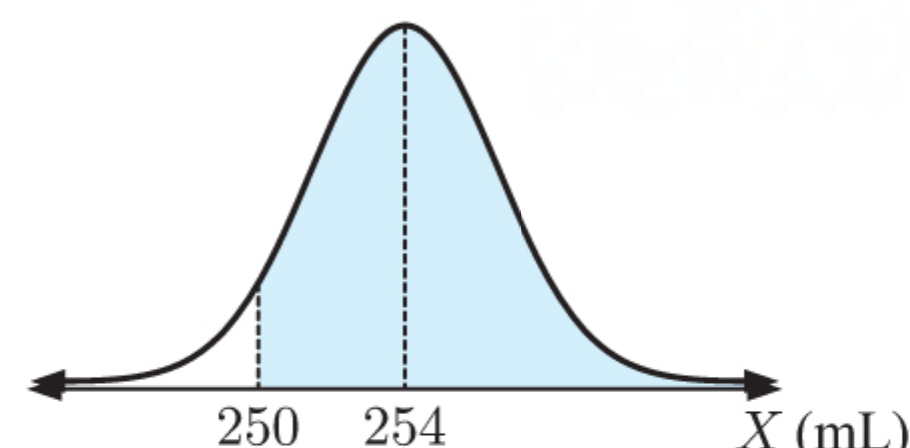


$$P(X \geq 250) \approx 0.959$$

\therefore the operator's guarantee that at least 95% of drinks will have volume at least 250 mL is valid.

ii Suppose $X \sim N(254, (2.5)^2)$.

<pre> Normal C.D Data :Variable Lower :250 Upper :9×10⁹⁹ σ :2.5 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.9452007 z:Low=-1.6 z:Up =3.6×10⁹⁹ </pre>
--	---



$$P(X \geq 250) \approx 0.945$$

\therefore about 94.5% of drinks will have volume at least 250 mL.

So, the operator's guarantee is no longer valid.

60 a Let the volume of sauce in a randomly selected bottle be X mL.

$$X \sim N(500, (2.5)^2)$$

$$\therefore P(X < 495) \approx 0.022750 \approx 0.0228$$

<pre> Math Deg Norm1 d/c Real NormCD(-9×10⁹⁹,495,2. 0.02275013195 </pre>

b Let Y be the number of bottles which require extra sauce.

$$Y \sim B(200, 0.022750) \quad \{\text{from a}\}$$

$$\therefore P(Y \geq 8) \approx 0.0884$$

<pre> Math Deg Norm1 d/c Real NormCD(-9×10⁹⁹,495,2. 0.02275013195 BinomialCD(8,200,200 0.08837407712 </pre>
--

61 a Let the completion time of a randomly selected run be X seconds.

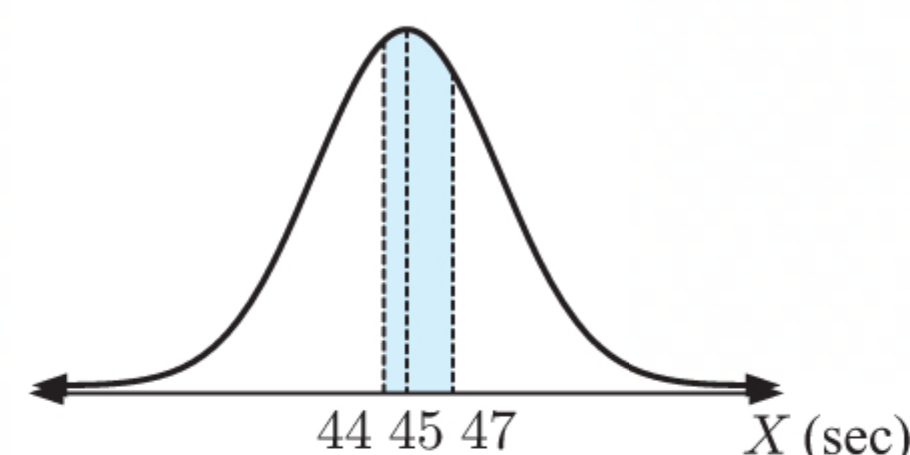
$$X \sim N(45, 4^2)$$

i $P(X < 40) \approx 0.106$

ii $P(\text{two consecutive runs under 40 seconds})$
 $= [P(X < 40)]^2$
 ≈ 0.0112

<pre> Math Deg Norm1 d/c Real NormCD(-9×10⁹⁹,40,4,4 0.1056497737 Ans² 0.01116187468 </pre>
--

b	<pre> Normal C.D Data :Variable Lower :44 Upper :47 σ :4 μ :45 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.29016878 z:Low=-0.25 z:Up =0.5 </pre>
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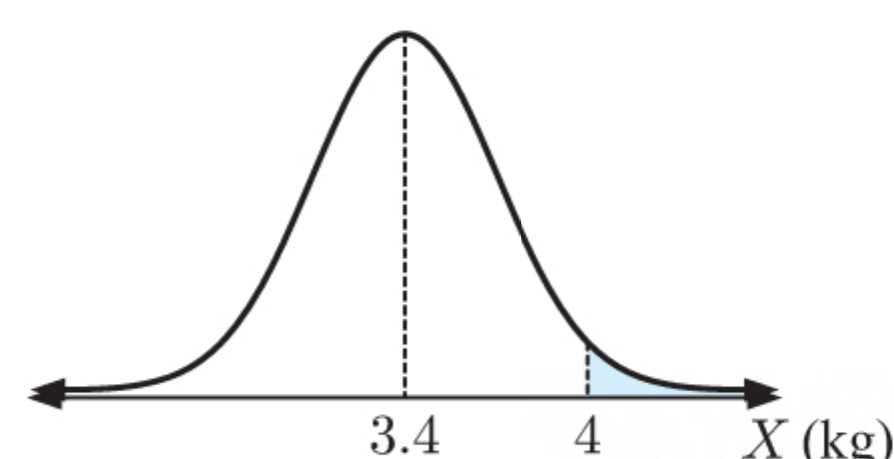
$$P(44 \leq X \leq 47) \approx 0.290$$

\therefore we expect about $0.290 \times 60 \approx 17$ runs to take between 44 seconds and 47 seconds.

62 Let the birth weight of a randomly selected baby be X kg.

$$\text{So, } X \sim N(3.4, (0.3)^2).$$

a i	<pre> Normal C.D Data :Variable Lower :4 Upper :9×10⁹⁹ σ :0.3 μ :3.4 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.02275013 z:Low=2 z:Up =9.99×10⁹⁹ </pre>
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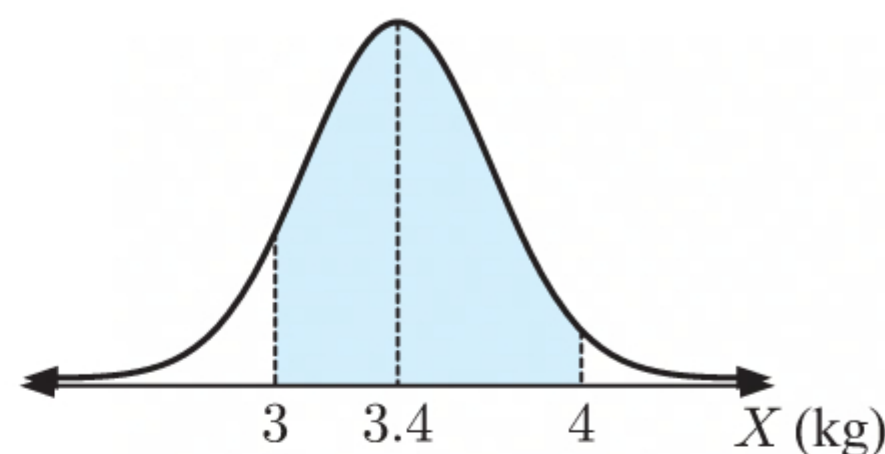
$$P(X > 4) \approx 0.0228$$

\therefore about 2.28% of babies have birth weights in excess of 4 kg.

ii	<div> <div> Normal C.D Data : Variable Lower : 3 Upper : 4 σ : 0.3 μ : 3.4 Save Res: None [None] LIST </div> <div> Normal C.D p = 0.88603864 z: Low = -1.3333333 z: Up = 2 </div> </div>
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$$P(3 \leq X \leq 4) \approx 0.886$$

∴ about 88.6% of babies have birth weights between 3 kg and 4 kg.

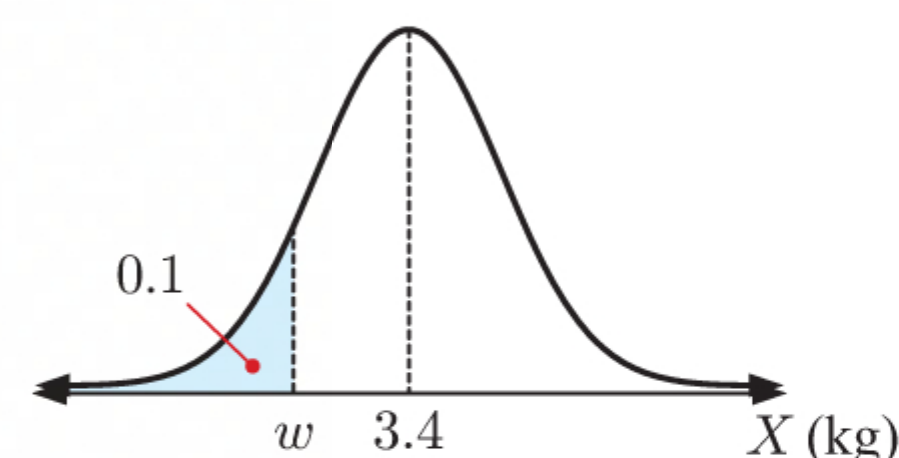


b	<div> <div> Inverse Normal Data : Variable Tail : Left Area : 0.1 σ : 0.3 μ : 3.4 Save Res: None [None] LIST </div> <div> Inverse Normal xInv = 3.01553453 </div> </div>
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$$P(X \leq w) = 0.1$$

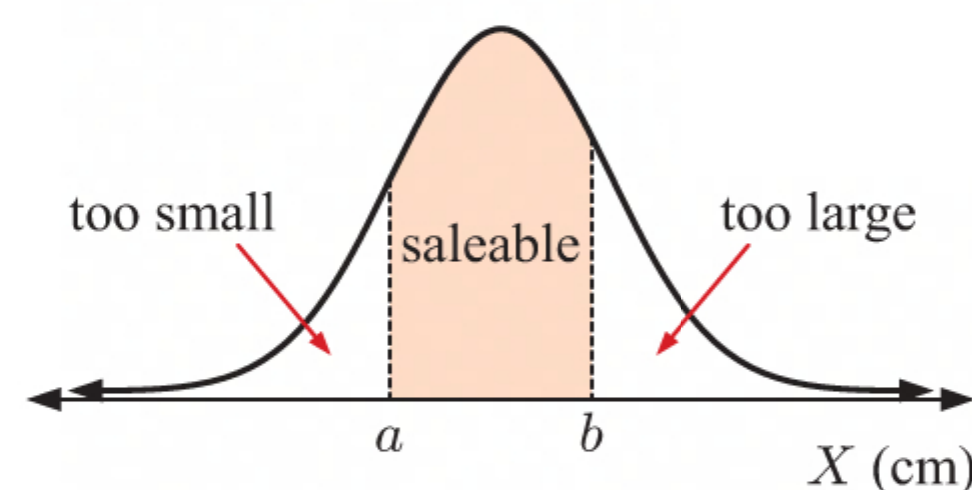
$$\therefore w \approx 3.02$$

∴ the weight below which a baby is classified as having *low birth weight* is about 3.02 kg.



63 Let the length of a randomly selected zucchini be X cm.

So, $X \sim N(24.3, (6.83)^2)$.



a 15% of zucchinis are too small.

$$P(X < a) = 0.15$$

$$\therefore a \approx 17.2$$

<div> <div> Inverse Normal Data : Variable Tail : Left Area : 0.15 σ : 6.83 μ : 24.3 Save Res: None [None] LIST </div> <div> Inverse Normal xInv = 17.2211599 </div> </div>	
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20% of zucchinis are too large.

$$P(X > b) = 0.2$$

$$\therefore P(X \leq b) = 0.8$$

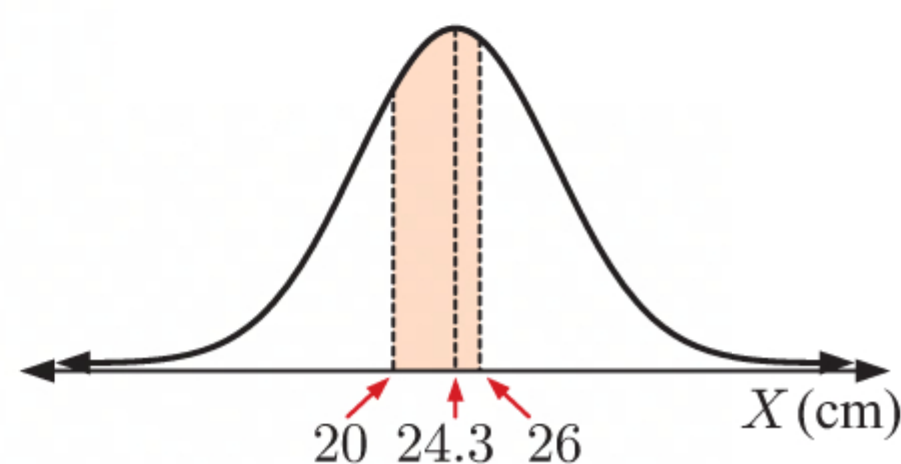
$$\therefore b \approx 30.0$$

<div> <div> Inverse Normal Data : Variable Tail : Left Area : 0.8 σ : 6.83 μ : 24.3 Save Res: None [None] LIST </div> <div> Inverse Normal xInv = 30.048273 </div> </div>	
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b i $P(\text{saleable length}) = 1 - 0.2 - 0.15$
 $= 0.65$

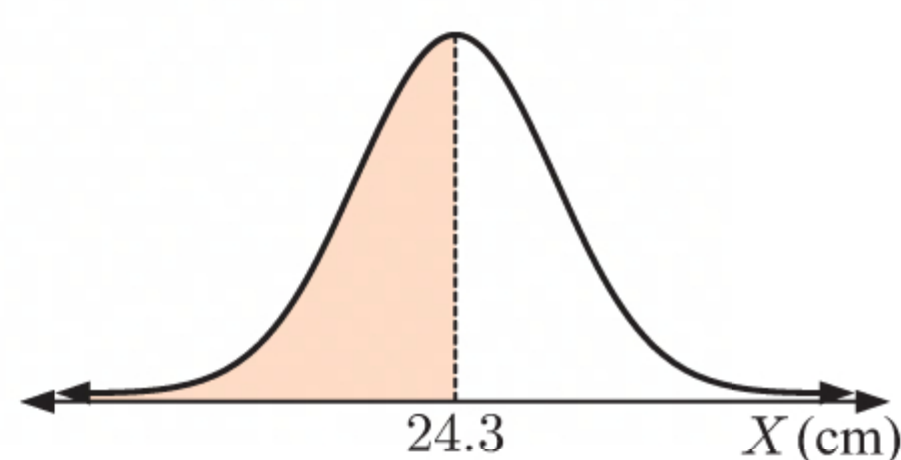
ii	<div> <div> Normal C.D Data : Variable Lower : 20 Upper : 26 σ : 6.83 μ : 24.3 Save Res: None [None] LIST </div> <div> Normal C.D p = 0.33379545 z: Low = -0.6295754 z: Up = 0.2489019 </div> </div>
----	--

$$P(20 \leq X \leq 26) \approx 0.334$$



iii	<div> <div> Normal C.D Data : Variable Lower : -9×10^9 Upper : 24.3 σ : 6.83 μ : 24.3 Save Res: None [None] LIST </div> <div> Normal C.D p = 0.5 z: Low = -1.318×10^9 z: Up = 0 </div> </div>
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$$P(X < 24.3) = 0.5$$

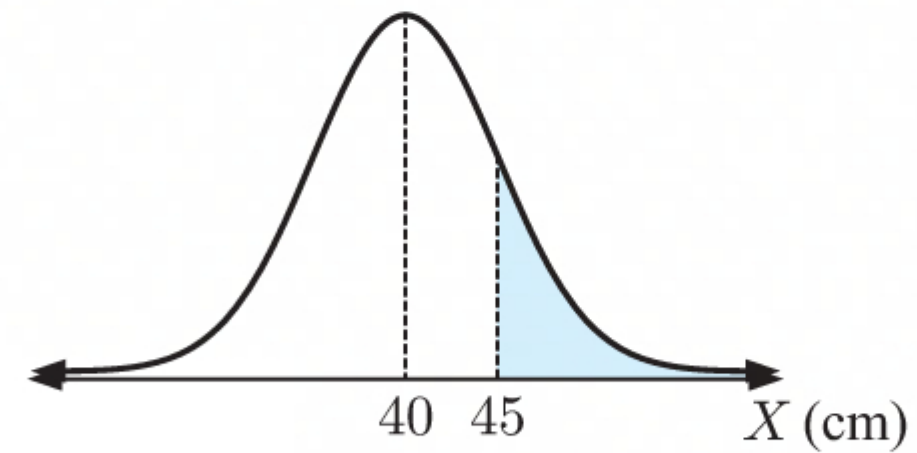


64 Let the length of a randomly selected adult fish be X cm.

So, $X \sim N(40, 5^2)$.

a i

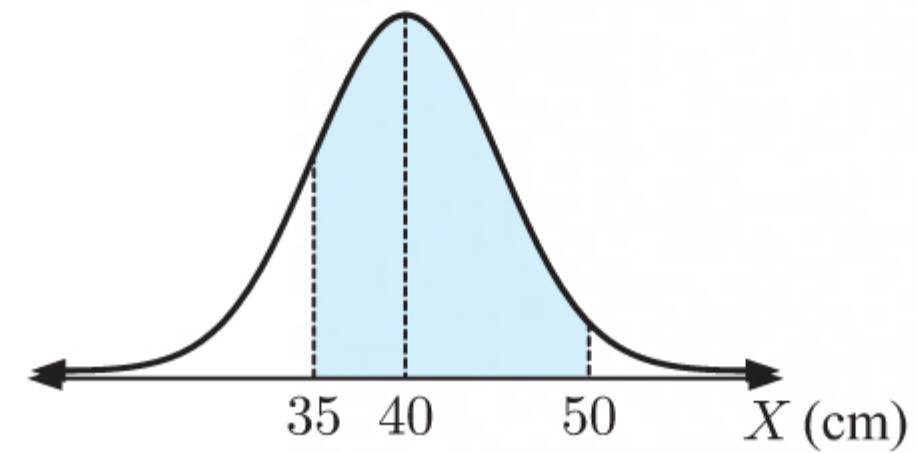
Deg Norm1 d/c Real Normal C.D Data : Variable Lower : 45 Upper : 9×10^9 σ : 5 μ : 40 Save Res: None None LIST	Deg Norm1 d/c Real Normal C.D p = 0.15865525 z: Low = 1 z: Up = 1.8×10^9
--	--



$$P(X > 45) \approx 0.159$$

ii

Deg Norm1 d/c Real Normal C.D Data : Variable Lower : 35 Upper : 50 σ : 5 μ : 40 Save Res: None None LIST	Deg Norm1 d/c Real Normal C.D p = 0.81859461 z: Low = -1 z: Up = 2
---	---

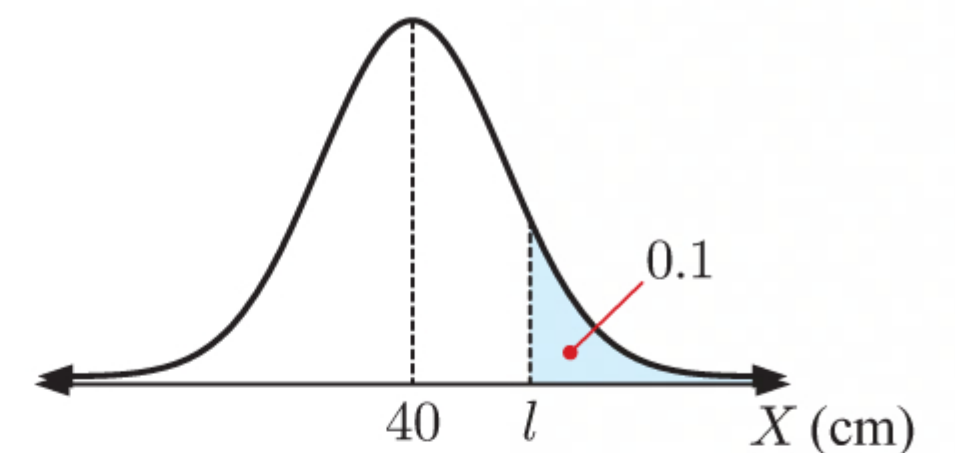


$$P(35 \leq X \leq 50) \approx 0.819$$

b

Deg Norm1 d/c Real Inverse Normal Data : Variable Tail : Left Area : 0.9 σ : 5 μ : 40 Save Res: None None LIST	Deg Norm1 d/c Real Inverse Normal xInv = 46.4077578
--	--

$$\begin{aligned}
 P(X > l) &= 0.1 \\
 \therefore P(X \leq l) &= 0.9 \\
 \therefore l &= 46.4 \\
 \therefore \text{the minimum length of the longest 10\% of fish is about 46.4 cm.}
 \end{aligned}$$



c

Deg Norm1 d/c Real Normal C.D Data : Variable Lower : 40 Upper : 44 σ : 5 μ : 40 Save Res: None None LIST	Deg Norm1 d/c Real Normal C.D p = 0.2881446 z: Low = 0 z: Up = 0.8	Deg Norm1 d/c Real Normal C.D Data : Variable Lower : -9×10^9 Upper : 48 σ : 5 μ : 40 Save Res: None None LIST	Deg Norm1 d/c Real Normal C.D p = 0.9452007 z: Low = -1.8×10^9 z: Up = 1.6
---	---	---	--

$$\begin{aligned}
 P(40 \leq X \leq 44 \mid X < 48) &= \frac{P((40 \leq X \leq 44) \cap (X < 48))}{P(X < 48)} \\
 &= \frac{P(40 \leq X \leq 44)}{P(X < 48)} \\
 &\approx \frac{0.288}{0.945} \\
 &\approx 0.305
 \end{aligned}$$

65 $X \sim N(\mu, \sigma^2)$

a

$$\begin{aligned}
 P(X < 56) &= 0.8 \\
 \therefore P\left(\frac{X - \mu}{\sigma} < \frac{56 - \mu}{\sigma}\right) &= 0.8 \\
 \therefore P\left(Z < \frac{56 - \mu}{\sigma}\right) &= 0.8 \quad \left\{Z = \frac{X - \mu}{\sigma}\right\} \\
 \therefore \frac{56 - \mu}{\sigma} &\approx 0.842 \quad \{Z \sim N(0, 1^2)\} \\
 \therefore 56 - \mu &\approx 0.842\sigma \quad \dots (*)
 \end{aligned}$$

Math Deg Norm1 d/c Real InvNormCD(0.8, 1, 0) 0.8416212336 Npd Ncd InvN

So, a score of 56 is about 0.842 standard deviations from the mean.

b If $\sigma = 4$, then (*) gives

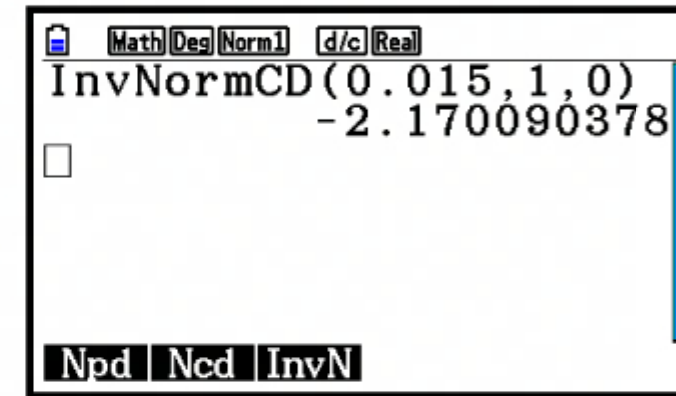
$$\begin{aligned}
 56 - \mu &\approx 0.842 \times 4 \\
 \therefore \mu &\approx 56 - 0.842 \times 4 \\
 \therefore \mu &\approx 52.6
 \end{aligned}$$

66 Let the length of a randomly selected steel rod be X cm.

So, $X \sim N(13.8, \sigma^2)$.

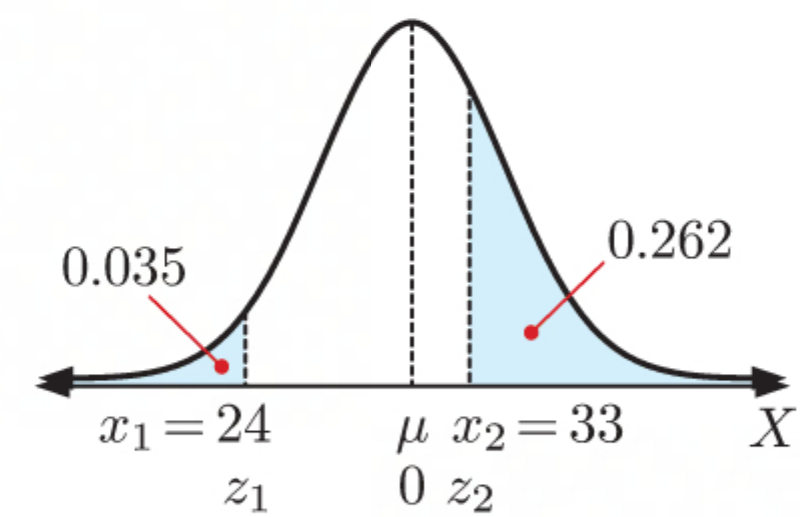
$$\begin{aligned} \mathbf{a} \quad P(13.2 < X < 13.8) &= P(X < 13.8) - P(X < 13.2) \\ &= 0.5 - 0.015 \\ &= 0.485 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(X < 13.2) &= 0.015 \\ \therefore P\left(\frac{X - 13.8}{\sigma} < \frac{13.2 - 13.8}{\sigma}\right) &= 0.015 \\ \therefore P\left(Z < \frac{-0.6}{\sigma}\right) &= 0.015 \quad \left\{Z = \frac{X - 13.8}{\sigma}\right\} \\ \therefore \frac{-0.6}{\sigma} &\approx -2.17 \quad \{Z \sim N(0, 1^2)\} \\ \therefore \sigma &\approx \frac{-0.6}{-2.17} \\ \therefore \sigma &\approx 0.276 \end{aligned}$$

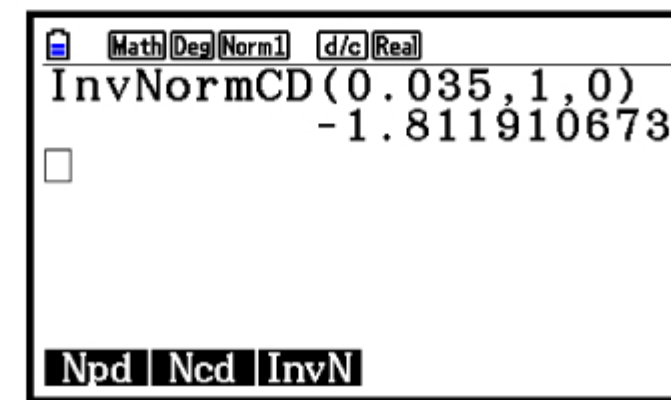


67 $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .

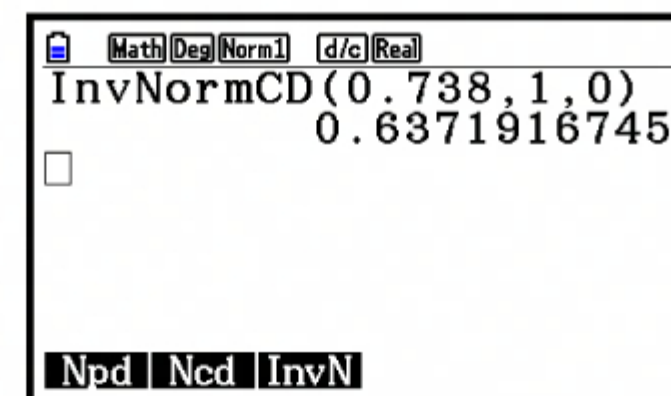
We start by finding z_1 and z_2 which correspond to $x_1 = 24$ and $x_2 = 33$.



$$\begin{aligned} \text{Now } P(X \leq x_1) &= 0.035 \\ \therefore P\left(\frac{X - \mu}{\sigma} \leq \frac{24 - \mu}{\sigma}\right) &= 0.035 \\ \therefore P\left(Z \leq \frac{24 - \mu}{\sigma}\right) &= 0.035 \quad \left\{Z = \frac{X - \mu}{\sigma}\right\} \\ \therefore z_1 = \frac{24 - \mu}{\sigma} &\approx -1.8119 \quad \{Z \sim N(0, 1^2)\} \\ \therefore 24 - \mu &\approx -1.8119\sigma \quad \dots (1) \end{aligned}$$



$$\begin{aligned} \text{and } P(X \geq x_2) &= 0.262 \\ \therefore P(X < x_2) &= 0.738 \\ \therefore P\left(\frac{X - \mu}{\sigma} < \frac{33 - \mu}{\sigma}\right) &= 0.738 \\ \therefore P\left(Z < \frac{33 - \mu}{\sigma}\right) &= 0.738 \quad \left\{Z = \frac{X - \mu}{\sigma}\right\} \\ \therefore z_2 = \frac{33 - \mu}{\sigma} &\approx 0.6372 \quad \{Z \sim N(0, 1^2)\} \\ \therefore 33 - \mu &\approx 0.6372\sigma \quad \dots (2) \end{aligned}$$



Solving (1) and (2) simultaneously, $\mu \approx 30.7$ and $\sigma \approx 3.67$.

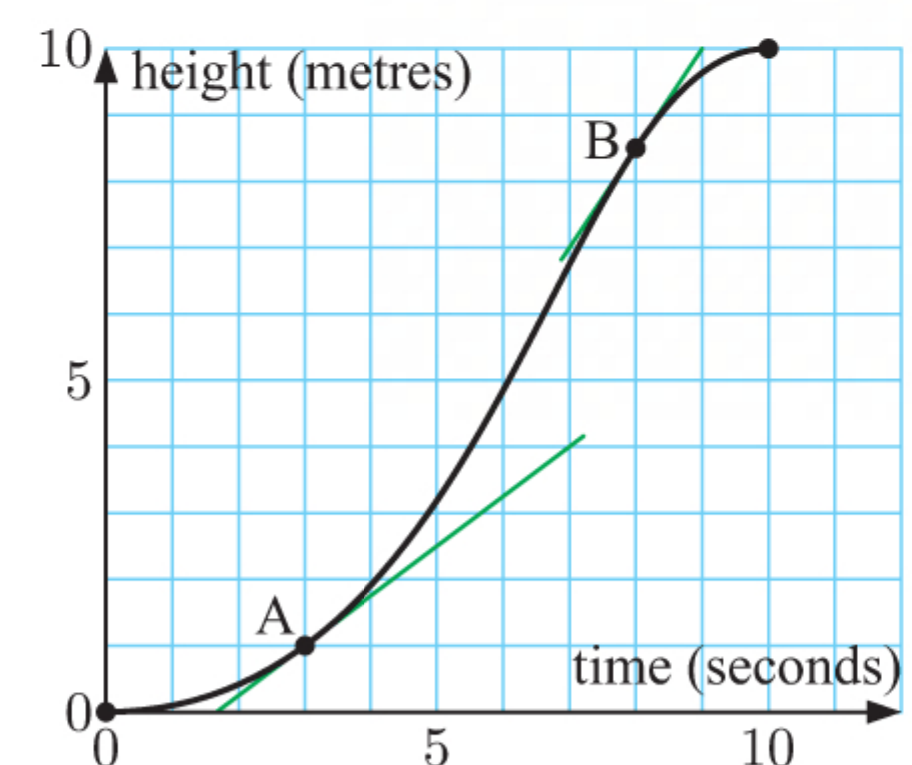
TOPIC 5 SKILL BUILDER QUESTIONS

1 a	x	50	100	200	500	1000
	$\frac{\ln x}{x}$	$\approx 0.078\,24$	$\approx 0.046\,05$	$\approx 0.026\,49$	$\approx 0.012\,43$	$\approx 0.006\,91$

b We predict that $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$.

2 a The tangent at A has gradient $\frac{4-1}{7-3} = \frac{3}{4}$.
 \therefore the elevator's instantaneous speed after 3 seconds is 0.75 m s^{-1} .

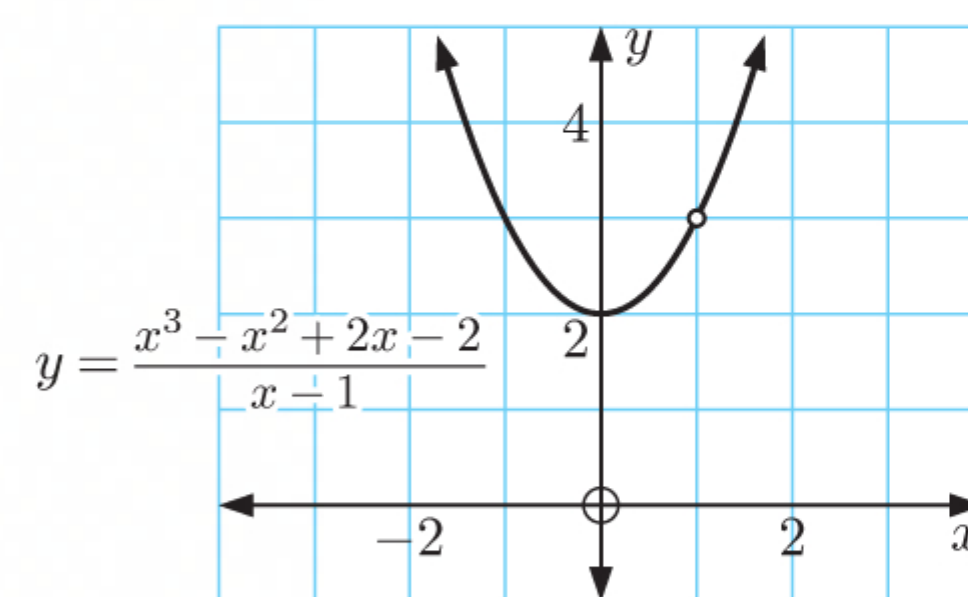
b The tangent at B has gradient $\frac{10-7}{9-7} = \frac{3}{2}$.
 \therefore the elevator's instantaneous speed after 8 seconds is 1.5 m s^{-1} .



- 3 a** When $x = 1$, the denominator of $\frac{x^3 - x^2 + 2x - 2}{x - 1}$ is zero. So, the function is undefined at $x = 1$.

- b** From the graph, we can see that $y \rightarrow 3$ as $x \rightarrow 1$ from either direction.

$$\text{So, } \lim_{x \rightarrow 1} f(x) = 3.$$



4 a $y = 5x^3 - 3x^2 + 4x + 7$
 $\therefore \frac{dy}{dx} = 5(3x^2) - 3(2x) + 4(1)$
 $= 15x^2 - 6x + 4$

b $y = \frac{3}{\sqrt{x}} - 2\sqrt{x}$
 $= 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = 3\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
 $= -\frac{3}{2}x^{-\frac{3}{2}} - x^{-\frac{1}{2}}$
 $= -\frac{3}{2x\sqrt{x}} - \frac{1}{\sqrt{x}}$

c $y = \frac{2x - x^2}{\sqrt{x}}$
 $= \frac{2x - x^2}{x^{\frac{1}{2}}}$
 $= 2x^{\frac{1}{2}} - x^{\frac{3}{2}}$
 $\therefore \frac{dy}{dx} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \frac{3}{2}x^{\frac{1}{2}}$
 $= x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$
 $= \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}$

5 a $f(x) = ax + \frac{b}{x^2}$, $f(1) = 8$, and $f'(1) = -7$
 $= ax + bx^{-2}$
 $\therefore f'(x) = a + b(-2x^{-3})$
 $= a - \frac{2b}{x^3}$

But $f'(1) = -7$, so $a - \frac{2b}{(1)^3} = -7$
 $\therefore a - 2b = -7$
 $\therefore a = 2b - 7 \quad \dots (*)$

and $f(1) = 8$, so $a(1) + \frac{b}{(1)^2} = 8$
 $\therefore a + b = 8$
 $\therefore 2b - 7 + b = 8 \quad \{\text{using } (*)\}$
 $\therefore 3b - 7 = 8$
 $\therefore 3b = 15$
 $\therefore b = 5$

and so $a = 2(5) - 7 \quad \{\text{using } (*)\}$
 $= 3$

6 a $y = (x^2 - 3x)^5$
 $\therefore \frac{dy}{dx} = 5(x^2 - 3x)^4(2x - 3) \quad \{\text{chain rule}\}$

c $y = \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3) \quad \{\text{chain rule}\}$
 $= \frac{2x - 3}{2\sqrt{x^2 - 3x}}$

7 a $y = (3 - x)^3$
 $\therefore \frac{dy}{dx} = 3(3 - x)^2 \times (-1) \quad \{\text{chain rule}\}$
 $= -3(3 - x)^2$

When $x = 2$, $\frac{dy}{dx} = -3(3 - 2)^2 = -3$.

So, the tangent has gradient -3 .

b $f(x) = ax^b$, $f(2) = \frac{32}{b}$, and $f'(1) = 8$

$$\therefore f'(x) = abx^{b-1}$$

But $f'(1) = 8$, so $ab(1)^{b-1} = 8$

$$\therefore ab = 8 \quad \dots (*)$$

and $f(2) = \frac{32}{b}$, so $a(2)^b = \frac{32}{b}$

$$\therefore 2^b = \frac{32}{ab}$$

$$\therefore 2^b = \frac{32}{8} \quad \{\text{using } (*)\}$$

$$\therefore 2^b = 4$$

$$\therefore 2^b = 2^2$$

$$\therefore b = 2 \quad \{\text{equating indices}\}$$

and so $a(2) = 8 \quad \{\text{using } (*)\}$

$$\therefore a = 4$$

b $y = \frac{3}{(x^2 + 3)^3} = 3(x^2 + 3)^{-3}$

$$\therefore \frac{dy}{dx} = 3 \times (-3)(x^2 + 3)^{-4} \times (2x) \quad \{\text{chain rule}\}$$

$$= -18x(x^2 + 3)^{-4}$$

$$= \frac{-18x}{(x^2 + 3)^4}$$

b $y = \frac{1}{(x + 2)^4} = (x + 2)^{-4}$

$$\therefore \frac{dy}{dx} = -4(x + 2)^{-5}$$

$$= \frac{-4}{(x + 2)^5}$$

When $x = -1$, $\frac{dy}{dx} = \frac{-4}{(-1 + 2)^5} = -4$.

So, the tangent has gradient -4 .

$$\begin{aligned} \text{c} \quad y &= \sqrt{x + \frac{1}{x^2} + 1} = (x + x^{-2} + 1)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(x + x^{-2} + 1)^{-\frac{1}{2}} \times (1 - 2x^{-3}) \quad \{\text{chain rule}\} \\ &= \frac{1 - \frac{2}{x^3}}{2\sqrt{x + \frac{1}{x^2} + 1}} \end{aligned}$$

$$\begin{aligned} \text{When } x = 4, \quad \frac{dy}{dx} &= \frac{1 - \frac{2}{4^3}}{2\sqrt{4 + \frac{1}{4^2} + 1}} \\ &= \frac{1 - \frac{2}{64}}{2\sqrt{5 + \frac{1}{16}}} \\ &= \frac{\frac{31}{32}}{2\sqrt{\frac{81}{16}}} \\ &= \frac{\frac{31}{32}}{2 \times \frac{9}{4}} \\ &= \frac{31}{2 \times 32} \times \frac{4}{9} \\ &= \frac{31}{144} \end{aligned}$$

So, the tangent has gradient $\frac{31}{144}$.

$$\begin{aligned} 8 \quad f(x) &= (ax + b)^c \\ \therefore f'(x) &= c(ax + b)^{c-1}(a) \quad \{\text{chain rule}\} \\ &= ac(ax + b)^{c-1} \end{aligned}$$

Now $f'(x) = 81x^2 + 108x + 36$ is a polynomial of degree 2, so we must have $c - 1 = 2$
 $\therefore c = 3$

$$\begin{aligned} \therefore 3a(ax + b)^2 &= 81x^2 + 108x + 36 \\ \therefore 3a(a^2x^2 + 2abx + b^2) &= 81x^2 + 108x + 36 \\ \therefore 3a^3x^2 + 6a^2bx + 3ab^2 &= 81x^2 + 108x + 36 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients gives:} \quad 3a^3 &= 81 \quad \dots (1) \\ 6a^2b &= 108 \quad \dots (2) \\ 3ab^2 &= 36 \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \text{From (1), } 3a^3 &= 81 \\ \therefore a^3 &= 27 \\ \therefore a &= 3 \end{aligned}$$

$$\begin{aligned} \text{Substituting } a = 3 \text{ into (2) gives } 6(3)^2b &= 108 \\ \therefore 54b &= 108 \\ \therefore b &= 2 \end{aligned}$$

$$\text{Check: } 3ab^2 = 3(3)(2)^2 = 36 \quad \checkmark$$

$$\begin{aligned} 9 \quad \text{a} \quad y &= x^2\sqrt{x^2 + 2x} = x^2(x^2 + 2x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= 2x(x^2 + 2x)^{\frac{1}{2}} + x^2 \times \frac{1}{2}(x^2 + 2x)^{-\frac{1}{2}}(2x + 2) \quad \{\text{product rule}\} \\ &= 2x\sqrt{x^2 + 2x} + \frac{x^2(x + 1)}{\sqrt{x^2 + 2x}} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= \sqrt{x}(2x + 3)^4 = x^{\frac{1}{2}} \times (2x + 3)^4 \\ \therefore \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}}(2x + 3)^4 + x^{\frac{1}{2}} \times 4(2x + 3)^3(2) \quad \{\text{product rule}\} \\ &= \frac{(2x + 3)^4}{2\sqrt{x}} + 8\sqrt{x}(2x + 3)^3 \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= (2x + 1)^3(x - 5)^2 \\ \therefore \frac{dy}{dx} &= 3(2x + 1)^2(2)(x - 5)^2 + (2x + 1)^3 \times 2(x - 5)(1) \quad \{\text{product rule}\} \\ &= 6(2x + 1)^2(x - 5)^2 + 2(2x + 1)^3(x - 5) \end{aligned}$$

$$\begin{aligned}
 10 \quad y &= (x-2)^2(2x-1) \\
 \therefore \frac{dy}{dx} &= 2(x-2)(2x-1) + (x-2)^2(2) \quad \{\text{product rule}\} \\
 &= 2(x-2)[(2x-1) + (x-2)] \\
 &= 2(x-2)(3x-3) \\
 &= 6(x-2)(x-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= 36 \text{ where } 6(x-2)(x-1) = 36 \\
 \therefore (x-2)(x-1) &= 6 \\
 \therefore x^2 - 3x + 2 &= 6 \\
 \therefore x^2 - 3x - 4 &= 0 \\
 \therefore (x+1)(x-4) &= 0 \\
 \therefore x &= -1 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 11 \quad a \quad y &= \frac{x^3}{x^2-1} \\
 \therefore \frac{dy}{dx} &= \frac{3x^2(x^2-1) - (2x)x^3}{(x^2-1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} \\
 &= \frac{x^4 - 3x^2}{(x^2-1)^2}
 \end{aligned}$$

$$\text{When } x = 2, \frac{dy}{dx} = \frac{(2)^4 - 3(2)^2}{(2^2 - 1)^2} = \frac{16 - 12}{9} = \frac{4}{9}.$$

So, the tangent has gradient $\frac{4}{9}$.

$$\begin{aligned}
 12 \quad a \quad f(x) &\text{ is defined when } x \geq 0 \text{ and } x \neq 3. \\
 \therefore \text{ the domain is } &\{x \mid x \geq 0, x \neq 3\}.
 \end{aligned}$$

$$\begin{aligned}
 c \quad i \quad f'(x) &= 0 \\
 \therefore -3 - x &= 0 \\
 \therefore -x &= 3 \\
 \therefore x &= -3
 \end{aligned}$$

But $f'(-3)$ is undefined, so $f'(x)$ is never zero.

$$\begin{aligned}
 13 \quad a \quad f(t) &= 20te^{-0.1t} \\
 \therefore f'(t) &= 20(1)e^{-0.1t} + 20t(-0.1e^{-0.1t}) \quad \{\text{product rule}\} \\
 &= 20e^{-0.1t} - 2te^{-0.1t}
 \end{aligned}$$

$$\begin{aligned}
 b \quad f(t) &= \frac{100}{1+7e^{-\frac{t}{4}}} = 100\left(1+7e^{-\frac{t}{4}}\right)^{-1} \\
 \therefore f'(t) &= -100\left(1+7e^{-\frac{t}{4}}\right)^{-2} \times \left(-\frac{7}{4}e^{-\frac{t}{4}}\right) \quad \{\text{chain rule}\} \\
 &= \frac{175e^{-\frac{t}{4}}}{\left(1+7e^{-\frac{t}{4}}\right)^2}
 \end{aligned}$$

$$\begin{aligned}
 c \quad f(t) &= \frac{t+9}{e^t} \\
 \therefore f'(t) &= \frac{(1)e^t - e^t(t+9)}{(e^t)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1 - (t+9)}{e^t} \\
 &= \frac{-t-8}{e^t}
 \end{aligned}$$

$$\begin{aligned}
 b \quad y &= \frac{\sqrt{x}}{2x+5} = \frac{x^{\frac{1}{2}}}{2x+5} \\
 \therefore \frac{dy}{dx} &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(2x+5) - (2) \times x^{\frac{1}{2}}}{(2x+5)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{2x+5-4x}{2\sqrt{x}(2x+5)^2} \\
 &= \frac{5-2x}{2\sqrt{x}(2x+5)^2}
 \end{aligned}$$

When $x = 4$,

$$\frac{dy}{dx} = \frac{5-2(4)}{2\sqrt{4}(2(4)+5)^2} = \frac{-3}{2 \times 2 \times 169} = \frac{-3}{676}.$$

So, the tangent has gradient $-\frac{3}{676}$.

$$\begin{aligned}
 b \quad f(x) &= \frac{\sqrt{x}}{x-3} = \frac{x^{\frac{1}{2}}}{x-3} \\
 \therefore f'(x) &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(x-3) - (1)x^{\frac{1}{2}}}{(x-3)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{x-3-2x}{2\sqrt{x}(x-3)^2} \\
 &= \frac{-3-x}{2\sqrt{x}(x-3)^2}
 \end{aligned}$$

$$\begin{aligned}
 ii \quad f'(x) &\text{ is undefined when } x \leq 0, \text{ and when} \\
 2\sqrt{x}(x-3)^2 &= 0 \\
 \therefore \sqrt{x} &= 0 \quad \text{or} \quad (x-3)^2 = 0 \\
 \therefore x &= 0 \quad \text{or} \quad x = 3
 \end{aligned}$$

That is, when $x \leq 0$ or $x = 3$.

$$14 \quad f(x) = e^{ax+2} + x^2 \quad \text{and} \quad f(2) = f'(2)$$

$$\therefore f'(x) = ae^{ax+2} + 2x$$

$$\text{Now } f(2) = e^{2a+2} + 4 \quad \text{and} \quad f'(2) = ae^{2a+2} + 4$$

$$\therefore e^{2a+2} + 4 = ae^{2a+2} + 4$$

$$\therefore e^{2a+2} = ae^{2a+2}$$

$$\therefore (a-1)e^{2a+2} = 0$$

$$\therefore a-1 = 0 \quad \{e^x > 0 \text{ for all } x\}$$

$$\therefore a = 1$$

$$15 \quad \mathbf{a} \quad f(x) = \ln(2x+3)$$

$$\therefore f'(x) = \frac{2}{2x+3}$$

$$\mathbf{b} \quad f(x) = [\ln(x^2+5)]^2$$

$$\begin{aligned} \therefore f'(x) &= 2\ln(x^2+5) \times \left(\frac{2x}{x^2+5}\right) \quad \{\text{chain rule}\} \\ &= \frac{4x\ln(x^2+5)}{x^2+5} \end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{1}{\ln(\ln x)} = [\ln(\ln x)]^{-1}$$

$$\begin{aligned} \therefore f'(x) &= -[\ln(\ln x)]^{-2} \times \left(\frac{\frac{1}{x}}{\ln x}\right) \quad \{\text{chain rule}\} \\ &= \frac{-1}{x \ln x [\ln(\ln x)]^2} \end{aligned}$$

$$16 \quad \mathbf{a} \quad \frac{d}{dx} \left(\ln \left(\frac{x-4}{x^2+4} \right) \right) = \frac{d}{dx} (\ln(x-4) - \ln(x^2+4))$$

$$= \frac{1}{x-4} - \frac{2x}{x^2+4}$$

$$\begin{aligned} \mathbf{b} \quad \frac{d}{dx} \left(\ln \left(x\sqrt{x^2+4} \right) \right) &= \frac{d}{dx} (\ln x + \ln((x^2+4)^{\frac{1}{2}})) \\ &= \frac{d}{dx} \left(\ln x + \frac{1}{2} \ln(x^2+4) \right) \\ &= \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2+4} \right) \\ &= \frac{1}{x} + \frac{x}{x^2+4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{d}{dx} \left(\ln \left(\frac{\sqrt{x^2+1}}{(x+3)(x-2)} \right) \right) &= \frac{d}{dx} \left(\ln((x^2+1)^{\frac{1}{2}}) - \ln((x+3)(x-2)) \right) \\ &= \frac{d}{dx} \left(\frac{1}{2} \ln(x^2+1) - \ln(x+3) - \ln(x-2) \right) \\ &= \frac{1}{2} \left(\frac{2x}{x^2+1} \right) - \frac{1}{x+3} - \frac{1}{x-2} \\ &= \frac{x}{x^2+1} - \frac{1}{x+3} - \frac{1}{x-2} \end{aligned}$$

$$17 \quad \mathbf{a} \quad \frac{d}{dx} (3 \sin(x-4)) = 3 \cos(x-4)$$

$$\begin{aligned} \mathbf{b} \quad \frac{d}{dx} (12x - 2 \cos \frac{x}{3}) &= 12 - 2(-\sin \frac{x}{3} \times \frac{1}{3}) \\ &= 12 + \frac{2}{3} \sin \frac{x}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{d}{dx} (x^2 \sin 3x) &= (2x) \sin 3x + x^2 (3 \cos 3x) \quad \{\text{product rule}\} \\ &= 2x \sin 3x + 3x^2 \cos 3x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \frac{d}{dx} ((\sin x)e^{\cos x}) &= (\cos x)e^{\cos x} + (\sin x)((-\sin x)e^{\cos x}) \quad \{\text{product rule}\} \\ &= (\cos x)e^{\cos x} - (\sin^2 x)e^{\cos x} \\ &= e^{\cos x} (\cos x - \sin^2 x) \end{aligned}$$

$$\begin{aligned} 18 \quad \mathbf{a} \quad f(x) &= \sqrt{\sin(2x+1)} = (\sin(2x+1))^{\frac{1}{2}} \\ \therefore f'(x) &= \frac{1}{2} (\sin(2x+1))^{-\frac{1}{2}} (2 \cos(2x+1)) \quad \{\text{chain rule}\} \\ &= \frac{\cos(2x+1)}{\sqrt{\sin(2x+1)}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= \cos \frac{x}{2} \sin \frac{x}{3} \\ \therefore f'(x) &= \left(-\frac{1}{2} \sin \frac{x}{2}\right) \sin \frac{x}{3} + \cos \frac{x}{2} \left(\frac{1}{3} \cos \frac{x}{3}\right) \quad \{\text{product rule}\} \\ &= -\frac{1}{2} \sin \frac{x}{2} \sin \frac{x}{3} + \frac{1}{3} \cos \frac{x}{2} \cos \frac{x}{3} \end{aligned}$$

$$\mathbf{c} \quad f(x) = \ln\left(\frac{\sin x}{x}\right) = \ln(\sin x) - \ln x$$

$$\begin{aligned}\therefore f'(x) &= \frac{\cos x}{\sin x} - \frac{1}{x} \\ &= \frac{1}{\tan x} - \frac{1}{x}\end{aligned}$$

$$\begin{aligned}\mathbf{19} \quad \mathbf{a} \quad f(x) &= \cos^4 x \\ \therefore f'(x) &= 4 \cos^3 x (-\sin x) \quad \{\text{chain rule}\} \\ &= -4 \cos^3 x \sin x\end{aligned}$$

$$\begin{aligned}f'\left(\frac{3\pi}{4}\right) &= -4 \cos^3\left(\frac{3\pi}{4}\right) \sin \frac{3\pi}{4} \\ &= -4\left(-\frac{1}{\sqrt{2}}\right)^3 \left(\frac{1}{\sqrt{2}}\right) \\ &= 1\end{aligned}$$

So, the gradient of the tangent is 1.

$$\begin{aligned}\mathbf{b} \quad f(x) &= \frac{3 \sin^2 x}{\cos 2x} \\ \therefore f'(x) &= \frac{3(2 \sin x \cos x) \cos 2x - (-2 \sin 2x)(3 \sin^2 x)}{\cos^2 2x} \\ &= \frac{3 \sin 2x \cos 2x + 6 \sin 2x \sin^2 x}{\cos^2 2x} \\ &= \frac{3 \sin 2x (\cos 2x + 2 \sin^2 x)}{\cos^2 2x} \\ &= \frac{3 \sin 2x (1 - \cancel{2 \sin^2 x} + \cancel{2 \sin^2 x})}{\cos^2 2x} \\ &\quad \{\text{double angle formula}\} \\ &= \frac{3 \sin 2x}{\cos^2 2x}\end{aligned}$$

$$\therefore f'\left(-\frac{\pi}{3}\right) = \frac{3 \sin\left(-\frac{2\pi}{3}\right)}{\cos^2\left(-\frac{2\pi}{3}\right)} = \frac{3\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)^2} = -6\sqrt{3}$$

So, the gradient of the tangent is $-6\sqrt{3}$.

$$\mathbf{20} \quad \mathbf{a} \quad y = \frac{3}{x^2} = 3x^{-2}$$

$$\therefore \frac{dy}{dx} = -6x^{-3}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= 18x^{-4} \\ &= \frac{18}{x^4}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad y &= \frac{x+3}{6-x} \\ \therefore \frac{dy}{dx} &= \frac{(1)(6-x) - (-1)(x+3)}{(6-x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{6-x+x+3}{(6-x)^2} \\ &= \frac{9}{(6-x)^2} \\ &= 9(6-x)^{-2} \\ \therefore \frac{d^2y}{dx^2} &= -18(6-x)^{-3} \times (-1) \quad \{\text{chain rule}\} \\ &= \frac{18}{(6-x)^3}\end{aligned}$$

$$\mathbf{21} \quad f(x) = \ln(\cos x)$$

$$\begin{aligned}\mathbf{a} \quad f\left(\frac{\pi}{4}\right) &= \ln\left(\cos \frac{\pi}{4}\right) \\ &= \ln\left(\frac{1}{\sqrt{2}}\right) \\ &= \ln\left(2^{-\frac{1}{2}}\right) \\ &= -\frac{1}{2} \ln 2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad f'(x) &= \frac{-\sin x}{\cos x} \quad \{\text{from } \mathbf{b}\} \\ \therefore f''(x) &= \frac{-\cos x(\cos x) - (-\sin x)(-\sin x)}{\cos^2 x} \quad \{\text{quotient rule}\} \\ &= \frac{-\cos^2 x - \sin^2 x}{\cos^2 x} \\ &= \frac{-1}{\cos^2 x} \\ \therefore f''\left(\frac{\pi}{4}\right) &= \frac{-1}{\cos^2\left(\frac{\pi}{4}\right)} \\ &= \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= -2\end{aligned}$$

$$\mathbf{b} \quad y = 2x^3 + 3x^2 + 2$$

$$\therefore \frac{dy}{dx} = 6x^2 + 6x$$

$$\therefore \frac{d^2y}{dx^2} = 12x + 6$$

$$\begin{aligned}\mathbf{b} \quad f'(x) &= \frac{-\sin x}{\cos x} \quad \{\text{chain rule}\} \\ &= -\tan x \\ \therefore f'\left(\frac{\pi}{4}\right) &= -\tan \frac{\pi}{4} \\ &= -1\end{aligned}$$

22 a $y = x^3 - 5$

Now $\frac{dy}{dx} = 3x^2$, so at $x = 1$,

$$\frac{dy}{dx} = 3(1)^2 = 3$$

The tangent at $(1, -4)$ has equation $y = 3(x - 1) - 4$

$$\therefore y = 3x - 7$$

b The tangent cuts the x -axis where $y = 0$

$$\therefore 3x - 7 = 0$$

$$\therefore x = \frac{7}{3}$$

23 a $g(x) = -x \cos x$

$$\therefore g'(x) = (-1) \cos x - x(-\sin x) \quad \{\text{product rule}\}$$

$$= -\cos x + x \sin x$$

b Since $g\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} \cos \frac{\pi}{3} = -\frac{\pi}{3} \times \frac{1}{2} = -\frac{\pi}{6}$, the point of contact is $\left(\frac{\pi}{3}, -\frac{\pi}{6}\right)$.

Now $g'\left(\frac{\pi}{3}\right) = -\cos \frac{\pi}{3} + \frac{\pi}{3} \sin \frac{\pi}{3}$

$$= -\frac{1}{2} + \frac{\pi}{3} \times \frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2} + \frac{\pi}{2\sqrt{3}}$$

The tangent has equation $y = g'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + g\left(\frac{\pi}{3}\right)$

$$\therefore y = \left(-\frac{1}{2} + \frac{\pi}{2\sqrt{3}}\right)\left(x - \frac{\pi}{3}\right) - \frac{\pi}{6}$$

$$\therefore y = \left(-\frac{1}{2} + \frac{\pi}{2\sqrt{3}}\right)x + \frac{\pi}{6} - \frac{\pi^2}{6\sqrt{3}} - \frac{\pi}{6}$$

$$\therefore y = \left(-\frac{1}{2} + \frac{\pi}{2\sqrt{3}}\right)x - \frac{\pi^2}{6\sqrt{3}}$$

24 a $f(x) = -x^2 + 4x$

$$\therefore f'(x) = -2x + 4$$

b Since $f(k) = -k^2 + 4k$, the point of contact is $(k, -k^2 + 4k)$.

Now $f'(k) = -2k + 4$.

The tangent has equation $y = f'(k)(x - k) + f(k)$

$$\therefore y = (-2k + 4)(x - k) - k^2 + 4k$$

$$\therefore y = (-2k + 4)x + 2k^2 - 4k - k^2 + 4k$$

$$\therefore y = (-2k + 4)x + k^2$$

c The tangent passes through $(4, 9)$, so $(-2k + 4)(4) + k^2 = 9$

$$\therefore -8k + 16 + k^2 = 9$$

$$\therefore k^2 - 8k + 7 = 0$$

$$\therefore (k - 7)(k - 1) = 0$$

$$\therefore k = 1 \text{ or } 7$$

The gradient of the tangent is positive, so $-2k + 4 > 0$.

Now if $k = 1$, $-2k + 4 = -2 + 4 = 2$ ✓

if $k = 7$, $-2k + 4 = -14 + 4 = -10$ ✗

So $k = 1$.

25 Since $f(2) = \ln(2(2) + 3) = \ln 7$, the point of contact is $(2, \ln 7)$.

Now $f'(x) = \frac{2}{2x + 3}$, so at $x = 2$ the tangent has gradient $f'(2) = \frac{2}{2(2) + 3} = \frac{2}{7}$.

The tangent has equation $y = f'(2)(x - 2) + f(2)$

$$\therefore y = \frac{2}{7}(x - 2) + \ln 7$$

$$\therefore y = \frac{2}{7}x - \frac{4}{7} + \ln 7$$

26 $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$

a The curves meet where $\sqrt{3x+1} = \sqrt{5x-x^2}$
 $\therefore 3x+1 = 5x-x^2$ {squaring both sides}
 $\therefore x^2 - 2x + 1 = 0$
 $\therefore (x-1)^2 = 0$
 $\therefore x = 1$

Now when $x = 1$, $y = \sqrt{3(1)+1} = \sqrt{4} = 2$

So the curves meet at $(1, 2)$.

b For $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}} \times 3$ {chain rule}
 $= \frac{3}{2\sqrt{3x+1}}$

When $x = 1$, $\frac{dy}{dx} = \frac{3}{2\sqrt{3(1)+1}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$.

For $y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x)$ {chain rule}
 $= \frac{5-2x}{2\sqrt{5x-x^2}}$

When $x = 1$, $\frac{dy}{dx} = \frac{5-2(1)}{2\sqrt{5(1)-(1)^2}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$.

So, the tangents to the curves have the same gradient at the intersection point.

c The equation of the common tangent is $y = \frac{3}{4}(x-1) + 2$
 $\therefore y = \frac{3}{4}x - \frac{3}{4} + 2$
 $\therefore y = \frac{3}{4}x + \frac{5}{4}$

27 a $y = \frac{a}{x} - x^2 + 1 = ax^{-1} - x^2 + 1$

$\therefore \frac{dy}{dx} = -ax^{-2} - 2x$
 $= \frac{-a}{x^2} - 2x$

Now the gradient of the tangent at $x = 2$ is -5 .

$\therefore \frac{-a}{2^2} - 2(2) = -5$
 $\therefore \frac{-a}{4} - 4 = -5$
 $\therefore \frac{-a}{4} = -1$
 $\therefore a = 4$

b $y = \frac{a}{x} - x^2 + 1 = \frac{4}{x} - x^2 + 1$ {from **a**}

When $x = 2$, $y = \frac{4}{2} - 2^2 + 1$
 $= 2 - 4 + 1$
 $= -1$

So, the point of contact is $(2, -1)$.

The equation of tangent is $y = -5(x-2) + (-1)$
 $\therefore y = -5x + 10 - 1$
 $\therefore y = -5x + 9$

28 a $f(x) = \frac{x+2}{\sqrt{x-1}}$

$f(x)$ is undefined when $x-1 < 0$ and $x = 1$
 $\therefore x < 1$

So, the domain of $f(x)$ is $\{x \mid x > 1\}$.

b $f(10) = \frac{10+2}{\sqrt{10-1}} = \frac{12}{\sqrt{9}} = \frac{12}{3} = 4$, so the point of contact is $(10, 4)$.

Now $f(x) = \frac{x+2}{\sqrt{x-1}} = \frac{x+2}{(x-1)^{\frac{1}{2}}}$ has derivative

$$\begin{aligned} f'(x) &= \frac{(1)(x-1)^{\frac{1}{2}} - \frac{1}{2}(x-1)^{-\frac{1}{2}}(x+2)}{x-1} \quad \{\text{quotient rule}\} \\ &= \frac{\sqrt{x-1} - \frac{x+2}{2\sqrt{x-1}}}{x-1} \\ &= \frac{2x-2-x-2}{2(x-1)^{\frac{3}{2}}} \\ &= \frac{x-4}{2(x-1)^{\frac{3}{2}}} \end{aligned}$$

$$\therefore f'(10) = \frac{10-4}{2(10-1)^{\frac{3}{2}}} = \frac{6}{2 \times 9^{\frac{3}{2}}} = \frac{6}{2 \times 27} = \frac{1}{9}$$

So, the normal has gradient -9 .

The normal has equation $y = -9(x-10) + 4$

$$\therefore y = -9x + 90 + 4$$

$$\therefore y = -9x + 94$$

29 a $y = x^3 + ax^2 + bx + 3$

The function passes through $(1, 8)$, so $(1)^3 + a(1)^2 + b(1) + 3 = 8$

$$\therefore 1 + a + b + 3 = 8$$

$$\therefore a + b + 4 = 8$$

$$\therefore a + b = 4$$

$$\therefore a = 4 - b \quad \dots (*)$$

Now $\frac{dy}{dx} = 3x^2 + 2ax + b$, and at $(1, 8)$ the tangent has equation $y = 2x + 6$ which has gradient 2.

$$\therefore 3(1)^2 + 2a(1) + b = 2$$

$$\therefore 3 + 2a + b = 2$$

$$\therefore 2a + b = -1$$

$$\therefore 2(4-b) + b = -1 \quad \{\text{using } (*)\}$$

$$\therefore 8 - 2b + b = -1$$

$$\therefore -b = -9$$

$$\therefore b = 9$$

$$\therefore a = 4 - 9 = -5$$

b From **a**, $y = x^3 - 5x^2 + 9x + 3$ and

$$\frac{dy}{dx} = 3x^2 - 10x + 9$$

Now when $x = -1$, $y = (-1)^3 - 5(-1)^2 + 9(-1) + 3 = -1 - 5 - 9 + 3 = -12$

$$\text{and } \frac{dy}{dx} = 3(-1)^2 - 10(-1) + 9 = 3 + 10 + 9 = 22$$

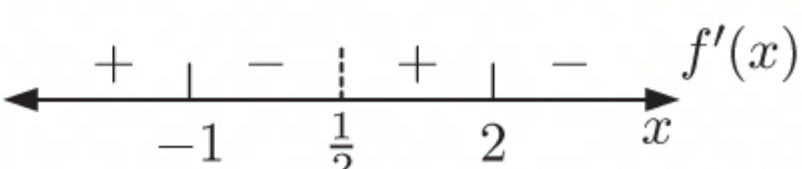
So the point of contact is $(-1, -12)$ and the gradient of the normal is $-\frac{1}{22}$.

The equation of the normal is $y = -\frac{1}{22}(x - (-1)) - 12$

$$\therefore y = -\frac{1}{22}x - \frac{265}{22}$$

$$30 \quad a \quad f(x) = \ln\left(\frac{1-2x}{x^2+2}\right) = \ln(1-2x) - \ln(x^2+2)$$

$$\begin{aligned} \therefore f'(x) &= \frac{-2}{1-2x} - \frac{2x}{x^2+2} \\ &= \frac{-2(x^2+2) - 2x(1-2x)}{(1-2x)(x^2+2)} \\ &= \frac{-2x^2 - 4 - 2x + 4x^2}{(x^2+2)(1-2x)} \\ &= \frac{2x^2 - 2x - 4}{(x^2+2)(1-2x)} \\ &= \frac{2(x-2)(x+1)}{(x^2+2)(1-2x)} \end{aligned}$$

b The sign diagram of $f'(x)$ is 

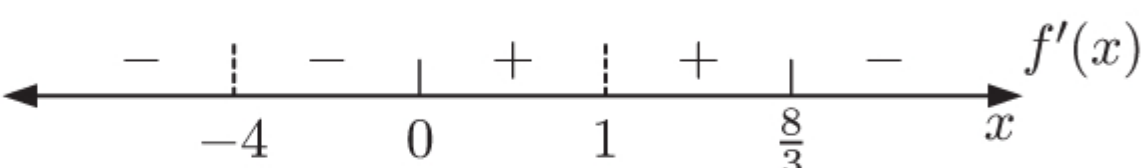
$f(x)$ is decreasing whenever $f'(x) \leq 0$.

But $f(x)$ is undefined when $1-2x \leq 0$ which is when $x \geq \frac{1}{2}$.

$\therefore f(x)$ is decreasing for $-1 \leq x < \frac{1}{2}$.

$$31 \quad a \quad f(x) = \frac{3x-4}{(x-1)(x+4)} = \frac{3x-4}{x^2+3x-4}$$

$$\begin{aligned} \therefore f'(x) &= \frac{3(x^2+3x-4) - (2x+3)(3x-4)}{(x^2+3x-4)^2} \quad \{\text{quotient rule}\} \\ &= \frac{3x^2 + 9x - 12 - 6x^2 + 8x - 9x + 12}{(x-1)^2(x+4)^2} \\ &= \frac{-3x^2 + 8x}{(x-1)^2(x+4)^2} \\ &= \frac{-x(3x-8)}{(x-1)^2(x+4)^2} \end{aligned}$$

$f'(x)$ has sign diagram 

b i $y = f(x)$ is increasing where $f'(x) \geq 0$

$$\therefore 0 \leq x < 1 \text{ and } 1 < x \leq \frac{8}{3}$$

ii $y = f(x)$ is decreasing where $f'(x) \leq 0$

$$\therefore x < -4, -4 < x \leq 0, \text{ and } x \geq \frac{8}{3}$$

$$32 \quad a \quad y = xe^{-x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x} \end{aligned}$$

Stationary points occur where $\frac{dy}{dx} = 0$

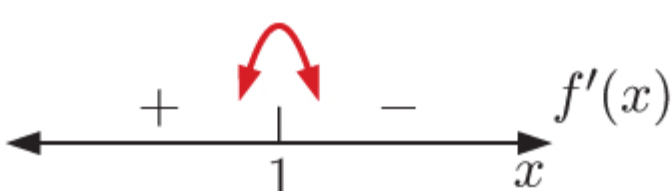
$$\therefore e^{-x} - xe^{-x} = 0$$

$$\therefore e^{-x}(1-x) = 0$$

$$\therefore 1-x = 0 \quad \{e^{-x} > 0\}$$

$$\therefore x = 1$$

$$\text{When } x = 1, y = (1)e^{-1} = \frac{1}{e}$$

Now $f'(x)$ has sign diagram 

So, $\left(1, \frac{1}{e}\right)$ is a local maximum.

$$b \quad y = \frac{x-3}{x^2-5}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(1)(x^2-5) - (2x)(x-3)}{(x^2-5)^2} \quad \{\text{quotient rule}\} \\ &= \frac{x^2 - 5 - 2x^2 + 6x}{(x^2-5)^2} \\ &= \frac{-x^2 + 6x - 5}{(x^2-5)^2} \end{aligned}$$

Stationary points occur where $\frac{dy}{dx} = 0$

$$\therefore -x^2 + 6x - 5 = 0$$

$$\therefore x^2 - 6x + 5 = 0$$

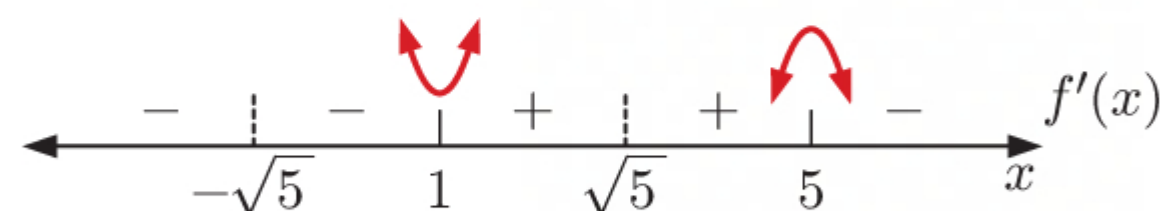
$$\therefore (x - 5)(x - 1) = 0$$

$$\therefore x = 1 \text{ or } 5$$

When $x = 1$, $y = \frac{1-3}{1^2-5} = \frac{-2}{-4} = \frac{1}{2}$.

When $x = 5$, $y = \frac{5-3}{5^2-5} = \frac{2}{20} = \frac{1}{10}$.

Now $f'(x)$ has sign diagram



So, $(1, \frac{1}{2})$ is a local minimum and $(5, \frac{1}{10})$ is a local maximum.

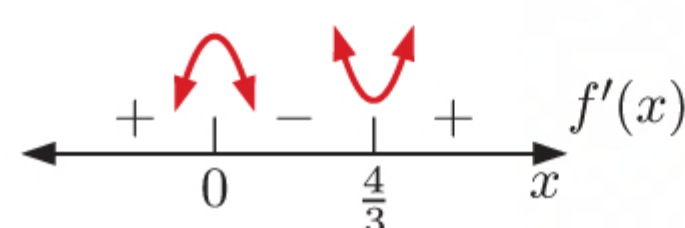
33 a $f(x) = x^3 - 2x^2$, $-1 \leq x \leq 1$

$$\therefore f'(x) = 3x^2 - 4x$$

$$= x(3x - 4)$$

which is 0 when $x = 0$ or $\frac{4}{3}$.

The sign diagram of $f'(x)$ is



\therefore there is a local maximum at $x = 0$, and a local minimum at $x = \frac{4}{3}$.

Critical value (x)	$f(x)$
-1 (end point)	-3
0 (local maximum)	0
1 (end point)	-1

The greatest of these values is 0 when $x = 0$.

The least of these values is -3 when $x = -1$.

b $f(x) = x^2 - \frac{27}{x} = x^2 - 27x^{-1}$, $-6 \leq x \leq -1$

$$\therefore f'(x) = 2x + 27x^{-2}$$

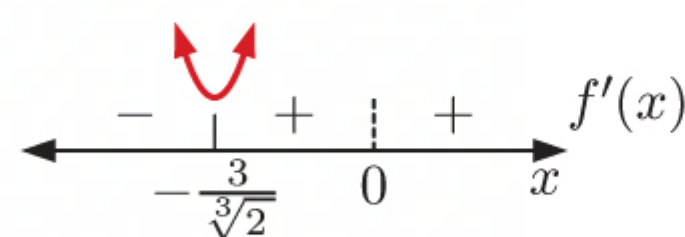
which is 0 when $2x + 27x^{-2} = 0$

$$\therefore 2x = -\frac{27}{x^2}$$

$$\therefore x^3 = -\frac{27}{2}$$

$$\therefore x = -\frac{3}{\sqrt[3]{2}}$$

The sign diagram of $f'(x)$ is



\therefore there is a local minimum at $x = -\frac{3}{\sqrt[3]{2}}$.

Critical value (x)	$f(x)$
-6 (end point)	40.5
$-\frac{3}{\sqrt[3]{2}}$ (local minimum)	≈ 17.0
-1 (end point)	28

The greatest of these values is 40.5 when $x = -6$.

The least of these values is ≈ 17.0 when $x = -\frac{3}{\sqrt[3]{2}}$.

$$\mathbf{c} \quad f(x) = x^3 - 6x^2 + 12x - 10, \quad 0 \leq x \leq 5$$

$$\therefore f'(x) = 3x^2 - 12x + 12$$

$$\text{which is 0 when } 3x^2 - 12x + 12 = 0$$

$$\therefore x^2 - 4x + 4 = 0$$

$$\therefore (x - 2)^2 = 0$$

$$\therefore x - 2 = 0$$

$$\therefore x = 2$$

The sign diagram of $f'(x)$ is

\therefore there is a stationary inflection at $x = 2$.

Critical value (x)	$f(x)$
0 (end point)	-10
5 (end point)	25

The greatest of these values is 25 when $x = 5$.

The least of these values is -10 when $x = 0$.

$$\mathbf{34} \quad f(t) = at^3 e^{bt^3}$$

$$\begin{aligned} \therefore f'(t) &= (3at^2)e^{bt^3} + at^3(3bt^2e^{bt^3}) \quad \{\text{product rule}\} \\ &= 3at^2e^{bt^3}(1 + bt^3) \end{aligned}$$

If $f(t)$ has minimum value -2 when $t = -1$, then

$$f(-1) = -2 \quad \text{and} \quad f'(-1) = 0$$

$$\therefore a(-1)^3 e^{b(-1)^3} = -2 \quad \therefore 3a(-1)^2 e^{b(-1)^3} (1 + b(-1)^3) = 0$$

$$\therefore -ae^{-b} = -2 \quad \therefore 3ae^{-b}(1 - b) = 0 \quad \dots (2)$$

$$\therefore ae^{-b} = 2 \quad \dots (1)$$

Substituting (1) into (2) gives: $3(2)(1 - b) = 0$

$$\therefore 6(1 - b) = 0$$

$$\therefore 1 - b = 0$$

$$\therefore b = 1$$

Substituting $b = 1$ into (1) gives: $ae^{-1} = 2$

$$\therefore a = 2e$$

So, $a = 2e$ and $b = 1$.

$$\mathbf{35} \quad \mathbf{a} \quad f(x) = \frac{e^{3x}}{kx}, \quad k \neq 0$$

$$\begin{aligned} \therefore f'(x) &= \frac{3e^{3x}(kx) - ke^{3x}}{(kx)^2} \quad \{\text{quotient rule}\} \\ &= \frac{ke^{3x}(3x - 1)}{k^2x^2} \\ &= \frac{e^{3x}(3x - 1)}{kx^2} \quad \{k \neq 0\} \end{aligned}$$

Stationary points occur where $f'(x) = 0$

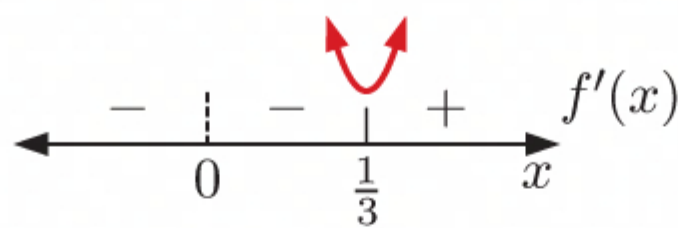
$$\therefore e^{3x}(3x - 1) = 0$$

$$\therefore 3x - 1 = 0 \quad \{e^{3x} > 0 \text{ for all } x\}$$

$$\therefore x = \frac{1}{3}$$

So, the stationary point has x -coordinate $\frac{1}{3}$.

- b i** If the stationary point is a local minimum, then the sign diagram of $f'(x)$ should be



This occurs when $k > 0$.

- ii** Using **b i**, if the stationary point is a local maximum, then $k < 0$.

Note: It is not possible for the stationary point to be an inflection point because the factor $(3x - 1)$ in $f'(x)$ is raised to an odd power. So, the sign of $f'(x)$ will always be different on either side of $x = \frac{1}{3}$.

- c** The stationary point has y -coordinate $-\frac{e}{2}$.

$$\therefore f\left(\frac{1}{3}\right) = -\frac{e}{2} \quad \{\text{using a}\}$$

$$\therefore \frac{e^{3(\frac{1}{3})}}{k(\frac{1}{3})} = -\frac{e}{2}$$

$$\therefore \frac{3e}{k} = -\frac{e}{2}$$

$$\therefore \frac{k}{2} = -3$$

$$\therefore k = -6$$

Since $k < 0$, the stationary point is a local maximum.

- d** $g(x) = -f(2x)$

$$f(x) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} -f(x) \xrightarrow[\text{horizontal stretch scale factor } \frac{1}{2}]{\text{reflection in } x\text{-axis}} -f(2x)$$

So, a reflection in the x -axis, followed by a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = f(x)$ onto $y = g(x)$.

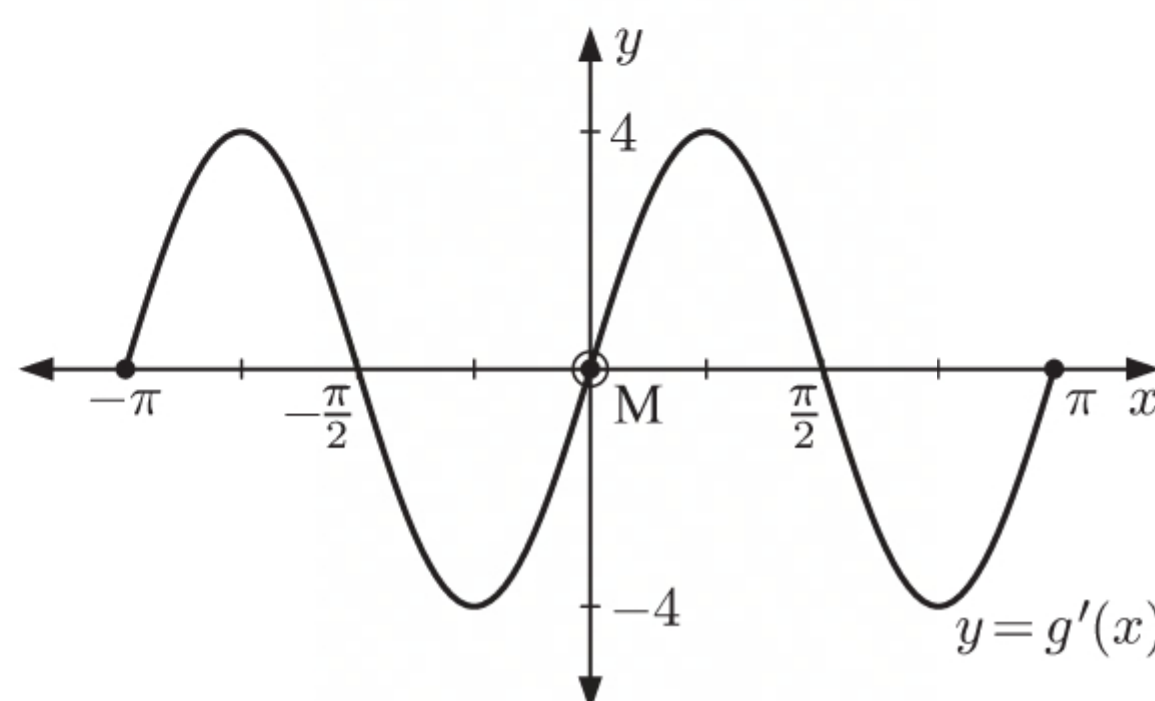
The stationary point of $f(x)$ is $\left(\frac{1}{3}, -\frac{e}{2}\right)$, so for the stationary point of $g(x)$:

$$\left(\frac{1}{3}, -\frac{e}{2}\right) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} \left(\frac{1}{3}, \frac{e}{2}\right) \xrightarrow[\text{horizontal stretch scale factor } \frac{1}{2}]{\text{reflection in } x\text{-axis}} \left(\frac{1}{6}, \frac{e}{2}\right)$$

So, the stationary point of $g(x)$ is $\left(\frac{1}{6}, \frac{e}{2}\right)$ which is a local minimum due to the reflection in the x -axis.

36 a $g(x) = 3 - 2 \cos 2x$
 $\therefore g'(x) = -2(-2 \sin 2x)$
 $= 4 \sin 2x$

b, d



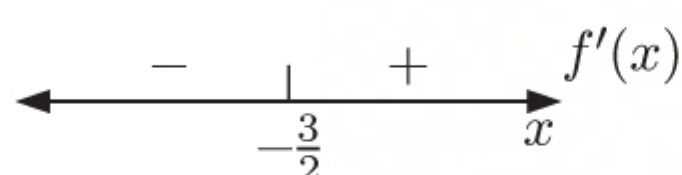
- c** From **b**, the graph of $y = g'(x)$ cuts the x -axis 5 times.

\therefore there are 5 solutions to $g'(x) = 0$ for $-\pi \leq x \leq \pi$.

37 a $f(x) = x^2 + 3x + 5$
 $\therefore f'(x) = 2x + 3$
 $\therefore f''(x) = 2$

- i** $f'(x) = 0$ when $x = -\frac{3}{2}$

The sign diagram of $f'(x)$ is



So, $f(x)$ is increasing for $x \geq -\frac{3}{2}$.

- ii** Using **i**, $f(x)$ is decreasing for $x \leq -\frac{3}{2}$.

- iii** $f''(x) = 2 > 0$ for all $x \in \mathbb{R}$.

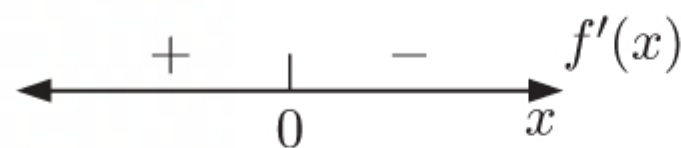
So, $f(x)$ is concave upwards for all $x \in \mathbb{R}$.

- iv** Using **iii**, there are no intervals in which $f(x)$ is concave downwards.

$$\begin{aligned}
 \mathbf{b} \quad & f(x) = e^{-x^2} \\
 \therefore & f'(x) = -2xe^{-x^2} \quad \{\text{chain rule}\} \\
 \therefore & f''(x) = -2e^{-x^2} - 2x(-2xe^{-x^2}) \quad \{\text{product rule}\} \\
 & = -2e^{-x^2} + 4x^2e^{-x^2} \\
 & = 2e^{-x^2}(2x^2 - 1)
 \end{aligned}$$

$$\mathbf{i} \quad f'(x) = 0 \text{ when } x = 0$$

The sign diagram of $f'(x)$ is

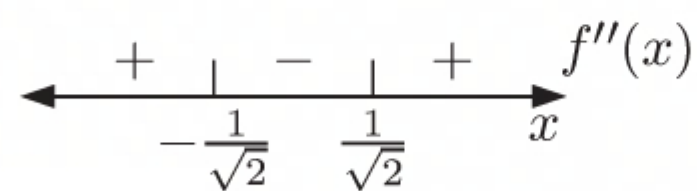


So, $f(x)$ is increasing for $x \leq 0$.

\mathbf{ii} Using \mathbf{i} , $f(x)$ is decreasing for $x \geq 0$.

$$\begin{aligned}
 \mathbf{iii} \quad & f''(x) = 0 \text{ when } 2e^{-x^2}(2x^2 - 1) = 0 \\
 & \therefore 2x^2 - 1 = 0 \quad \{e^{-x^2} > 0 \text{ for all } x\} \\
 & \therefore x^2 = \frac{1}{2} \\
 & \therefore x = \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$

The sign diagram for $f''(x)$ is



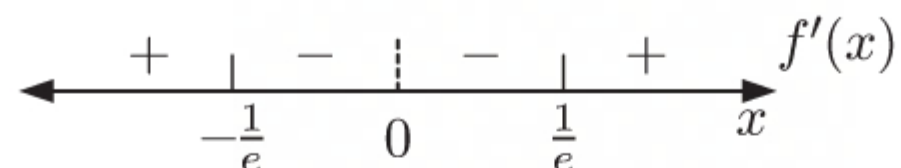
So, $f(x)$ is concave upwards for $x \leq -\frac{1}{\sqrt{2}}$ and $x \geq \frac{1}{\sqrt{2}}$.

\mathbf{iv} Using \mathbf{iii} , $f(x)$ is concave downwards for $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

$$\begin{aligned}
 \mathbf{c} \quad & f(x) = x \ln(x^2) \\
 \therefore & f'(x) = (1) \ln(x^2) + x \left(\frac{2x}{x^2} \right) \quad \{\text{product rule}\} \\
 & = \ln(x^2) + 2 \\
 \therefore & f''(x) = \frac{2x}{x^2} = \frac{2}{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & f'(x) = 0 \text{ when } \ln(x^2) + 2 = 0 \\
 & \therefore \ln(x^2) = -2 \\
 & \therefore x^2 = e^{-2} \\
 & \therefore x = \pm e^{-1} = \pm \frac{1}{e}
 \end{aligned}$$

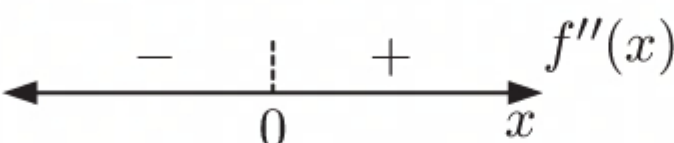
The sign diagram of $f'(x)$ is



So, $f(x)$ is increasing for $x \leq -\frac{1}{e}$ and $x \geq \frac{1}{e}$.

\mathbf{ii} Using \mathbf{i} , $f(x)$ is decreasing for $-\frac{1}{e} \leq x < 0$ and $0 < x \leq \frac{1}{e}$.

\mathbf{iii} The sign diagram of $f''(x)$ is



So, $f(x)$ is concave upwards for $x > 0$.

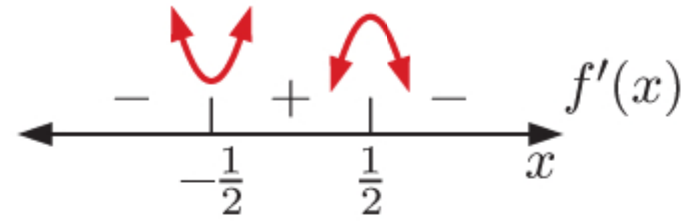
\mathbf{iv} Using \mathbf{iii} , $f(x)$ is concave downwards for $x < 0$.

$$\begin{aligned}
 \mathbf{38} \quad \mathbf{a} \quad & f(x) = xe^{1-2x^2} \\
 \therefore & f'(x) = (1)e^{1-2x^2} + x(-4xe^{1-2x^2}) \quad \{\text{product rule}\} \\
 & = e^{1-2x^2} - 4x^2e^{1-2x^2} \\
 & = e^{1-2x^2}(1 - 4x^2) \\
 \therefore & f''(x) = -4xe^{1-2x^2}(1 - 4x^2) + e^{1-2x^2}(-8x) \quad \{\text{product rule}\} \\
 & = -4xe^{1-2x^2}(1 - 4x^2 + 2) \\
 & = -4xe^{1-2x^2}(3 - 4x^2)
 \end{aligned}$$

b Stationary points occur where $f'(x) = 0$

$$\begin{aligned}\therefore e^{1-2x^2}(1-4x^2) &= 0 && \{\text{using a}\} \\ \therefore 1-4x^2 &= 0 && \{e^{1-2x^2} > 0 \text{ for all } x\} \\ \therefore x^2 &= \frac{1}{4} \\ \therefore x &= \pm \frac{1}{2}\end{aligned}$$

The sign diagram of $f'(x)$ is



Now $f(-\frac{1}{2}) = (-\frac{1}{2})e^{1-2(\frac{1}{4})} = -\frac{1}{2}e^{\frac{1}{2}} = -\frac{\sqrt{e}}{2}$ and

$$f(\frac{1}{2}) = (\frac{1}{2})e^{1-2(\frac{1}{4})} = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$$

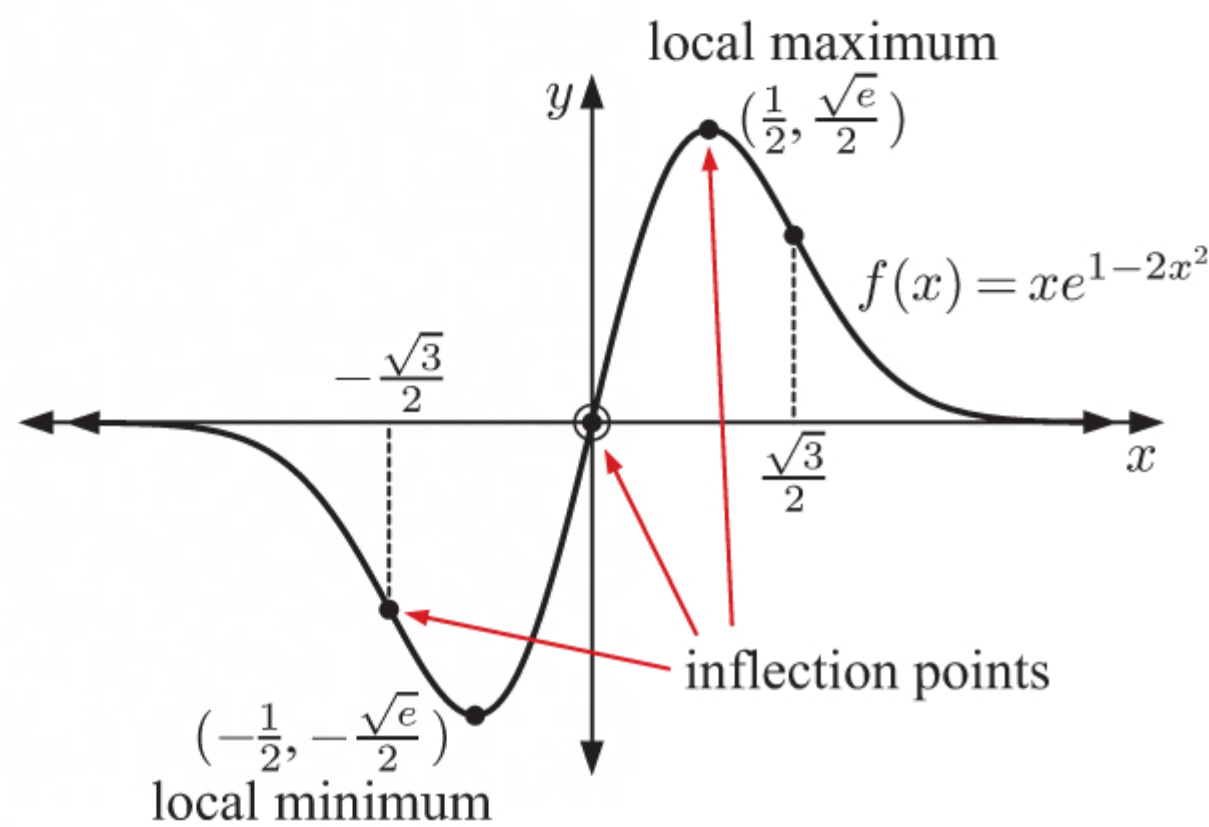
So $(-\frac{1}{2}, -\frac{\sqrt{e}}{2})$ is a local minimum and $(\frac{1}{2}, \frac{\sqrt{e}}{2})$ is a local maximum.

c Inflection points occur where $f''(x) = 0$

$$\begin{aligned}\therefore -4xe^{1-2x^2}(3-4x^2) &= 0 && \{\text{using a}\} \\ \therefore x(3-4x^2) &= 0 && \{e^{1-2x^2} > 0 \text{ for all } x\} \\ \therefore x &= 0 \text{ or } x^2 = \frac{3}{4} \\ \therefore x &= \pm \frac{\sqrt{3}}{2}\end{aligned}$$

So, the x -coordinates of the inflection points are $0, -\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{3}}{2}$.

d



39

$$f(x) = \frac{a \ln bx}{x}$$

$$\therefore f'(x) = \frac{\frac{ab}{bx}(x) - (1)a \ln bx}{x^2} \quad \{\text{quotient rule}\}$$

$$= \frac{a - a \ln bx}{x^2}$$

$$\therefore f''(x) = \frac{-\frac{ab}{bx}(x^2) - 2x(a - a \ln bx)}{x^4} \quad \{\text{quotient rule}\}$$

$$= \frac{-ax - 2x(a - a \ln bx)}{x^4}$$

$$= \frac{-a - 2a + 2a \ln bx}{x^3}$$

$$= \frac{-3a + 2a \ln bx}{x^3}$$

If $f(x)$ has an inflection point at $(\frac{e\sqrt{e}}{2}, \frac{9}{e\sqrt{e}})$, then

$$f\left(\frac{e\sqrt{e}}{2}\right) = \frac{9}{e\sqrt{e}} \quad \text{and} \quad f''\left(\frac{e\sqrt{e}}{2}\right) = 0$$

$$\therefore \frac{a \ln \left[b \left(\frac{e\sqrt{e}}{2} \right) \right]}{\frac{e\sqrt{e}}{2}} = \frac{9}{e\sqrt{e}} \quad \therefore \frac{-3a + 2a \ln \left[b \left(\frac{e\sqrt{e}}{2} \right) \right]}{\left(\frac{e\sqrt{e}}{2} \right)^3} = 0$$

$$\therefore a \ln \left(\frac{b}{2} e^{\frac{3}{2}} \right) = \frac{9}{2} \quad \dots (1)$$

$$\therefore -3a + 2a \ln \left(\frac{b}{2} e^{\frac{3}{2}} \right) = 0 \quad \dots (2)$$

$$\begin{aligned}\text{Substituting (1) into (2) gives: } -3a + 2\left(\frac{9}{2}\right) &= 0 \\ \therefore -3a + 9 &= 0 \\ \therefore -3a &= -9 \\ \therefore a &= 3\end{aligned}$$

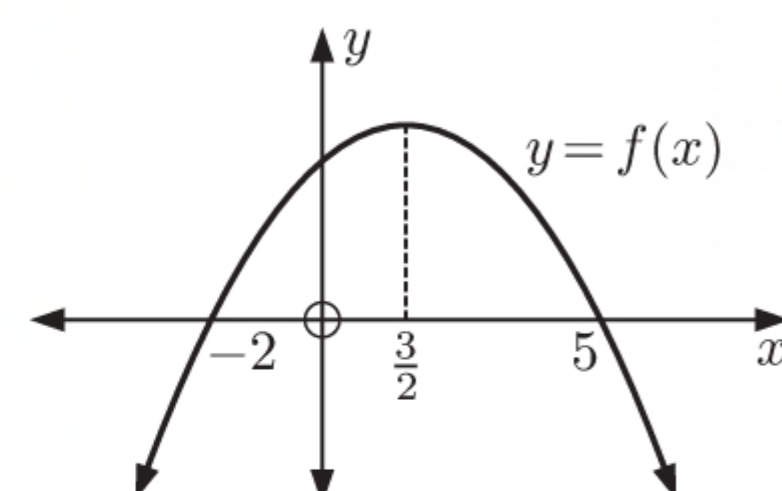
$$\begin{aligned}\text{Substituting } a = 3 \text{ into (1) gives: } 3 \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) &= \frac{9}{2} \\ \therefore \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) &= \frac{3}{2} \\ \therefore \ln\left(\frac{b}{2}\right) + \ln e^{\frac{3}{2}} &= \frac{3}{2} \\ \therefore \ln\left(\frac{b}{2}\right) + \frac{3}{2} &= \frac{3}{2} \\ \therefore \ln\left(\frac{b}{2}\right) &= 0 \\ \therefore \frac{b}{2} &= e^0 = 1 \\ \therefore b &= 2\end{aligned}$$

So, $a = 3$ and $b = 2$.

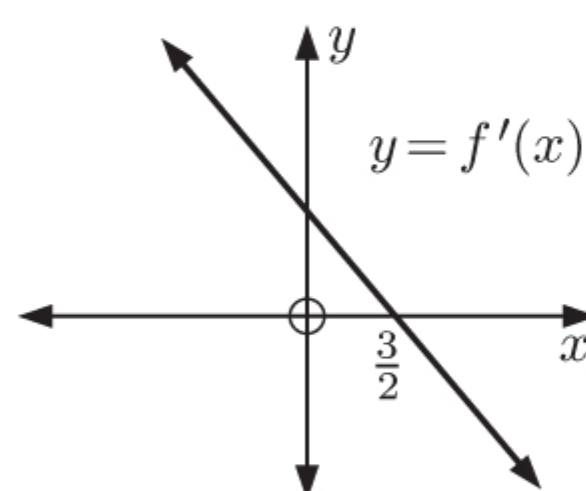
- 40 a** The turning point occurs at the average of the x -intercepts which is

$$x = \frac{5 + (-2)}{2} = \frac{3}{2}.$$

$f(x)$ is increasing for $x \leq \frac{3}{2}$, and decreasing for $x \geq \frac{3}{2}$.

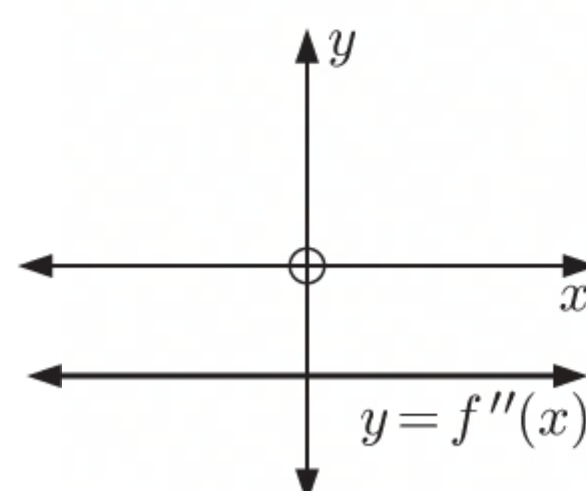


So, the graph of $y = f'(x)$ is:

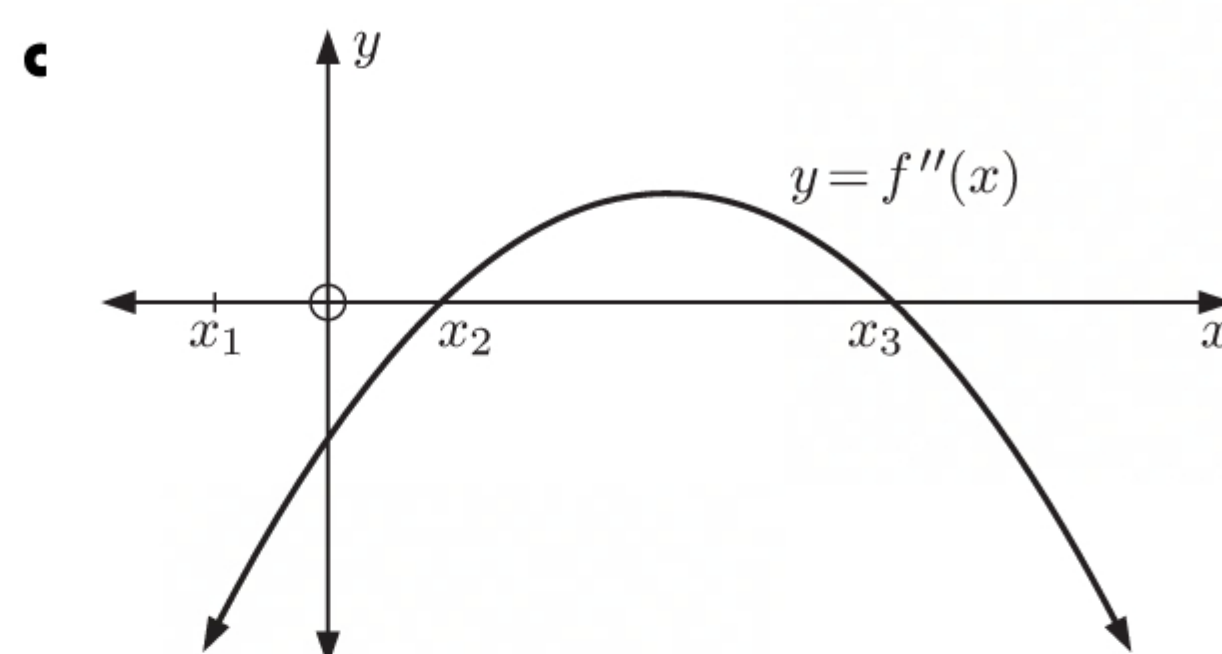
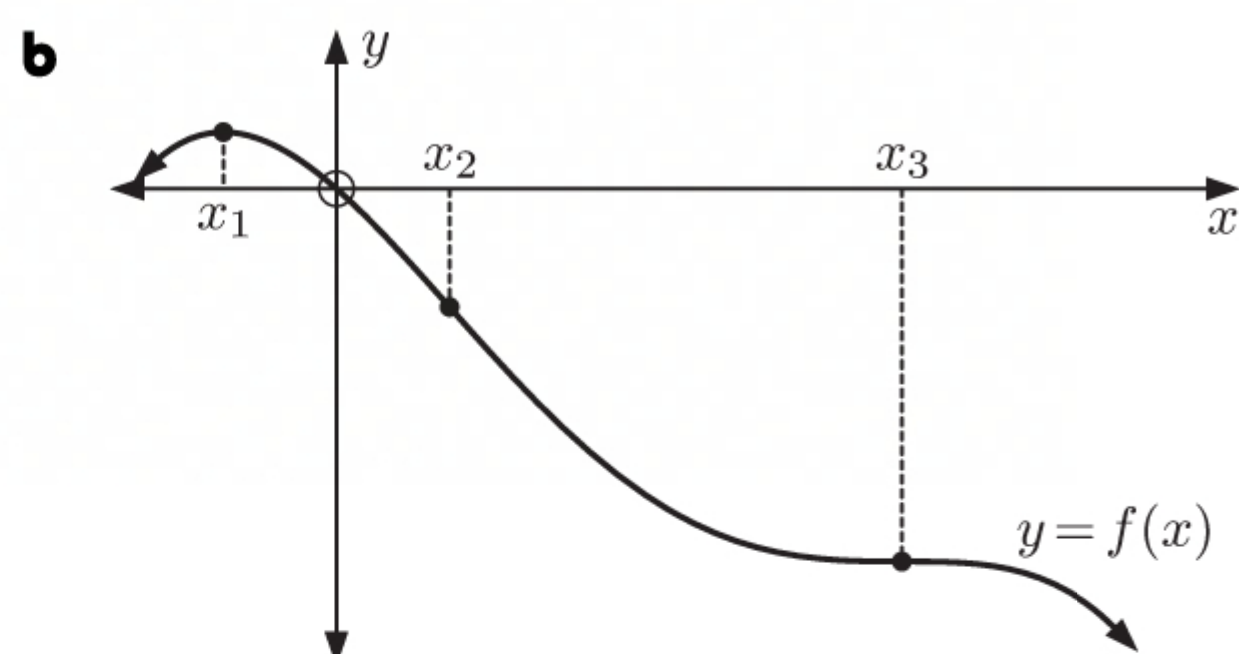
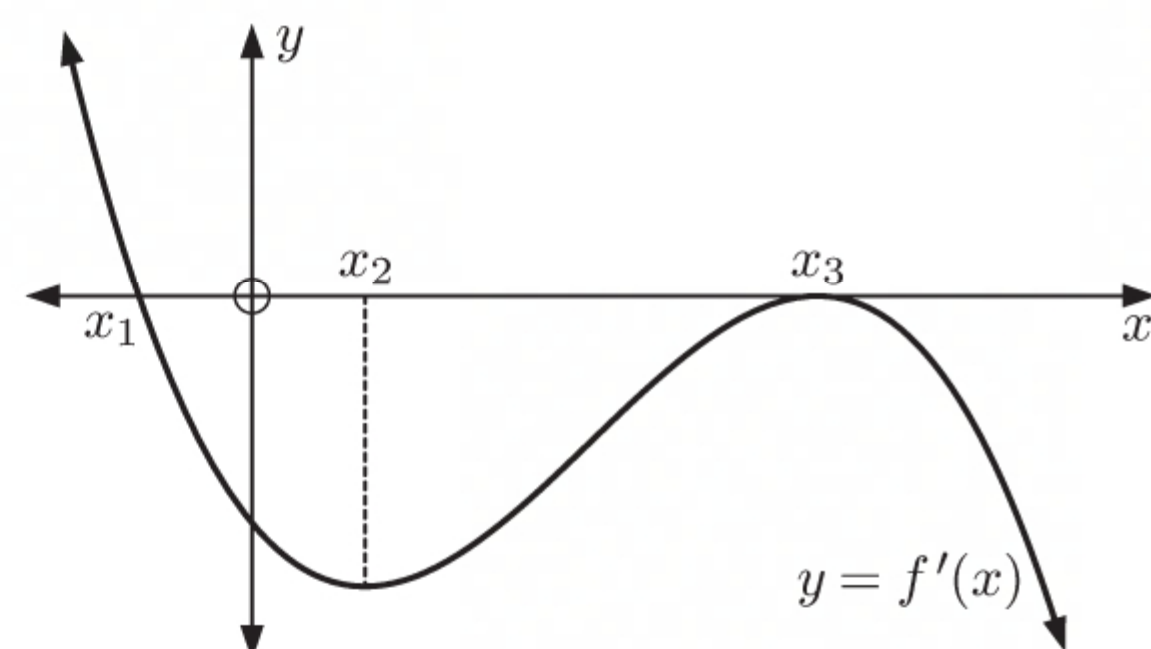


- b** $f'(x)$ has a constant negative gradient.

So, the graph of $y = f''(x)$ is:



- 41 a i** $f(x)$ is increasing when $f'(x) \geq 0$.
 $\therefore f(x)$ is increasing for $x \leq x_1$.
ii $f(x)$ is concave down when $f'(x)$ is decreasing.
 $\therefore f(x)$ is concave down for $x \leq x_2$ and $x \geq x_3$.



42 $V(t) = 10t^2 - \frac{1}{3}t^3, \quad 0 \leq t \leq 30$

a $V(5) = 10(5)^2 - \frac{1}{3}(5)^3$
 $= 250 - \frac{1}{3}(125)$
 $= 208\frac{1}{3} \text{ litres}$

This represents the volume of water in the tank after 5 minutes.

c $V'(t) = 0$
 $\therefore t(20 - t) = 0$
 $\therefore t = 0 \text{ or } 20$

b $V'(t) = 20t - t^2$
 $= t(20 - t) \text{ litres per minute}$

d $V'(5) = 5(20 - 5)$
 $= 5(15)$
 $= 75 \text{ litres per minute}$

$V'(25) = 25(20 - 25)$
 $= 25(-5)$
 $= -125 \text{ litres per minute}$

e $V'(t) = 75$
 $\therefore t(20 - t) = 75$
 $\therefore 20t - t^2 = 75$
 $\therefore t^2 - 20t + 75 = 0$
 $\therefore (t - 15)(t - 5) = 0$
 $\therefore t = 5 \text{ or } 15$

So, the volume is increasing by 75 litres per minute after 5 minutes and 15 minutes.

43 C has coordinates $(x, \cos x)$.

Let A be the area of rectangle ABCD.

$\therefore A = 2x \cos x \text{ units}^2, \quad 0 \leq x \leq \frac{\pi}{2}$

Now $\frac{dA}{dx} = 2 \cos x + 2x(-\sin x) \quad \{\text{product rule}\}$
 $= 2 \cos x - 2x \sin x$
 $= 2(\cos x - x \sin x)$

A is maximised when $\frac{dA}{dx} = 0$

$\therefore 2(\cos x - x \sin x) = 0$

$\therefore \cos x - x \sin x = 0$

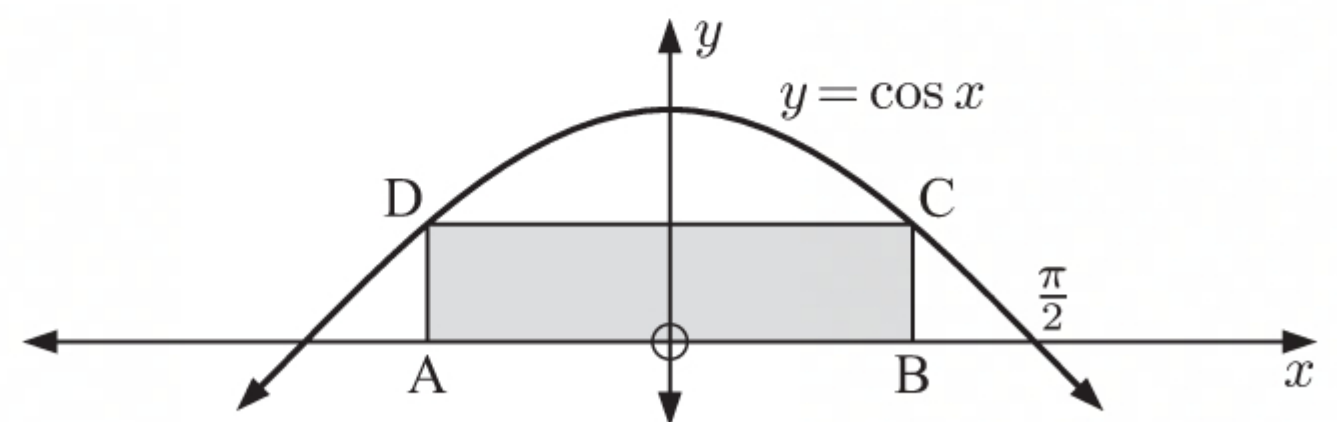
$\therefore \cos x = x \sin x$

$\therefore x \approx 0.860 \quad \{\text{using technology}\}$

The sign diagram of $\frac{dA}{dx}$ is

$\therefore A$ is maximised when $x \approx 0.860$.

\therefore when ABCD has maximum area, C has coordinates $(\approx 0.860, \approx \cos 0.860)$ which is $(\approx 0.860, \approx 0.652)$.



44 a $N = (8 - t)e^{t-6}, \quad 0 \leq t \leq 8$

$\frac{dN}{dt} = (-1)e^{t-6} + (8 - t)e^{t-6} \quad \{\text{product rule}\}$
 $= e^{t-6}(-1 + 8 - t)$
 $= (7 - t)e^{t-6}$

- b i** The turning point occurs when

$$\frac{dN}{dt} = 0$$

$$\therefore (7-t)e^{t-6} = 0$$

$$\therefore 7-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

$$\therefore t = 7$$

$$\text{When } t = 7, \quad N = (8-7)e^{7-6}$$

$$= (1)e^1$$

$$= e$$

\therefore the turning point has coordinates $(7, e)$.

iii $N = 0$

$$\therefore (8-t)e^{t-6} = 0$$

$$\therefore 8-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

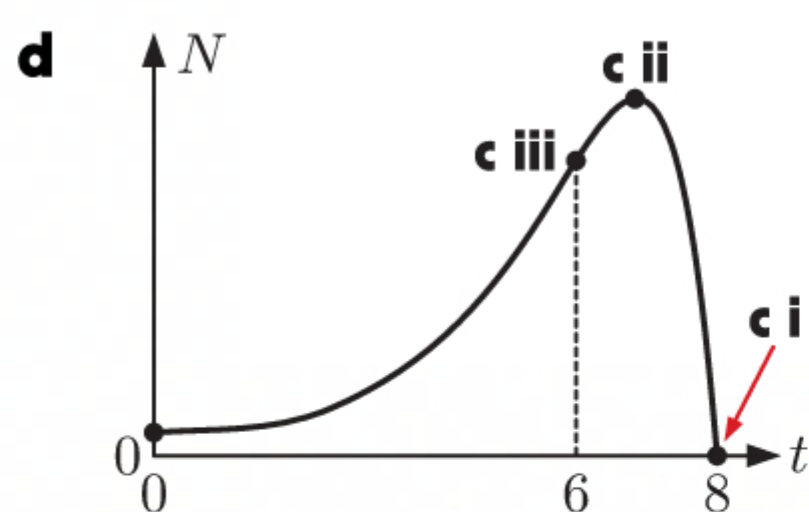
$$\therefore t = 8$$

\therefore the t -intercept has coordinates $(8, 0)$.

- c i** Using the t -intercept, the time when all the bacteria are dead is $t = 8$ hours.

- ii** Using the turning point, the maximum number of bacteria reached in the sample was $N = e \approx 2.71$ million bacteria.

- iii** Using the point of inflection, the time at which the rate of increase of the bacteria is a maximum is $t = 6$ hours.



45 $W(t) = 100e^{-\frac{t}{20}}, \quad t \geq 0$

a $W(0) = 100e^0$
 $= 100$

\therefore the initial amount of radioactive substance present is 100 grams.

c $W'(t) = 100\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)$
 $= -5e^{-\frac{t}{20}}$

Since $e^{-\frac{t}{20}} > 0$ for all t , $W'(t) < 0$ for all t .

So, the weight of the radioactive substance is decreasing for all $t \geq 0$.

e As $t \rightarrow \infty$, $e^{-\frac{t}{20}} \rightarrow 0$
 $\therefore W(t) \rightarrow 0$

ii $\frac{d^2N}{dt^2} = (-1)e^{t-6} + (7-t)e^{t-6} \quad \{\text{product rule}\}$
 $= e^{t-6}(-1+7-t)$
 $= (6-t)e^{t-6}$

The point of inflection occurs when

$$\frac{d^2N}{dt^2} = 0$$

$$\therefore (6-t)e^{t-6} = 0$$

$$\therefore 6-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

$$\therefore t = 6$$

$$\text{When } t = 6, \quad N = (8-6)e^{6-6}$$

$$= 2e^0$$

$$= 2$$

\therefore the point of inflection has coordinates $(6, 2)$.

b $W(t) = \frac{100}{2} = 50$

$$\therefore 100e^{-\frac{t}{20}} = 50$$

$$\therefore e^{-\frac{t}{20}} = \frac{50}{100} = \frac{1}{2}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{2}\right) = -\ln 2$$

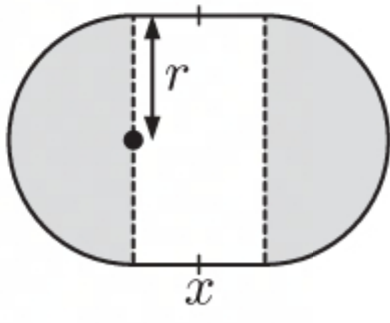
$$\therefore t = 20 \ln 2 \approx 13.9$$

\therefore it will take about 13.9 hours for half of the mass to decay.

d $W'(3) = -5e^{-\frac{3}{20}} \approx -4.30$

The weight of the radioactive substance is decreasing at about 4.30 grams per day after 3 days.

46



Let the length of the straight sides be x units and the radius of the semi-circular ends be r units.

$$\therefore A = 2xr + \pi r^2$$

$$\therefore 2xr = A - \pi r^2$$

$$\therefore x = \frac{A}{2r} - \frac{\pi r}{2} \quad \dots (*)$$

Let c be the cost of tiling the straight sides per unit length.

$$\begin{aligned} \therefore \text{the total tiling cost } C &= 2cx + 2\pi r \times 1.25c \\ &= 2cx + \frac{5c}{2}\pi r \\ &= 2c\left(\frac{A}{2r} - \frac{\pi r}{2}\right) + \frac{5c}{2}\pi r \quad \{\text{using } (*)\} \\ &= \frac{Ac}{r} - c\pi r + \frac{5c}{2}\pi r \\ &= \frac{Ac}{r} + \frac{3c}{2}\pi r \end{aligned}$$

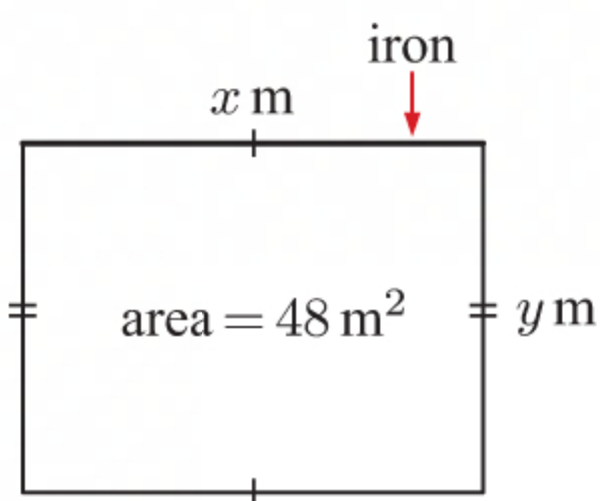
$$\begin{aligned} \text{Now } \frac{dC}{dr} &= -\frac{Ac}{r^2} + \frac{3c}{2}\pi \text{ which is 0 when } -\frac{Ac}{r^2} + \frac{3c}{2}\pi = 0 \\ \therefore -\frac{Ac}{r^2} &= -\frac{3c}{2}\pi \\ \therefore r^2 &= \frac{2A}{3\pi} \\ \therefore r &= \sqrt{\frac{2A}{3\pi}} \quad \{r > 0\} \end{aligned}$$

The sign diagram of $\frac{dC}{dr}$ is

So, C is minimised when $r = \sqrt{\frac{2A}{3\pi}}$.

$$\begin{aligned} \text{Now when } r &= \sqrt{\frac{2A}{3\pi}}, \text{ the shaded area} = \pi r^2 \\ &= \pi \left(\sqrt{\frac{2A}{3\pi}} \right)^2 \\ &= \frac{2}{3}A \end{aligned}$$

47 a



Let the length of the side adjacent to the corrugated iron fence be y m.

$$\text{Now area} = xy$$

$$\therefore xy = 48$$

$$\therefore y = \frac{48}{x}$$

$$\text{Cost of fencing} = 18(2y + x) + 30x$$

$$\therefore C = 18\left(\frac{2 \times 48}{x} + x\right) + 30x$$

$$\therefore C = \frac{2 \times 48 \times 18}{x} + 18x + 30x$$

$$\therefore C = \frac{2 \times 48 \times 18}{x} + 48x$$

$$\therefore C = 48\left(\frac{36}{x} + x\right) \text{ dollars}$$

$$\mathbf{b} \quad \frac{dC}{dx} = 48\left(\frac{-36}{x^2} + 1\right)$$

Now C is minimised when $\frac{dC}{dx} = 0$

$$\therefore 48\left(\frac{-36}{x^2} + 1\right) = 0$$

$$\therefore \frac{-36}{x^2} + 1 = 0$$

$$\therefore \frac{36}{x^2} = 1$$

$$\therefore x^2 = 36$$

$$\therefore x = 6 \quad \{x > 0\}$$

The sign diagram of $\frac{dC}{dx}$ is

$\therefore C$ is minimised when $x = 6$.

When $x = 6$, $y = \frac{48}{6} = 8$.

The dimensions that minimise the cost of fencing are $6 \text{ m} \times 8 \text{ m}$, where one of the 6 m sides is fenced with corrugated iron.

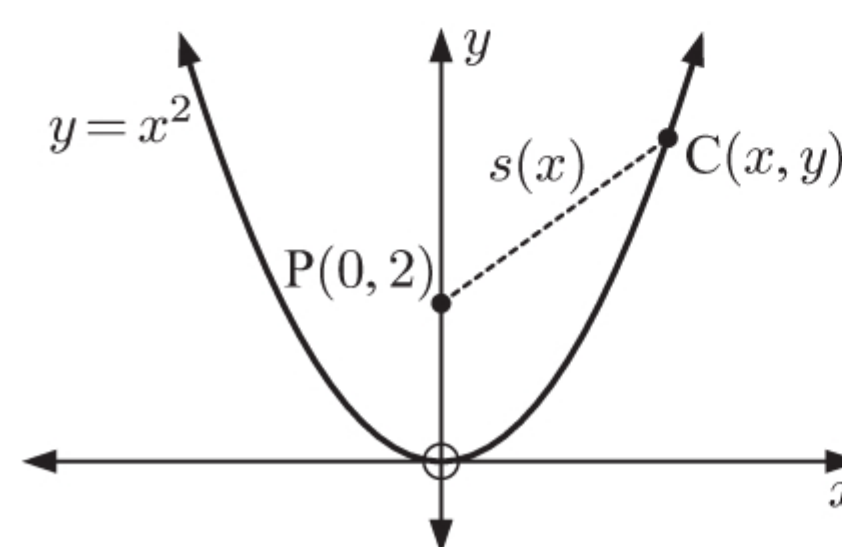
48 a C has coordinates (x, x^2) .

Now $s(x) = CP$

$$= \sqrt{(x-0)^2 + (x^2-2)^2}$$

$$= \sqrt{x^2 + x^4 - 4x^2 + 4}$$

$$= \sqrt{x^4 - 3x^2 + 4}$$



$$\mathbf{b} \quad s'(x) = \frac{1}{2}(x^4 - 3x^2 + 4)^{-\frac{1}{2}}(4x^3 - 6x) \quad \{\text{chain rule}\}$$

$$= \frac{2x^3 - 3x}{\sqrt{x^4 - 3x^2 + 4}}$$

which is 0 when $2x^3 - 3x = 0$

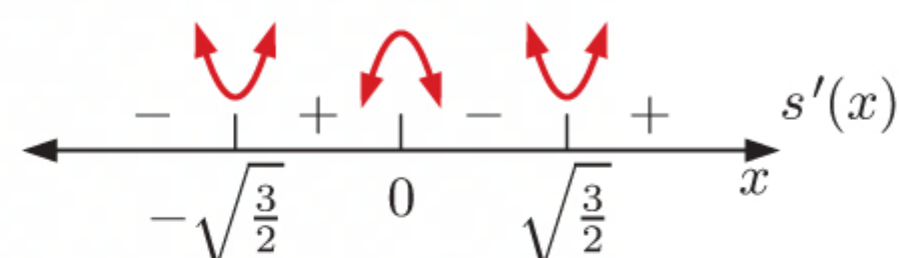
$$\therefore x(2x^2 - 3) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 2x^2 - 3 = 0$$

$$\therefore x^2 = \frac{3}{2}$$

$$\therefore x = \pm\sqrt{\frac{3}{2}}$$

The sign diagram of $s'(x)$ is



\therefore there is a local maximum at $x = 0$, and local minima at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$.

Critical point (x)	$s(x)$
-2 (end point)	≈ 2.83
$-\sqrt{\frac{3}{2}}$ (local minimum)	≈ 1.32
0 (local maximum)	2
$\sqrt{\frac{3}{2}}$ (local minimum)	≈ 1.32
2 (end point)	≈ 2.83

The greatest distance between the comet and the observer is ≈ 2.83 units when $x = \pm 2$.

The shortest distance between the comet and the observer is ≈ 1.32 units when $x = \pm\sqrt{\frac{3}{2}}$.

49 a Surface area = area of triangles + area of rectangles

$$= 2\left(\frac{1}{2}x^2\right) + 2(xy)$$

$$= x^2 + 2xy$$

$$\therefore x^2 + 2xy = 27$$

b From **a**, $x^2 + 2xy = 27$

$$\therefore 2xy = 27 - x^2$$

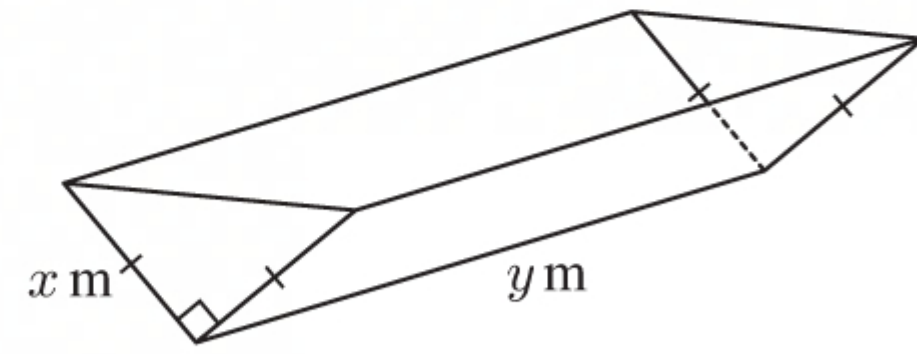
$$\therefore y = \frac{27}{2x} - \frac{x}{2}$$

Now volume = cross-sectional area \times depth

$$\therefore V = \frac{1}{2}x^2y$$

$$\therefore V = \frac{1}{2}x^2\left(\frac{27}{2x} - \frac{x}{2}\right)$$

$$\therefore V = \frac{27x}{4} - \frac{x^3}{4}$$



c $\frac{dV}{dx} = \frac{27}{4} - \frac{3}{4}x^2$

Now V is maximised when $\frac{dV}{dx} = 0$

$$\therefore \frac{27}{4} - \frac{3}{4}x^2 = 0$$

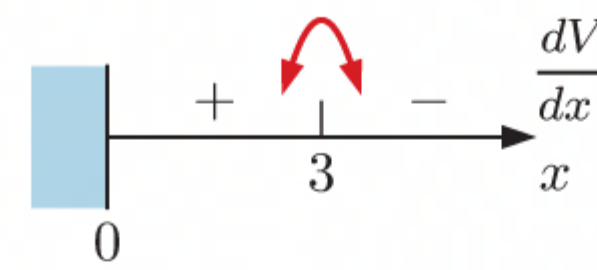
$$\therefore 27 - 3x^2 = 0$$

$$\therefore 3x^2 = 27$$

$$\therefore x^2 = 9$$

$$\therefore x = 3 \quad \{x > 0\}$$

The sign diagram of $\frac{dV}{dx}$ is



$\therefore V$ is maximised when $x = 3$.

When $x = 3$, $y = \frac{27}{2(3)} - \frac{3}{2}$

$$= \frac{9}{2} - \frac{3}{2}$$

$$= \frac{6}{2}$$

$$= 3$$

So, V is maximised when $x = y = 3$.

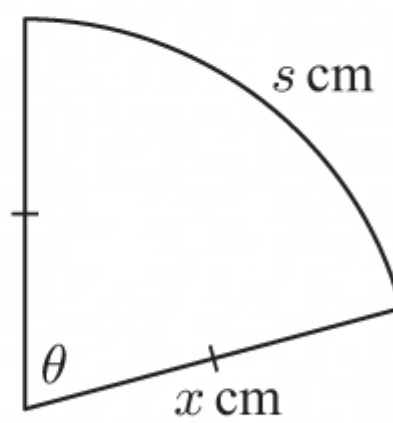
50 a Perimeter = $2x + s$

$$= 2x + \theta x$$

$$\therefore 2x + \theta x = 40$$

$$\therefore \theta x = 40 - 2x$$

$$\therefore \theta = \frac{40}{x} - 2$$



c $\frac{dA}{dx} = 20 - 2x$ which is 0 when $x = 10$.

The sign diagram of $\frac{dA}{dx}$ is

When $x = 10$, $\theta = \frac{40}{10} - 2 = 4 - 2 = 2$.

$\therefore A$ is a maximum when $x = 10$ and $\theta = 2$.

b Area $A = \frac{1}{2}\theta x^2$

$$= \frac{1}{2}\left(\frac{40}{x} - 2\right)x^2 \quad \{\text{using a}\}$$

$$= 20x - x^2 \text{ cm}^2$$

51 a $\int (3x^2 + 2x + 1) dx$

$$= \frac{3x^3}{3} + \frac{2x^2}{2} + x + c$$

$$= x^3 + x^2 + x + c$$

b $\int e^{4x} dx = \frac{1}{4}e^{4x} + c$

c $\int \cos(2x + 1) dx$

$$= \frac{1}{2} \sin(2x + 1) + c$$

$$\begin{aligned}
 \mathbf{52} \quad \mathbf{a} \quad f(x) &= \sqrt{xe^x} = (xe^x)^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}(xe^x)^{-\frac{1}{2}} \times \frac{d}{dx}(xe^x) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2\sqrt{xe^x}} \times ((1)e^x + xe^x) \quad \{\text{product rule}\} \\
 &= \frac{e^x + xe^x}{2\sqrt{xe^x}} \\
 &= \frac{e^x(1+x)}{2e^{\frac{x}{2}}\sqrt{x}} \\
 &= \frac{e^{\frac{x}{2}}(1+x)}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{53} \quad \frac{d}{dx}(x^2 \ln x) &= 2x \ln x + x^2 \left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\
 &= 2x \ln x + x
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \int x \ln x \, dx &= \int \frac{\frac{d}{dx}(x^2 \ln x) - x}{2} \, dx \\
 &= \frac{1}{2} \int \left(\frac{d}{dx}(x^2 \ln x) - x \right) \, dx \\
 &= \frac{1}{2} \left(x^2 \ln x - \frac{1}{2}x^2 \right) + c \\
 &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{54} \quad \mathbf{a} \quad \int (x\sqrt{x} - 5 \cos x) \, dx &= \int (x^{\frac{3}{2}} - 5 \cos x) \, dx \\
 &= \frac{2}{5}x^{\frac{5}{2}} - 5 \sin x + c \\
 &= \frac{2}{5}x^2\sqrt{x} - 5 \sin x + c
 \end{aligned}$$

$$\mathbf{55} \quad \mathbf{a} \quad \int (x-3)^2 \, dx = \frac{1}{3}(x-3)^3 + c$$

$$\begin{aligned}
 \mathbf{56} \quad f'(x) &= (x^2 + 2)^2 = x^4 + 4x^2 + 4, \quad f(1) = \frac{8}{15} \\
 \therefore f(x) &= \int (x^4 + 4x^2 + 4) \, dx \\
 &= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f(1) &= \frac{8}{15}, \quad \therefore \frac{1}{5}(1)^5 + \frac{4}{3}(1)^3 + 4(1) + c = \frac{8}{15} \\
 \therefore \frac{1}{5} + \frac{4}{3} + 4 + c &= \frac{8}{15} \\
 \therefore \frac{83}{15} + c &= \frac{8}{15} \\
 \therefore c &= -5
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x - 5$$

$$\mathbf{57} \quad f'(x) = \sqrt{4x+5} = (4x+5)^{\frac{1}{2}}, \quad f(0) = -\frac{\sqrt{5}}{6}$$

$$\begin{aligned}
 \mathbf{a} \quad f'(x) \text{ is defined when } 4x+5 &\geq 0 \\
 \therefore 4x &\geq -5 \\
 \therefore x &\geq -\frac{5}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{2e^{\frac{x}{2}}(1+x)}{\sqrt{x}} \, dx &= \int 4 \left(\frac{e^{\frac{x}{2}}(1+x)}{2\sqrt{x}} \right) \, dx \\
 &= \int 4f'(x) \, dx \quad \{\text{using a}\} \\
 &= 4 \int f'(x) \, dx \\
 &= 4f(x) + c \\
 &= 4\sqrt{xe^x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \left(\sin x + \frac{1}{\sqrt[3]{x}} \right) \, dx &= \int (\sin x + x^{-\frac{1}{3}}) \, dx \\
 &= -\cos x + \frac{3}{2}x^{\frac{2}{3}} + c \\
 &= -\cos x + \frac{3}{2}\sqrt[3]{x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{x^2+3x+5}{\sqrt[3]{x}} \, dx &= \int \frac{x^2+3x+5}{x^{\frac{1}{3}}} \, dx \\
 &= \int (x^{\frac{5}{3}} + 3x^{\frac{2}{3}} + 5x^{-\frac{1}{3}}) \, dx \\
 &= \frac{3}{8}x^{\frac{8}{3}} + \frac{9}{5}x^{\frac{5}{3}} + \frac{15}{2}x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= \int (4x+5)^{\frac{1}{2}} \, dx \\
 &= \frac{2}{3} \times \frac{1}{4}(4x+5)^{\frac{3}{2}} + c \\
 &= \frac{1}{6}(4x+5)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\text{Now } f(0) = -\frac{\sqrt{5}}{6},$$

$$\therefore \frac{1}{6}(5)^{\frac{3}{2}} + c = -\frac{\sqrt{5}}{6}$$

$$\therefore \frac{5\sqrt{5}}{6} + c = -\frac{\sqrt{5}}{6}$$

$$\therefore c = -\frac{6\sqrt{5}}{6} = -\sqrt{5}$$

$$\therefore f(x) = \frac{1}{6}(4x+5)^{\frac{3}{2}} - \sqrt{5}$$

58 a $f''(x) = e^x + 2x - 1, \quad f'(0) = 4, \quad f(0) = 1$

$$\begin{aligned}\therefore f'(x) &= \int (e^x + 2x - 1) dx \\ &= e^x + x^2 - x + c\end{aligned}$$

Now $f'(0) = 4, \quad \therefore e^0 + 0^2 - 0 + c = 4$

$$\therefore 1 + c = 4$$

$$\therefore c = 3$$

$$\therefore f'(x) = e^x + x^2 - x + 3$$

$$\begin{aligned}\therefore f(x) &= \int (e^x + x^2 - x + 3) dx \\ &= e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + d\end{aligned}$$

Now $f(0) = 1,$

$$\therefore e^0 + \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 3(0) + d = 1$$

$$\therefore 1 + d = 1$$

$$\therefore d = 0$$

$$\therefore f(x) = e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x$$

b $f''(x) = 2 + \sin x, \quad f'(\pi) = 1, \quad f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$

$$\begin{aligned}\therefore f'(x) &= \int (2 + \sin x) dx \\ &= 2x - \cos x + c\end{aligned}$$

Now $f'(\pi) = 1, \quad \therefore 2\pi - \cos \pi + c = 1$

$$\therefore 2\pi - (-1) + c = 1$$

$$\therefore 2\pi + 1 + c = 1$$

$$\therefore 2\pi + c = 0$$

$$\therefore c = -2\pi$$

$$\therefore f'(x) = 2x - \cos x - 2\pi$$

$$\begin{aligned}\therefore f(x) &= \int (2x - \cos x - 2\pi) dx \\ &= x^2 - \sin x - 2\pi x + d\end{aligned}$$

Now $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4},$

$$\therefore \frac{\pi^2}{4} - \sin \frac{\pi}{2} - 2\pi\left(\frac{\pi}{2}\right) + d = \frac{\pi^2}{4}$$

$$\therefore -1 - \pi^2 + d = 0$$

$$\therefore d = \pi^2 + 1$$

$$\therefore f(x) = x^2 - \sin x - 2\pi x + \pi^2 + 1$$

c $f''(x) = \frac{2}{\sqrt{x}} + 3x = 2x^{-\frac{1}{2}} + 3x, \quad f(1) = -\frac{19}{3}, \quad f(4) = \frac{64}{3}$

$$\begin{aligned}\therefore f'(x) &= \int (2x^{-\frac{1}{2}} + 3x) dx \\ &= 4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= \int \left(4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c\right) dx \\ &= \frac{8}{3}x^{\frac{3}{2}} + \frac{x^3}{2} + cx + d\end{aligned}$$

Now $f(1) = -\frac{19}{3}$

$$\therefore \frac{8}{3}(1)^{\frac{3}{2}} + \frac{1^3}{2} + c(1) + d = -\frac{19}{3}$$

$$\therefore \frac{8}{3} + \frac{1}{2} + c + d = -\frac{19}{3}$$

$$\therefore \frac{19}{6} + c + d = -\frac{19}{3}$$

$$\therefore c + d = -\frac{19}{2} \quad \dots (1)$$

$$4c + 4d = -38 \quad \{(1) \times 4\}$$

$$\underline{-4c - d = 32} \quad \{(2) \times -1\}$$

Adding, $3d = -6$

$$\therefore d = -2$$

Substituting $d = -2$ into (1) gives $c - 2 = -\frac{19}{2}$

$$\therefore c = -\frac{15}{2}$$

$$\begin{aligned}\therefore f(x) &= \frac{8}{3}x^{\frac{3}{2}} + \frac{1}{2}x^3 - \frac{15}{2}x - 2 \\ &= \frac{8}{3}x\sqrt{x} + \frac{1}{2}x^3 - \frac{15}{2}x - 2\end{aligned}$$

59 a $\int (3x - 5)^3 dx$

$$\begin{aligned}&= \frac{1}{3} \frac{(3x - 5)^4}{4} + c \\ &= \frac{1}{12} (3x - 5)^4 + c\end{aligned}$$

b $\int \frac{2}{\sqrt{4-x}} dx$

$$\begin{aligned}&= \int 2(4-x)^{-\frac{1}{2}} dx \\ &= 2 \times \frac{1}{-\frac{1}{2}} \frac{(4-x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -4(4-x)^{\frac{1}{2}} + c \\ &= -4\sqrt{4-x} + c\end{aligned}$$

c $\int (e^{2x} + 3e^{-x+2}) dx$

$$\begin{aligned}&= \frac{1}{2}e^{2x} + \frac{3}{-1}e^{-x+2} + c \\ &= \frac{1}{2}e^{2x} - 3e^{-x+2} + c\end{aligned}$$

$$\begin{aligned}
 \mathbf{60} \quad \mathbf{a} \quad & \int (2 \sin(x-3) + e^{3x}) dx \\
 &= 2(-\cos(x-3)) + \frac{1}{3}e^{3x} + c \\
 &= -2\cos(x-3) + \frac{1}{3}e^{3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{2}{5x-1} dx \\
 &= \frac{2}{5} \ln|5x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \cos(5-7x) dx \\
 &= -\frac{1}{7} \sin(5-7x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{61} \quad \mathbf{a} \quad & \int (2 \sin^2 x - 1) dx \\
 &= \int (-(1 - 2 \sin^2 x)) dx \\
 &= \int (-\cos 2x) dx \quad \{\text{double angle formula}\} \\
 &= -\frac{1}{2} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (\sin 2x - \cos 2x)^2 dx \\
 &= \int (\sin^2 2x - 2 \sin 2x \cos 2x + \cos^2 2x) dx \\
 &= \int (1 - 2 \sin 2x \cos 2x) dx \\
 &= \int (1 - \sin 4x) dx \quad \{\text{double angle formula}\} \\
 &= x + \frac{1}{4} \cos 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (\cos x + 2)^2 dx \\
 &= \int (\cos^2 x + 4 \cos x + 4) dx \\
 &= \int \left(\frac{1}{2} \cos 2x + \frac{1}{2} + 4 \cos x + 4\right) dx \quad \{\text{double angle formula}\} \\
 &= \int \left(\frac{1}{2} \cos 2x + 4 \cos x + \frac{9}{2}\right) dx \\
 &= \frac{1}{4} \sin 2x + 4 \sin x + \frac{9}{2}x + c
 \end{aligned}$$

$$\mathbf{62} \quad f'(x) = \frac{4}{5-x}, \quad f(4) = 6$$

$$\begin{aligned}
 \therefore f(x) &= \int \frac{4}{5-x} dx \\
 &= -4 \ln|5-x| + c
 \end{aligned}$$

$$\text{Now } f(4) = 6, \quad \therefore -4 \ln|5-4| + c = 6$$

$$\therefore -4 \ln 1 + c = 6$$

$$\therefore c = 6$$

$$\therefore f(x) = -4 \ln|5-x| + 6$$

$$\begin{aligned}
 \mathbf{63} \quad \mathbf{a} \quad & \int 3x^2(5+x^3)^4 dx \\
 &= \int u^4 \frac{du}{dx} dx \quad \left\{ u = 5+x^3 \quad \therefore \frac{du}{dx} = 3x^2 \right\} \\
 &= \int u^4 du \\
 &= \frac{1}{5} u^5 + c \\
 &= \frac{1}{5} (5+x^3)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{e^{\frac{1}{x}}}{x^2} dx \\
 &= \int e^u \left(-\frac{du}{dx}\right) dx \quad \left\{ u = \frac{1}{x} \quad \therefore \frac{du}{dx} = -\frac{1}{x^2} \right\} \\
 &= \int -e^u du \\
 &= -e^u + c \\
 &= -e^{\frac{1}{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int x e^{x^2+2} dx \\
 &= \int e^u \times \frac{1}{2} \frac{du}{dx} dx \quad \left\{ u = x^2 + 2 \quad \therefore \frac{du}{dx} = 2x \right\} \\
 &= \int \frac{1}{2} e^u du \\
 &= \frac{1}{2} e^u + c \\
 &= \frac{1}{2} e^{x^2+2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{2(\ln x)^2}{x} dx \\
 &= \int 2u^2 \frac{du}{dx} dx \quad \left\{ u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x} \right\} \\
 &= \int 2u^2 du \\
 &= \frac{2}{3} u^3 + c \\
 &= \frac{2}{3} (\ln x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
\mathbf{64} \quad \mathbf{a} \quad & \int \sqrt{x^2 + 3x - 1} (2x + 3) dx \\
&= \int \sqrt{u} \frac{du}{dx} dx \\
&\quad \left\{ u = x^2 + 3x - 1 \quad \therefore \frac{du}{dx} = 2x + 3 \right\} \\
&= \int u^{\frac{1}{2}} du \\
&= \frac{2}{3} u^{\frac{3}{2}} + c \\
&= \frac{2}{3} (x^2 + 3x - 1)^{\frac{3}{2}} + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int \frac{e^x + 2}{e^x + 2x} dx \\
&= \int \frac{1}{u} \frac{du}{dx} dx \\
&\quad \left\{ u = e^x + 2x \quad \therefore \frac{du}{dx} = e^x + 2 \right\} \\
&= \int \frac{1}{u} du \\
&= \ln|u| + c \\
&= \ln|e^x + 2x| + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad & \int \frac{6 - 8x}{2x^2 - 3x + 2} dx \\
&= \int \frac{1}{u} \left(-2 \frac{du}{dx} \right) dx \quad \left\{ u = 2x^2 - 3x + 2 \quad \therefore \frac{du}{dx} = 4x - 3 \right\} \\
&= \int -\frac{2}{u} du \\
&= -2 \ln|u| + c \\
&= -2 \ln|2x^2 - 3x + 2| + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{65} \quad \mathbf{a} \quad & \int \sin^4 x \cos x dx \\
&= \int u^4 \frac{du}{dx} dx \quad \left\{ u = \sin x \quad \therefore \frac{du}{dx} = \cos x \right\} \\
&= \int u^4 du \\
&= \frac{1}{5} u^5 + c \\
&= \frac{1}{5} \sin^5 x + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int 3x^3 \sin(x^4) dx \\
&= \int \sin u \left(\frac{3}{4} \frac{du}{dx} \right) dx \quad \left\{ u = x^4 \quad \therefore \frac{du}{dx} = 4x^3 \right\} \\
&= \int \frac{3}{4} \sin u du \\
&= -\frac{3}{4} \cos u + c \\
&= -\frac{3}{4} \cos(x^4) + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad & \int \frac{\sin 2x}{(3 - \cos 2x)^3} dx \\
&= \int \frac{1}{u^3} \left(\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = 3 - \cos 2x \quad \therefore \frac{du}{dx} = 2 \sin 2x \right\} \\
&= \int \frac{1}{2} u^{-3} du \\
&= \frac{1}{2} \times \frac{1}{-2} u^{-2} + c \\
&= -\frac{1}{4u^2} + c \\
&= -\frac{1}{4(3 - \cos 2x)^2} + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{66} \quad \mathbf{a} \quad \cos^4 x &= (\cos^2 x)^2 \\
&= \left(\frac{1}{2} \cos 2x + \frac{1}{2} \right)^2 && \{\text{double angle formula}\} \\
&= \frac{1}{4} \cos^2 2x + \frac{1}{2} \cos 2x + \frac{1}{4} \\
&= \frac{1}{4} \left(\frac{1}{2} \cos 4x + \frac{1}{2} \right) + \frac{1}{2} \cos 2x + \frac{1}{4} && \{\text{double angle formula}\} \\
&= \frac{1}{8} \cos 4x + \frac{1}{8} + \frac{1}{2} \cos 2x + \frac{1}{4} \\
&= \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \int \cos^4 x dx &= \int \left(\frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx \quad \{\text{using a}\} \\
&= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c
\end{aligned}$$

$$\begin{aligned}
 \mathbf{67} \quad & \int_0^{\frac{\pi}{2}} (\sin 3x + 5 \cos x) dx \\
 &= \left[-\frac{1}{3} \cos 3x + 5 \sin x \right]_0^{\frac{\pi}{2}} \\
 &= \left(-\frac{1}{3} \cos \frac{3\pi}{2} + 5 \sin \frac{\pi}{2} \right) - \left(-\frac{1}{3} \cos 0 + 5 \sin 0 \right) \\
 &= 0 + 5 - \left(-\frac{1}{3} + 0 \right) \\
 &= 5 + \frac{1}{3} \\
 &= \frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{68} \quad & \int_a^{2a} \sqrt{x} dx = 2 \\
 & \therefore \int_a^{2a} x^{\frac{1}{2}} dx = 2 \\
 & \therefore \left[\frac{2}{3} x^{\frac{3}{2}} \right]_a^{2a} = 2 \\
 & \therefore \frac{2}{3} (2a)^{\frac{3}{2}} - \frac{2}{3} a^{\frac{3}{2}} = 2 \\
 & \therefore 2^{\frac{3}{2}} \times a^{\frac{3}{2}} - a^{\frac{3}{2}} = 3 \\
 & \therefore a^{\frac{3}{2}} (2^{\frac{3}{2}} - 1) = 3 \\
 & \therefore a^{\frac{3}{2}} = \frac{3}{2^{\frac{3}{2}} - 1} \\
 & \therefore a = \left(\frac{3}{2^{\frac{3}{2}} - 1} \right)^{\frac{2}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{69} \quad & y = x\sqrt{4-x} = x(4-x)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= (1)(4-x)^{\frac{1}{2}} + x \left(\frac{1}{2} (4-x)^{-\frac{1}{2}} (-1) \right) \quad \{\text{product rule}\} \\
 &= \sqrt{4-x} - \frac{x}{2\sqrt{4-x}} \\
 &= \frac{2(4-x) - x}{2\sqrt{4-x}} \\
 &= \frac{8-2x-x}{2\sqrt{4-x}} \\
 &= \frac{8-3x}{2\sqrt{4-x}} \quad \dots (*) \\
 \therefore \int_0^2 \frac{8-3x}{\sqrt{4-x}} dx &= [x\sqrt{4-x}]_0^2 \quad \{\text{using } (*)\} \\
 &= 2\sqrt{4-2} - 0\sqrt{4-0} \\
 &= 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{70} \quad \mathbf{a} \quad & \int_1^5 \frac{2x^3+1}{x^2} dx \\
 &= \int_1^5 (2x + x^{-2}) dx \\
 &= \left[x^2 - \frac{1}{x} \right]_1^5 \\
 &= \left(25 - \frac{1}{5} \right) - (1 - 1) \\
 &= \frac{124}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_{-1}^1 e^x (2 - 3e^{-x})^2 dx \\
 &= \int_{-1}^1 e^x (4 - 12e^{-x} + 9e^{-2x}) dx \\
 &= \int_{-1}^1 (4e^x - 12 + 9e^{-x}) dx \\
 &= [4e^x - 12x - 9e^{-x}]_{-1}^1 \\
 &= (4e^1 - 12 - 9e^{-1}) - (4e^{-1} + 12 - 9e^1) \\
 &= 13e - 24 - 13e^{-1} \\
 &= 13 \left(e - \frac{1}{e} \right) - 24
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_0^2 \frac{3}{5-2x} dx \\
 &= \left[\frac{3}{-2} \ln |5-2x| \right]_0^2 \\
 &= -\frac{3}{2} \ln |5-4| + \frac{3}{2} \ln |5-0| \\
 &= -\frac{3}{2} \ln 1 + \frac{3}{2} \ln 5 \\
 &= \frac{3}{2} \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_{-2}^{\frac{\pi}{4}-2} \sin^2(x+2) dx \\
 &= \int_{-2}^{\frac{\pi}{4}-2} \left(\frac{1}{2} - \frac{1}{2} \cos(2(x+2)) \right) dx \\
 &= \int_{-2}^{\frac{\pi}{4}-2} \left(\frac{1}{2} - \frac{1}{2} \cos(2x+4) \right) dx \\
 &= \left[\frac{1}{2}x - \frac{1}{4} \sin(2x+4) \right]_{-2}^{\frac{\pi}{4}-2} \\
 &= \left(\frac{1}{2} \left(\frac{\pi}{4} - 2 \right) - \frac{1}{4} \sin \left(2 \left(\frac{\pi}{4} - 2 \right) + 4 \right) \right) - \left(\frac{-2}{2} - \frac{1}{4} \sin(-4+4) \right) \\
 &= \frac{\pi}{8} - 1 - \frac{1}{4} \sin \left(\frac{\pi}{2} - 4 + 4 \right) + 1 + \frac{1}{4} \sin 0 \\
 &= \frac{\pi}{8} - \frac{1}{4} \sin \frac{\pi}{2} \\
 &= \frac{\pi}{8} - \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
\mathbf{71} \quad \mathbf{a} \quad & \int (2x+3)(x^2+3x+4)^3 dx \\
&= \int u^3 \frac{du}{dx} dx \\
&\quad \left\{ u = x^2 + 3x + 4 \quad \therefore \frac{du}{dx} = 2x + 3 \right\} \\
&= \int u^3 du \\
&= \frac{1}{4}u^4 + c \\
&= \frac{1}{4}(x^2 + 3x + 4)^4 + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int_0^1 (2x+3)(x^2+3x+4)^3 dx \\
&= \left[\frac{1}{4}(x^2+3x+4)^4 \right]_0^1 \quad \{\text{using a}\} \\
&= \frac{1}{4}(1^2+3(1)+4)^4 - \frac{1}{4}(0^2+3(0)+4)^4 \\
&= \frac{1}{4}(8)^4 - \frac{1}{4}(4)^4 \\
&= 1024 - 64 \\
&= 960
\end{aligned}$$

$$\begin{aligned}
\mathbf{72} \quad \mathbf{a} \quad & \int 8xe^{x^2+1} dx \\
&= \int e^u \left(4 \frac{du}{dx} \right) dx \quad \left\{ u = x^2 + 1 \quad \therefore \frac{du}{dx} = 2x \right\} \\
&= \int 4e^u du \\
&= 4e^u + c \\
&= 4e^{x^2+1} + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int_{-2}^2 8xe^{x^2+1} dx \\
&= \left[4e^{x^2+1} \right]_{-2}^2 \quad \{\text{using a}\} \\
&= 4e^{2^2+1} - 4e^{(-2)^2+1} \\
&= 4e^5 - 4e^5 \\
&= 0
\end{aligned}$$

$$\mathbf{73} \quad \int_0^2 f(x) dx = 2$$

$$\begin{aligned}
\mathbf{a} \quad & \int_2^0 f(x) dx = - \int_0^2 f(x) dx \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad & \int_0^2 (kf(x) + 2) dx = 10 \\
\therefore & \int_0^2 kf(x) dx + \int_0^2 2 dx = 10 \\
\therefore & k \int_0^2 f(x) dx + 2(2-0) = 10 \\
& \therefore k(2) + 4 = 10 \\
& \therefore 2k + 4 = 10 \\
& \therefore 2k = 6 \\
& \therefore k = 3
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int_0^2 (3 + 2f(x)) dx \\
&= \int_0^2 3 dx + \int_0^2 2f(x) dx \\
&= 3(2-0) + 2 \int_0^2 f(x) dx \\
&= 6 + 2(2) \\
&= 6 + 4 \\
&= 10
\end{aligned}$$

$$\begin{aligned}
\mathbf{74} \quad & \int \frac{x}{x^2+1} dx = \int \frac{1}{u} \left(\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = x^2 + 1 \quad \therefore \frac{du}{dx} = 2x \right\} \\
&= \int \frac{1}{2u} du \\
&= \frac{1}{2} \ln|u| + c \\
&= \frac{1}{2} \ln|x^2+1| + c \quad \dots (*)
\end{aligned}$$

$$\text{Now } \int_0^a \frac{x}{x^2+1} dx = 3$$

$$\therefore \left[\frac{1}{2} \ln|x^2+1| \right]_0^a = 3 \quad \{\text{using (*)}\}$$

$$\therefore \frac{1}{2} \ln|a^2+1| - \frac{1}{2} \ln|0^2+1| = 3$$

$$\therefore \frac{1}{2} \ln|a^2+1| - \frac{1}{2} \ln 1 = 3$$

$$\therefore \frac{1}{2} \ln|a^2+1| - 0 = 3$$

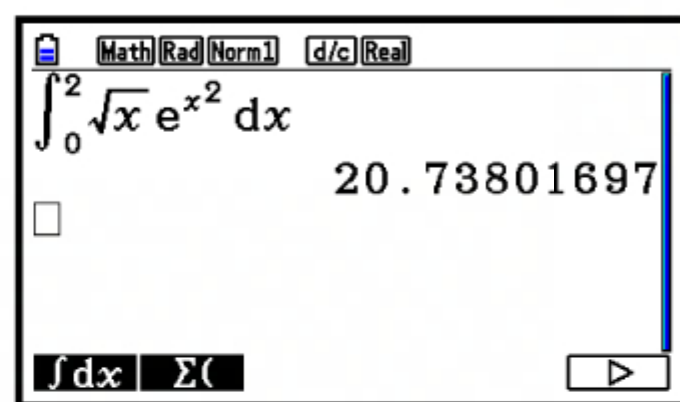
$$\therefore \ln(a^2+1) = 6 \quad \{a^2+1 > 0\}$$

$$\therefore a^2+1 = e^6$$

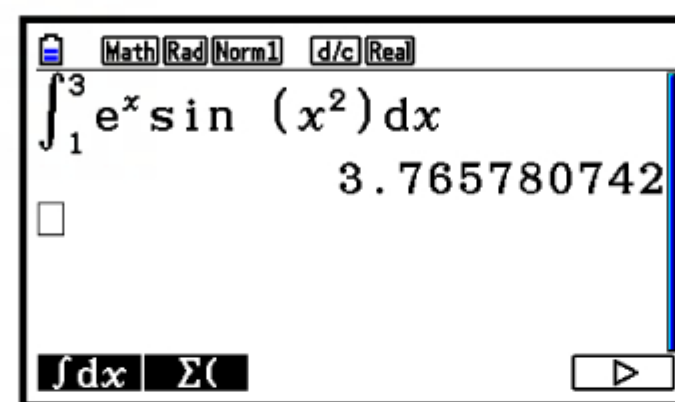
$$\therefore a^2 = e^6 - 1$$

$$\therefore a = \sqrt{e^6 - 1} \quad \{a > 0\}$$

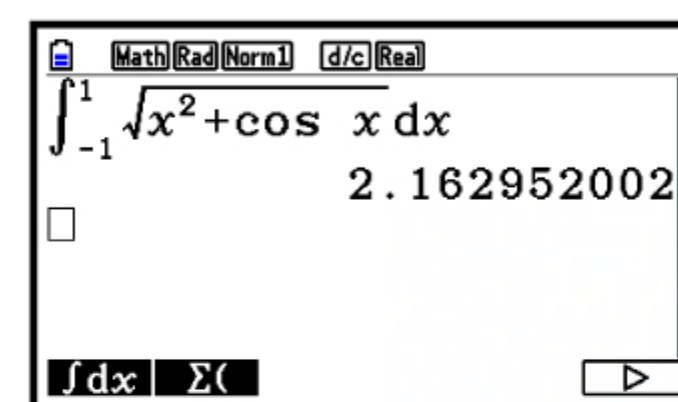
75 a $\int_0^2 \sqrt{x} e^{x^2} dx \approx 20.7$



b $\int_1^3 e^x \sin(x^2) dx \approx 3.77$

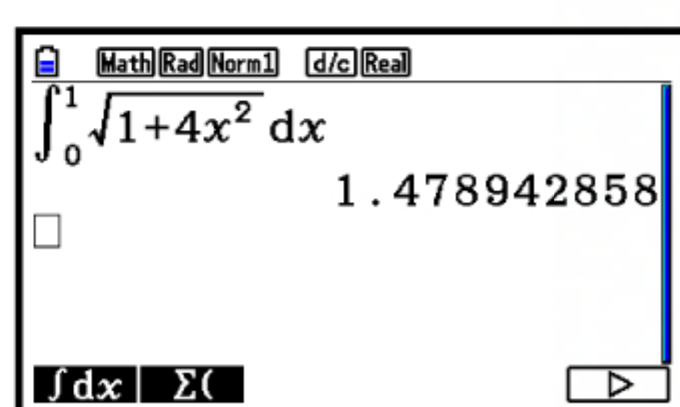


c $\int_{-1}^1 \sqrt{x^2 + \cos x} dx \approx 2.16$



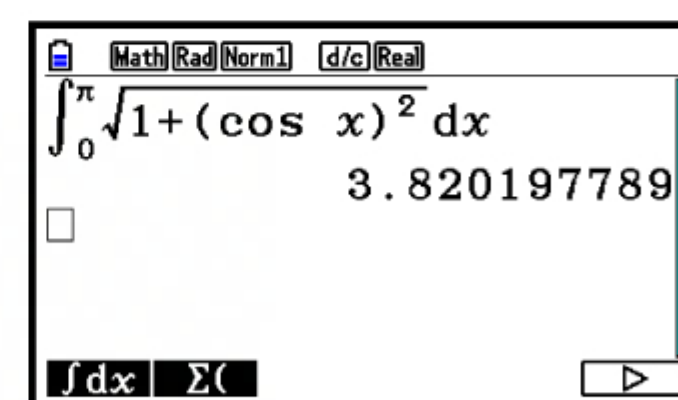
76 a $f(x) = x^2, \quad 0 \leq x \leq 1$
 $\therefore f'(x) = 2x$

So, $L = \int_0^1 \sqrt{1 + (2x)^2} dx$
 $= \int_0^1 \sqrt{1 + 4x^2} dx$
 ≈ 1.48 units



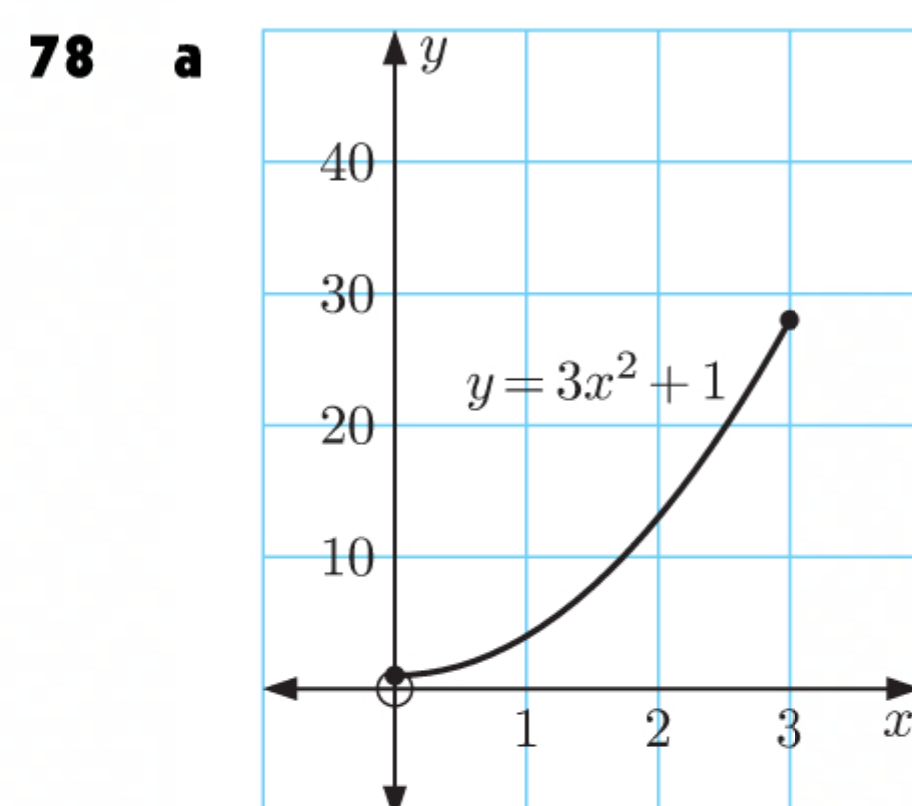
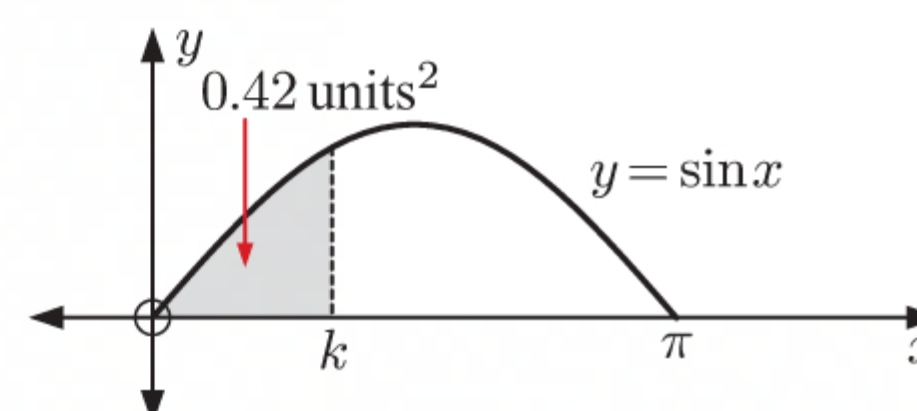
b $f(x) = \sin x, \quad 0 \leq x \leq \pi$
 $\therefore f'(x) = \cos x$

So, $L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$
 ≈ 3.82 units



77 Shaded area = 0.42 units²

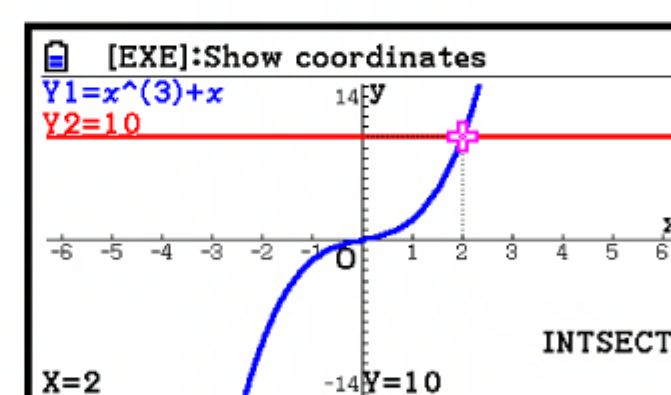
$\therefore \int_0^k \sin x dx = 0.42$
 $\therefore [-\cos x]_0^k = 0.42$
 $\therefore -\cos k + \cos 0 = 0.42$
 $\therefore -\cos k + 1 = 0.42$
 $\therefore -\cos k = -0.58$
 $\therefore \cos k = 0.58$
 $\therefore k = \cos^{-1}(0.58) \approx 0.95 \quad \{2 \text{ d.p.}\}$

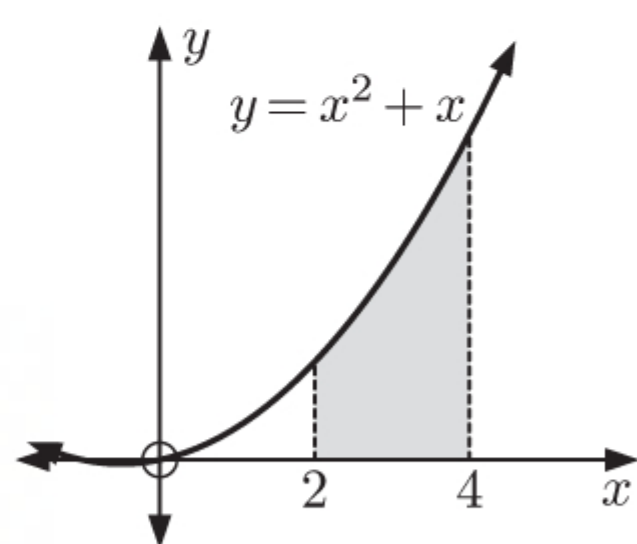


b Area = $\int_0^3 (3x^2 + 1) dx$
 $= [x^3 + x]_0^3$
 $= 3^3 + 3 - 0$
 $= 27 + 3$
 $= 30$ units²

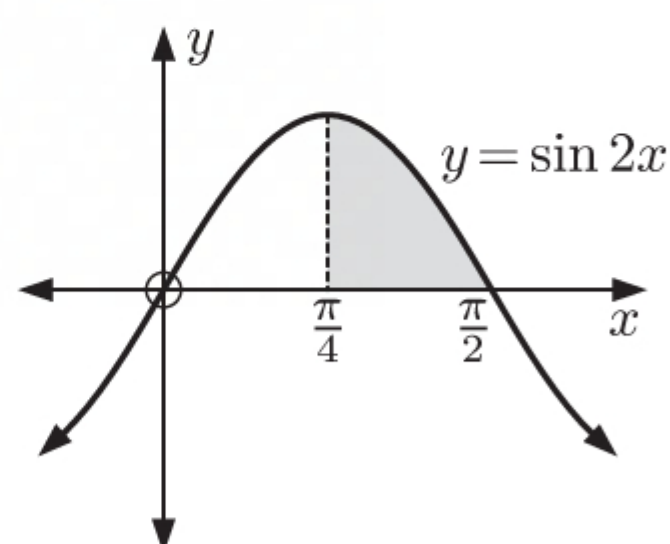
c Area = 10 units²

$\therefore \int_0^k (3x^2 + 1) dx = 10$
 $\therefore [x^3 + x]_0^k = 10$
 $\therefore k^3 + k = 10$
 $\therefore k = 2 \quad \{\text{using technology}\}$

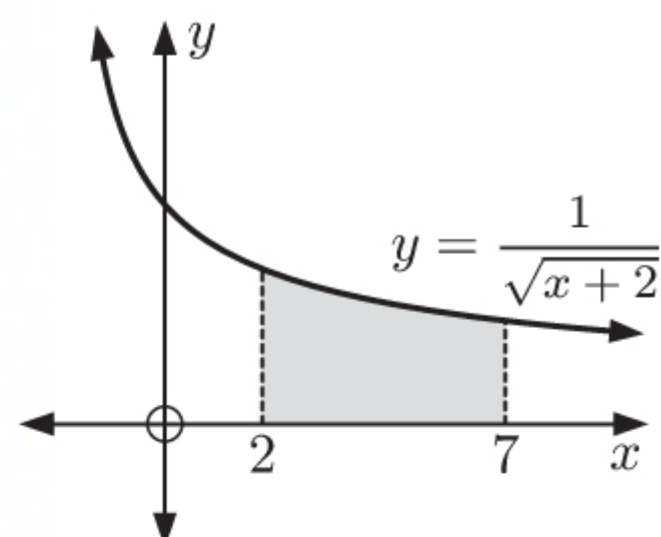


79 a

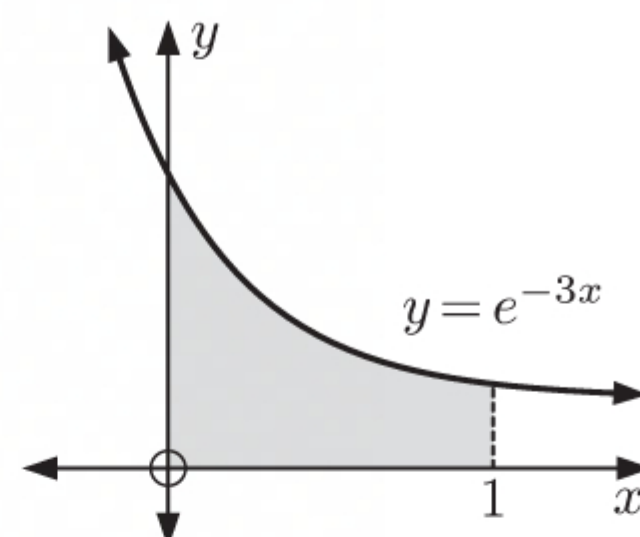
$$\begin{aligned}
 \text{Area} &= \int_2^4 (x^2 + x) dx \\
 &= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_2^4 \\
 &= \left(\frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 \right) - \left(\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 \right) \\
 &= \frac{64}{3} + 8 - \frac{8}{3} - 2 \\
 &= \frac{56}{3} + 6 \\
 &= \frac{74}{3} \\
 &= 24\frac{2}{3} \text{ units}^2
 \end{aligned}$$

b

$$\begin{aligned}
 \text{Area} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x dx \\
 &= \left[-\frac{1}{2} \cos 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= -\frac{1}{2} \cos \pi + \frac{1}{2} \cos \frac{\pi}{2} \\
 &= -\frac{1}{2}(-1) + \frac{1}{2}(0) \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

c

$$\begin{aligned}
 \text{Area} &= \int_2^7 \frac{1}{\sqrt{x+2}} dx \\
 &= \int_2^7 (x+2)^{-\frac{1}{2}} dx \\
 &= \left[2(x+2)^{\frac{1}{2}} \right]_2^7 \\
 &= 2\sqrt{7+2} - 2\sqrt{2+2} \\
 &= 2\sqrt{9} - 2\sqrt{4} \\
 &= 2(3) - 2(2) \\
 &= 2 \text{ units}^2
 \end{aligned}$$

d

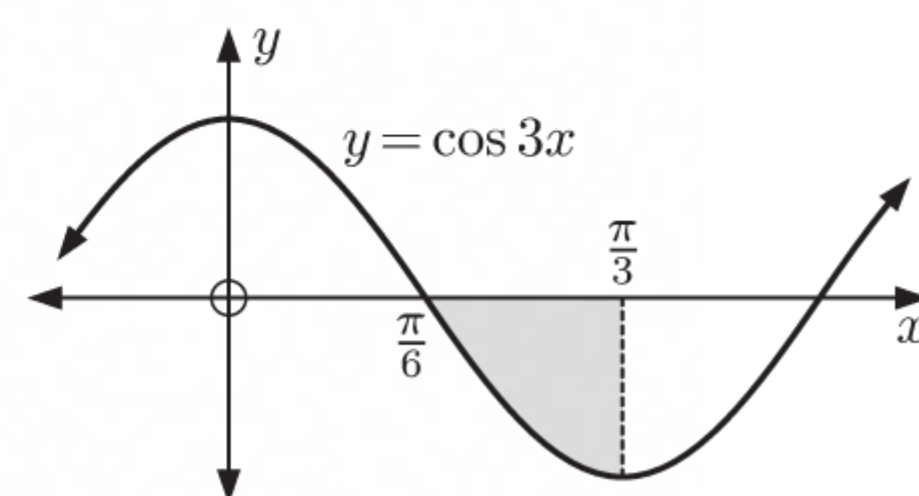
$$\begin{aligned}
 \text{Area} &= \int_0^1 e^{-3x} dx \\
 &= \left[-\frac{1}{3} e^{-3x} \right]_0^1 \\
 &= -\frac{1}{3} e^{-3} + \frac{1}{3} e^0 \\
 &= -\frac{1}{3} \frac{1}{e^3} + \frac{1}{3} \\
 &= \frac{1}{3} \left(1 - \frac{1}{e^3} \right) \text{ units}^2
 \end{aligned}$$

80 The first x -intercept of $y = \cos 3x$ is given by $\cos 3x = 0$

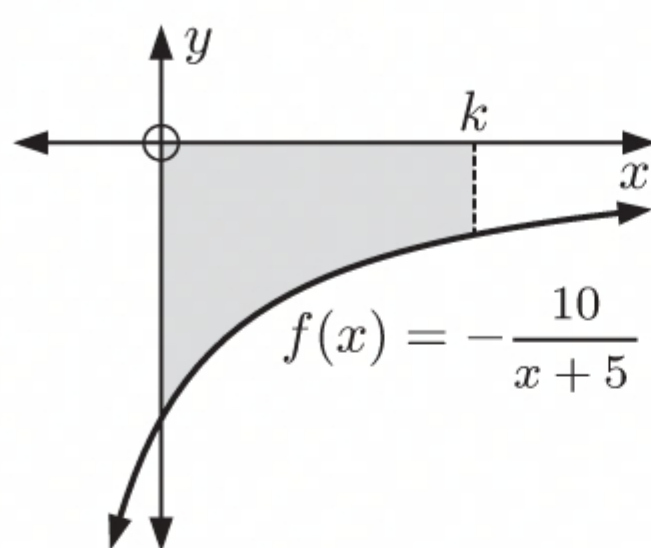
$$\therefore 3x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{6}$$

$$\begin{aligned}
 \therefore \text{area} &= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx \\
 &= - \left[\frac{1}{3} \sin 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= - \left(\frac{1}{3} \sin \pi - \frac{1}{3} \sin \frac{\pi}{2} \right) \\
 &= - \left(\frac{1}{3}(0) - \frac{1}{3}(1) \right) \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$



81



$$\text{Area} = 10 \ln 3 \text{ units}^2$$

$$\therefore -\int_0^k \frac{10}{x+5} dx = 10 \ln 3$$

$$\therefore \int_0^k \frac{10}{x+5} dx = 10 \ln 3$$

$$\therefore [10 \ln|x+5|]_0^k = 10 \ln 3$$

$$\therefore 10 \ln|k+5| - 10 \ln 5 = 10 \ln 3$$

$$\therefore \ln|k+5| - \ln 5 = \ln 3$$

$$\therefore \ln\left|\frac{k+5}{5}\right| = \ln 3$$

$$\therefore \left|\frac{k+5}{5}\right| = 3$$

$$\therefore |k+5| = 15$$

$$\therefore k+5 = \pm 15$$

$$\therefore k = 10 \text{ or } -20$$

But for $k = -20$, $f(x)$ is not defined for all x such that $k \leq x \leq 0$ as $f(x)$ is not defined for $x = -5$.

So, the only possible value of k is 10.

82 Suppose $\alpha > 0$, then the parabola has equation $f(x) = -\frac{\alpha}{\pi^2}(x-\pi)(x+\pi)$

$$= -\frac{\alpha}{\pi^2}(x^2 - \pi^2)$$

$$\text{Now area} = 4 \text{ units}^2$$

$$\therefore \int_{-\pi}^{\pi} f(x) dx = 4$$

$$\therefore \int_{-\pi}^{\pi} -\frac{\alpha}{\pi^2}(x^2 - \pi^2) dx = 4$$

$$\therefore -\frac{\alpha}{\pi^2} \int_{-\pi}^{\pi} (x^2 - \pi^2) dx = 4$$

$$\therefore \int_{-\pi}^{\pi} (x^2 - \pi^2) dx = -\frac{4\pi^2}{\alpha}$$

$$\therefore \left[\frac{1}{3}x^3 - \pi^2 x\right]_{-\pi}^{\pi} = -\frac{4\pi^2}{\alpha}$$

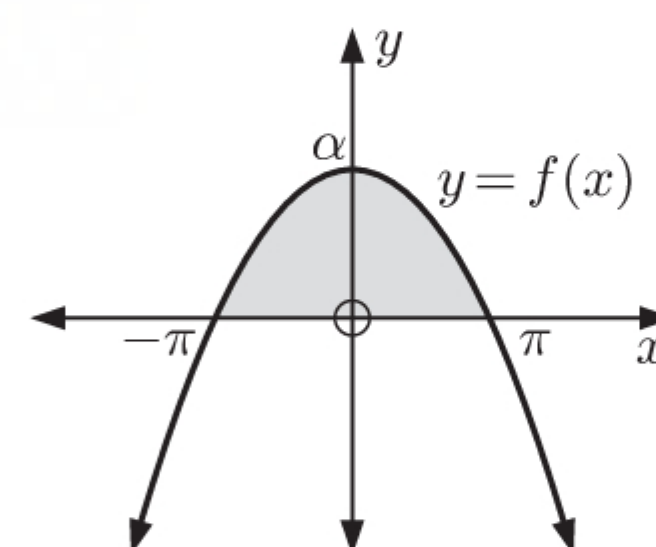
$$\therefore \left(\frac{1}{3}\pi^3 - \pi^2(\pi)\right) - \left(\frac{1}{3}(-\pi)^3 - \pi^2(-\pi)\right) = -\frac{4\pi^2}{\alpha}$$

$$\therefore \frac{\pi^3}{3} - \pi^3 - \left(-\frac{\pi^3}{3} + \pi^3\right) = -\frac{4\pi^2}{\alpha}$$

$$\therefore \frac{\pi^3}{3} - \pi^3 + \frac{\pi^3}{3} - \pi^3 = -\frac{4\pi^2}{\alpha}$$

$$\therefore -\frac{4\pi^3}{3} = -\frac{4\pi^2}{\alpha}$$

$$\therefore \alpha = \frac{3}{\pi}$$



Similarly, if $\alpha < 0$, $\alpha = -\frac{3}{\pi}$.

83 $f(x) = x^3 - x^2 - 4x + 4$

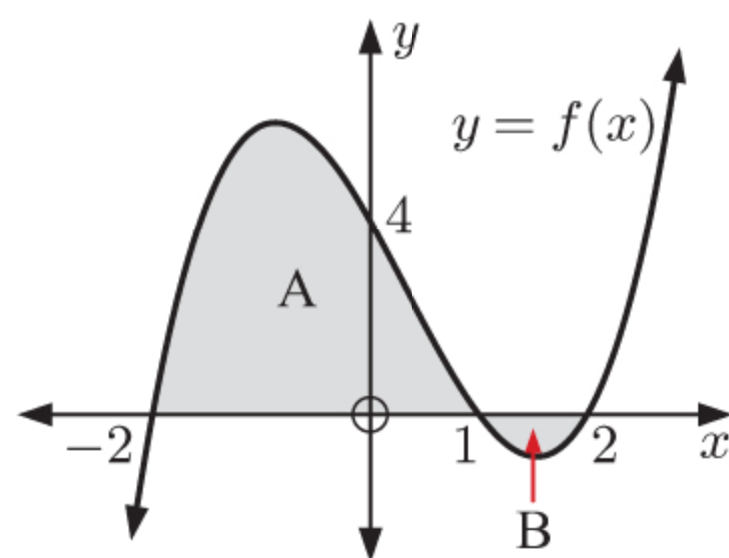
$$\mathbf{a} \int_{-2}^2 f(x) dx = \int_{-2}^2 (x^3 - x^2 - 4x + 4) dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x\right]_{-2}^2$$

$$= \left(\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2)\right) - \left(\frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2)\right)$$

$$= -\frac{8}{3} + 8 - \frac{8}{3} + 8$$

$$= \frac{32}{3}$$

b

For $1 \leq x \leq 2$, the graph of $y = f(x)$ is below the x -axis, so $\int_1^2 f(x) dx$ is negative.

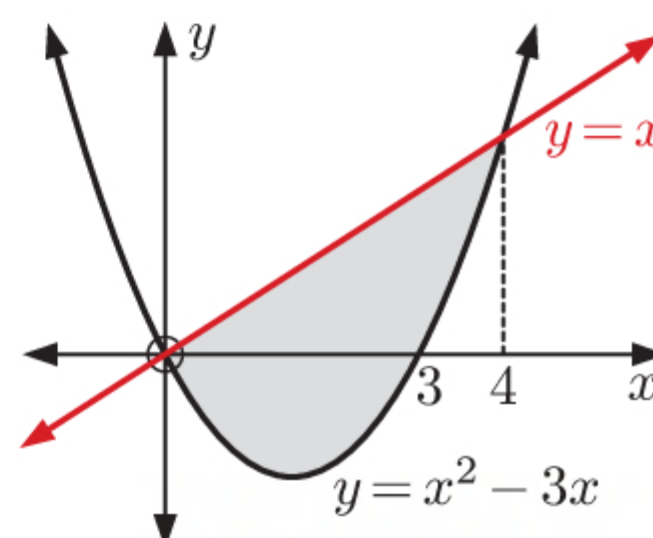
$\int_{-2}^2 f(x) dx$ is the *difference* of areas A and B.

$$\begin{aligned}
 \text{c Shaded area} &= \int_{-2}^1 f(x) dx + \int_1^2 -f(x) dx \\
 &= \int_{-2}^1 f(x) dx - \int_1^2 f(x) dx \\
 &= \int_{-2}^1 (x^3 - x^2 - 4x + 4) dx - \int_1^2 (x^3 - x^2 - 4x + 4) dx \\
 &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_{-2}^1 - \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_1^2 \\
 &= \left(\frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) - \left(\frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2) \right) \\
 &\quad - \left(\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) \right) + \left(\frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) \\
 &= \left(\frac{1}{4} - \frac{1}{3} - 2 + 4 \right) - \left(\frac{16}{4} - \frac{8}{3} - 8 - 8 \right) - \left(\frac{16}{4} - \frac{8}{3} - 8 + 8 \right) + \left(\frac{1}{4} - \frac{1}{3} - 2 + 4 \right) \\
 &= \left(-\frac{1}{12} + 2 \right) - (-4) - (-4) + \left(-\frac{1}{12} + 2 \right) \\
 &= -\frac{1}{6} + 12 \\
 &= \frac{71}{6} = 11\frac{5}{6} \text{ units}^2
 \end{aligned}$$

84 a $y = x^2 - 3x$ meets $y = x$ where $x^2 - 3x = x$
 $\therefore x^2 - 4x = 0$
 $\therefore x(x - 4) = 0$
 $\therefore x = 0$ or 4

Since $x \geq x^2 - 3x$ on the interval $0 \leq x \leq 4$,

$$\begin{aligned}
 \text{area} &= \int_0^4 [(x) - (x^2 - 3x)] dx \\
 &= \int_0^4 (-x^2 + 4x) dx \\
 &= \left[-\frac{x^3}{3} + 2x^2 \right]_0^4 \\
 &= \left(-\frac{4^3}{3} + 2(4)^2 \right) - 0 \\
 &= -\frac{64}{3} + 32 \\
 &= \frac{32}{3} \\
 &= 10\frac{2}{3} \text{ units}^2
 \end{aligned}$$



b $y = 4 - x^2$ meets $y = -2x - 4$ where $4 - x^2 = -2x - 4$

$$\therefore -x^2 + 2x + 8 = 0$$

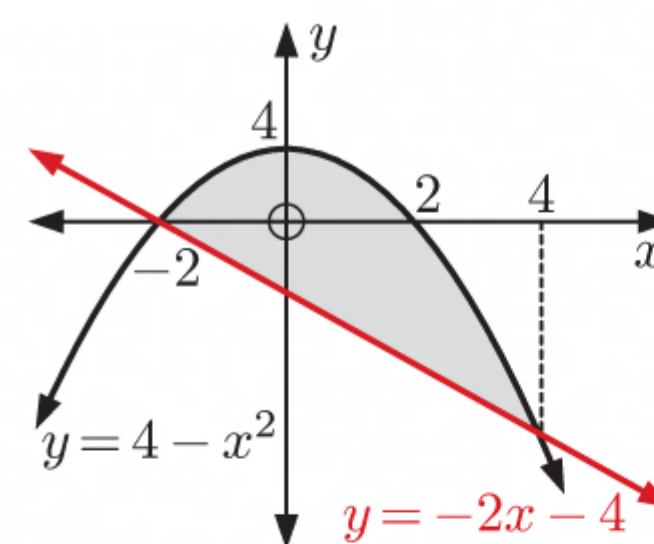
$$\therefore x^2 - 2x - 8 = 0$$

$$\therefore (x - 4)(x + 2) = 0$$

$$\therefore x = -2 \text{ or } 4$$

Since $4 - x^2 \geq -2x - 4$ on the interval $-2 \leq x \leq 4$,

$$\begin{aligned} \text{area} &= \int_{-2}^4 [(4 - x^2) - (-2x - 4)] dx \\ &= \int_{-2}^4 (-x^2 + 2x + 8) dx \\ &= \left[-\frac{x^3}{3} + x^2 + 8x \right]_{-2}^4 \\ &= \left(-\frac{4^3}{3} + 4^2 + 8(4) \right) - \left(-\frac{(-2)^3}{3} + (-2)^2 + 8(-2) \right) \\ &= \left(-\frac{64}{3} + 16 + 32 \right) - \left(\frac{8}{3} + 4 - 16 \right) \\ &= -\frac{72}{3} + 60 \\ &= 36 \text{ units}^2 \end{aligned}$$



c $y = x^2 + 2x - 3$ meets $y = x - 1$ where $x^2 + 2x - 3 = x - 1$

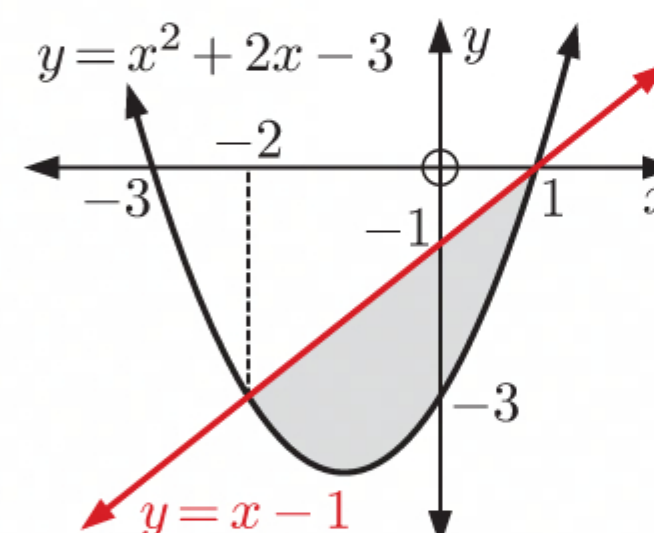
$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

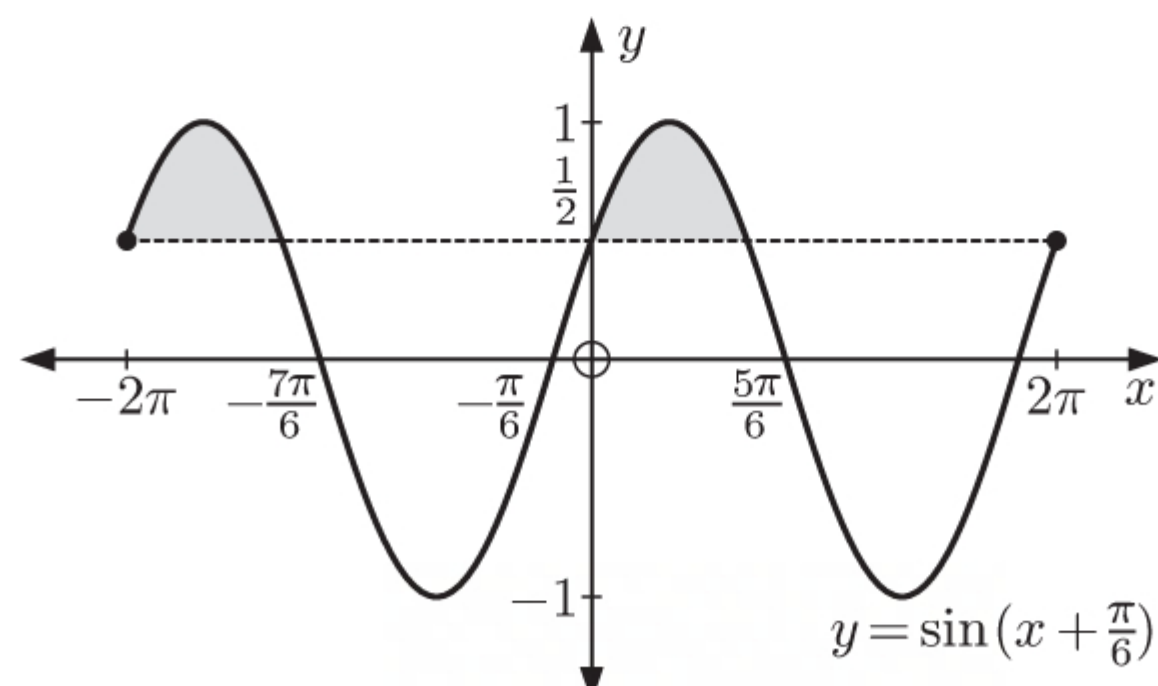
$$\therefore x = -2 \text{ or } 1$$

Since $x - 1 \geq x^2 + 2x - 3$ on the interval $-2 \leq x \leq 1$,

$$\begin{aligned} \text{area} &= \int_{-2}^1 [(x - 1) - (x^2 + 2x - 3)] dx \\ &= \int_{-2}^1 (-x^2 - x + 2) dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\ &= \left(-\frac{1^3}{3} - \frac{1^2}{2} + 2(1) \right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2) \right) \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\ &= \frac{7}{6} - \left(-\frac{10}{3} \right) \\ &= \frac{9}{2} \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$



85 a



b $y = \sin\left(x + \frac{\pi}{6}\right)$ meets $y = \frac{1}{2}$ where $\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$, $-2\pi \leq x \leq 2\pi$

$$\therefore x + \frac{\pi}{6} = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

$$\therefore x = -2\pi, -\frac{4\pi}{3}, 0, \frac{2\pi}{3}, 2\pi$$

Since $\sin(x + \frac{\pi}{6}) \geq \frac{1}{2}$ on the intervals $-2\pi \leq x \leq -\frac{4\pi}{3}$ and $0 \leq x \leq \frac{2\pi}{3}$,

$$\begin{aligned} \text{area} &= \int_{-2\pi}^{-\frac{4\pi}{3}} \left(\sin\left(x + \frac{\pi}{6}\right) - \frac{1}{2} \right) dx + \int_0^{\frac{2\pi}{3}} \left(\sin\left(x + \frac{\pi}{6}\right) - \frac{1}{2} \right) dx \\ &= \left[-\cos\left(x + \frac{\pi}{6}\right) - \frac{1}{2}x \right]_{-2\pi}^{-\frac{4\pi}{3}} + \left[-\cos\left(x + \frac{\pi}{6}\right) - \frac{1}{2}x \right]_0^{\frac{2\pi}{3}} \\ &= \left(-\cos\left(-\frac{4\pi}{3} + \frac{\pi}{6}\right) - \left(-\frac{2\pi}{3}\right) \right) - \left(-\cos\left(-2\pi + \frac{\pi}{6}\right) - (-\pi) \right) + \left(-\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) - \left(\frac{\pi}{3}\right) \right) - \left(-\cos\frac{\pi}{6} - 0 \right) \\ &= -\cos\left(-\frac{7\pi}{6}\right) + \frac{2\pi}{3} + \cos\left(-\frac{11\pi}{6}\right) - \pi - \cos\frac{5\pi}{6} - \frac{\pi}{3} + \cos\frac{\pi}{6} \\ &= -\left(-\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \pi - \left(-\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3} + \frac{\sqrt{3}}{2} \\ &= \left(2\sqrt{3} - \frac{2\pi}{3}\right) \text{ units}^2 \end{aligned}$$

86 $f(x) = \left(2 - \frac{1}{x}\right)e^{-x}, \quad x > 0$

a $f(x) = 0$ when $\left(2 - \frac{1}{x}\right)e^{-x} = 0$

$$\begin{aligned} \therefore 2 - \frac{1}{x} &= 0 \quad \{e^{-x} > 0 \text{ for all } x\} \\ \therefore \frac{1}{x} &= 2 \\ \therefore x &= \frac{1}{2} \end{aligned}$$

\therefore the zero of $f(x)$ is $\frac{1}{2}$.

b As $x \rightarrow 0^+$, $\frac{1}{x} \rightarrow \infty$ As $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0^+$

$$\begin{aligned} \therefore -\frac{1}{x} &\rightarrow -\infty & \therefore -\frac{1}{x} &\rightarrow 0^- \\ \therefore 2 - \frac{1}{x} &\rightarrow -\infty & \therefore 2 - \frac{1}{x} &\rightarrow 2^- \\ \therefore \left(2 - \frac{1}{x}\right)e^{-x} &\rightarrow -\infty \quad \{e^{-x} \approx 1 \text{ near } x = 0\} & \therefore \left(2 - \frac{1}{x}\right)e^{-x} &\rightarrow 0^+ \\ \therefore f(x) &\rightarrow -\infty & \therefore f(x) &\rightarrow 0^+ \end{aligned}$$

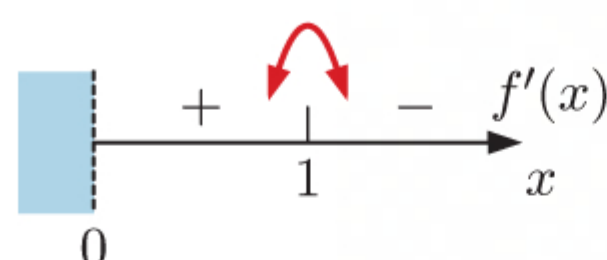
c $f(x) = \left(2 - \frac{1}{x}\right)e^{-x}$

$$\begin{aligned} \therefore f'(x) &= \left(\frac{1}{x^2}\right)e^{-x} - e^{-x}\left(2 - \frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= e^{-x}\left(\frac{1}{x^2} - 2 + \frac{1}{x}\right) \end{aligned}$$

Stationary point(s) occur where $f'(x) = 0$

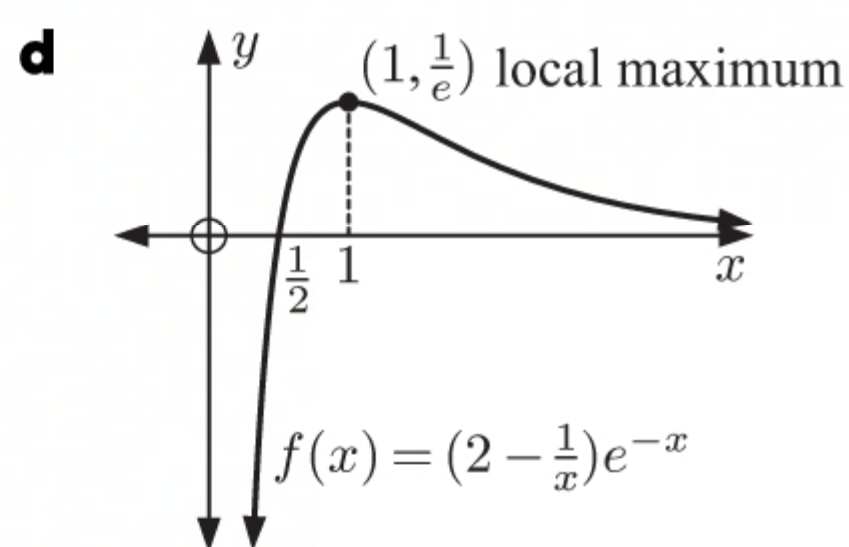
$$\begin{aligned} \therefore e^{-x}\left(\frac{1}{x^2} - 2 + \frac{1}{x}\right) &= 0 \\ \therefore \frac{1}{x^2} - 2 + \frac{1}{x} &= 0 \quad \{e^{-x} > 0 \text{ for all } x\} \\ \therefore 1 - 2x^2 + x &= 0 \\ \therefore 2x^2 - x - 1 &= 0 \\ \therefore 2x^2 - 2x + x - 1 &= 0 \\ \therefore 2x(x - 1) + (x - 1) &= 0 \\ \therefore (x - 1)(2x + 1) &= 0 \\ \therefore x &= 1 \text{ or } -\frac{1}{2} \\ \therefore x &= 1 \quad \{x > 0\} \end{aligned}$$

The sign diagram of $f'(x)$ is

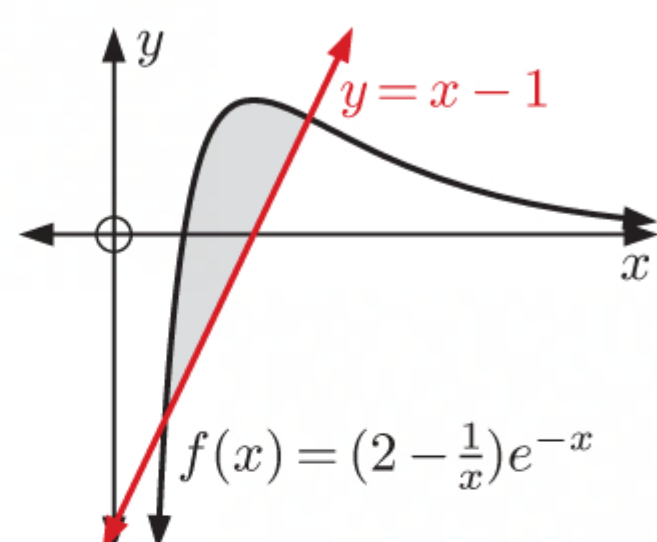


Now $f(1) = (2 - 1)e^{-1} = \frac{1}{e}$

$\therefore \left(1, \frac{1}{e}\right)$ is a local maximum.



e $y = (2 - \frac{1}{x})e^{-x}$ meets $y = x - 1$ where $(2 - \frac{1}{x})e^{-x} = x - 1$
 $\therefore x \approx 0.342$ or 1.33 {using technology}



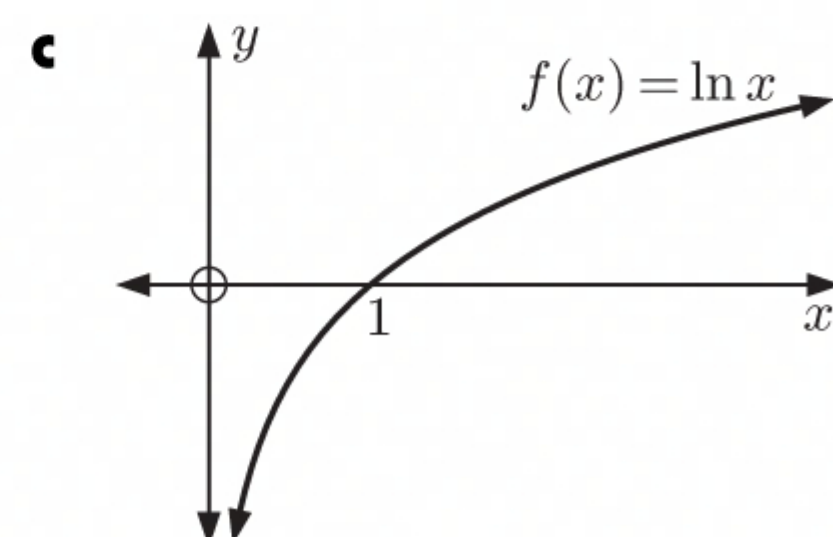
Since $(2 - \frac{1}{x})e^{-x} \geq x - 1$ on the interval $0.342 \leq x \leq 1.33$,

$$\begin{aligned} \text{area} &\approx \int_{0.342}^{1.33} \left[\left(2 - \frac{1}{x}\right)e^{-x} - (x - 1) \right] dx \\ &\approx 0.373 \quad \{3 \text{ d.p.}\} \end{aligned}$$

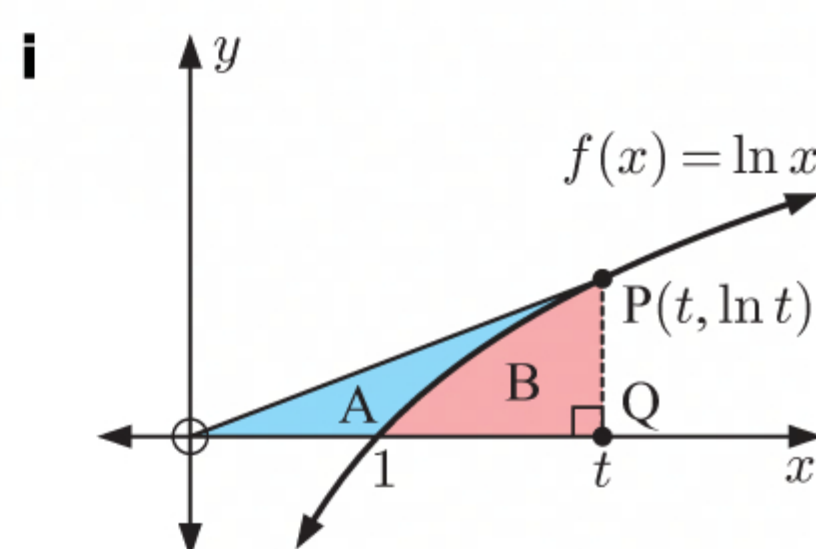
87 a i $f(x) = \ln x$
 $\therefore f'(x) = \frac{1}{x}$

ii $F(x) = x \ln x - x$
 $\therefore F'(x) = \left[(1) \ln x + x \left(\frac{1}{x} \right) \right] - 1$ {product rule}
 $= \ln x + 1 - 1$
 $= \ln x$

b $F'(x) = f(x)$, that is, $F(x)$ is the antiderivative of $f(x)$.



d $O(0, 0)$ and $P(t, \ln t)$, $1 < t \leq e$



Area A = area of triangle - area B

$$\begin{aligned} &= \frac{1}{2}t \ln t - \int_1^t \ln x \, dx \\ &= \frac{1}{2}t \ln t - [x \ln x - x]_1^t \quad \{\text{using a ii}\} \\ &= \frac{1}{2}t \ln t - [(t \ln t - t) - (1 \ln 1 - 1)] \\ &= \frac{1}{2}t \ln t - (t \ln t - t - 0 + 1) \\ &= \left(t - \frac{1}{2}t \ln t - 1 \right) \text{ units}^2 \end{aligned}$$

ii The area $A = t - \frac{1}{2}t \ln t - 1$

$$\begin{aligned} \therefore \frac{dA}{dt} &= 1 - \frac{1}{2}(\ln t + 1) \quad \{\text{product rule}\} \\ &= \frac{1}{2} - \frac{1}{2} \ln t \end{aligned}$$

Now A is maximised when $\frac{dA}{dt} = 0$

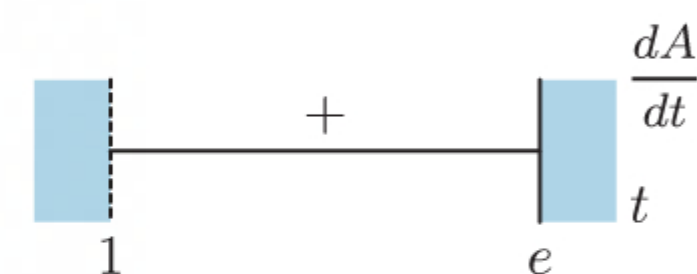
$$\therefore \frac{1}{2} - \frac{1}{2} \ln t = 0$$

$$\therefore \frac{1}{2} \ln t = \frac{1}{2}$$

$$\therefore \ln t = 1$$

$$\therefore t = e$$

Sign diagram of $\frac{dA}{dt}$:



$\therefore A$ is maximised when $t = e$.

e i The tangent has equation

$$y = f'(t)(x - t) + f(t)$$

$$\therefore y = \frac{1}{t}(x - t) + \ln t$$

$$\therefore y = \frac{x}{t} - 1 + \ln t$$

ii The tangent passes through the origin when

$$-1 + \ln t = 0$$

$$\therefore \ln t = 1$$

$$\therefore t = e$$

88 a $y = x^3 - 2x^2 - 3x$ meets $y = 5x - 4x^2$ where $x^3 - 2x^2 - 3x = 5x - 4x^2$

$$\therefore x^3 + 2x^2 - 8x = 0$$

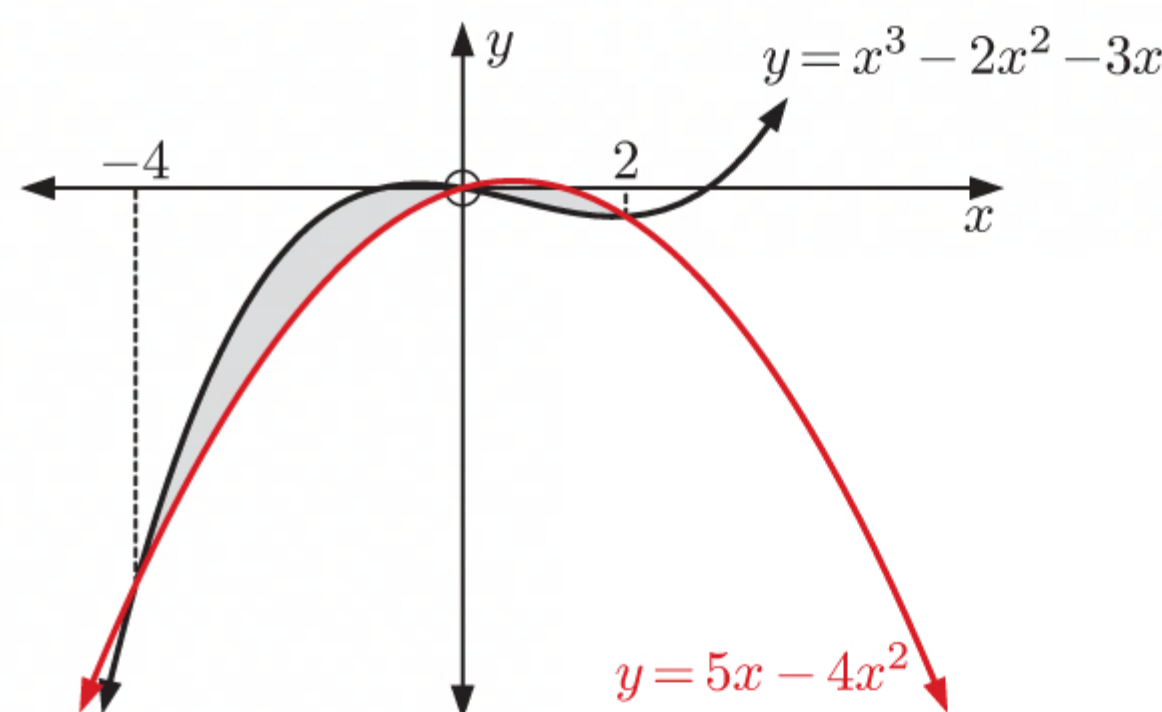
$$\therefore x(x^2 + 2x - 8) = 0$$

$$\therefore x(x+4)(x-2) = 0$$

$$\therefore x = -4, 0, \text{ or } 2$$

Since $x^3 - 2x^2 - 3x \geq 5x - 4x^2$ for $-4 \leq x \leq 0$, and $5x - 4x^2 \geq x^3 - 2x^2 - 3x$ for $0 \leq x \leq 2$,

$$\begin{aligned} \text{area} &= \int_{-4}^0 [(x^3 - 2x^2 - 3x) - (5x - 4x^2)] dx \\ &\quad + \int_0^2 [(5x - 4x^2) - (x^3 - 2x^2 - 3x)] dx \\ &= \int_{-4}^0 (x^3 + 2x^2 - 8x) dx + \int_0^2 (-x^3 - 2x^2 + 8x) dx \\ &= \int_{-4}^0 (x^3 + 2x^2 - 8x) dx - \int_0^2 (x^3 + 2x^2 - 8x) dx \\ &= \left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2 \right]_{-4}^0 - \left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2 \right]_0^2 \\ &= 0 - \left(\frac{(-4)^4}{4} + \frac{2(-4)^3}{3} - 4(-4)^2 \right) - \left(\frac{2^4}{4} + \frac{2(2)^3}{3} - 4(2)^2 \right) + 0 \\ &= -\left(64 - \frac{128}{3} - 64 \right) - \left(4 + \frac{16}{3} - 16 \right) \\ &= \frac{128}{3} - 4 - \frac{16}{3} + 16 \\ &= \frac{112}{3} + 12 \\ &= \frac{148}{3} \\ &= 49\frac{1}{3} \text{ units}^2 \end{aligned}$$



b $y = 2x^3 - 5x + 4$ meets $y = x^3 + 2x^2 - 2$ where

$$2x^3 - 5x + 4 = x^3 + 2x^2 - 2$$

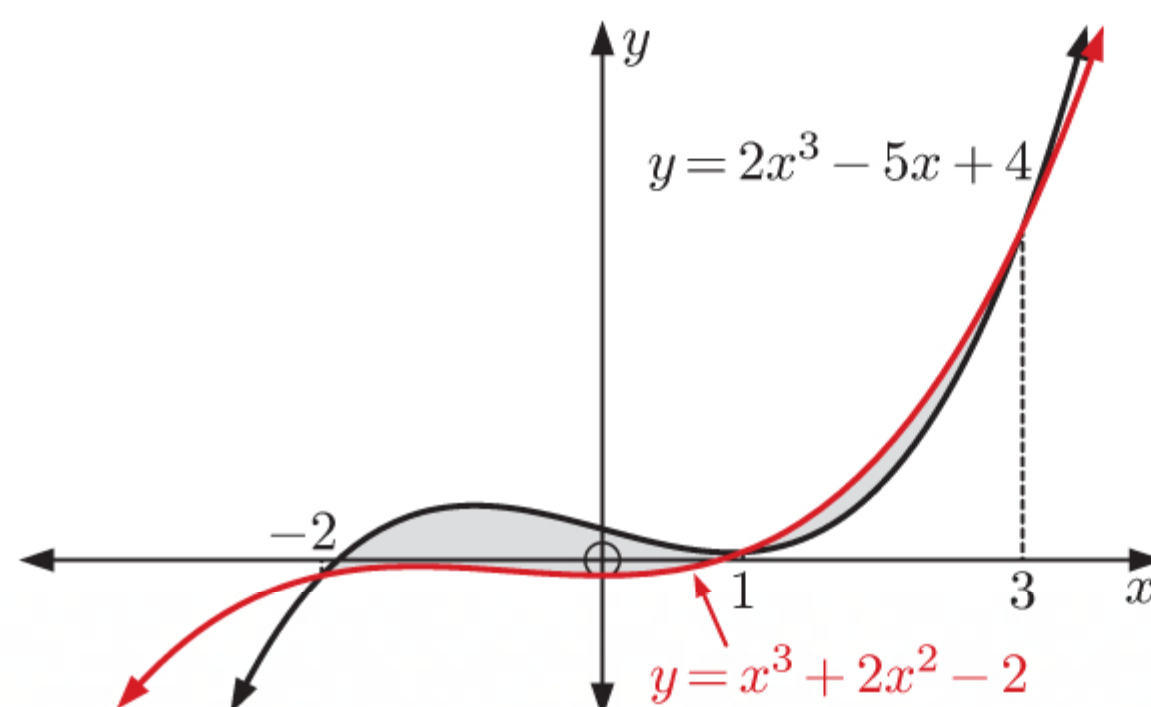
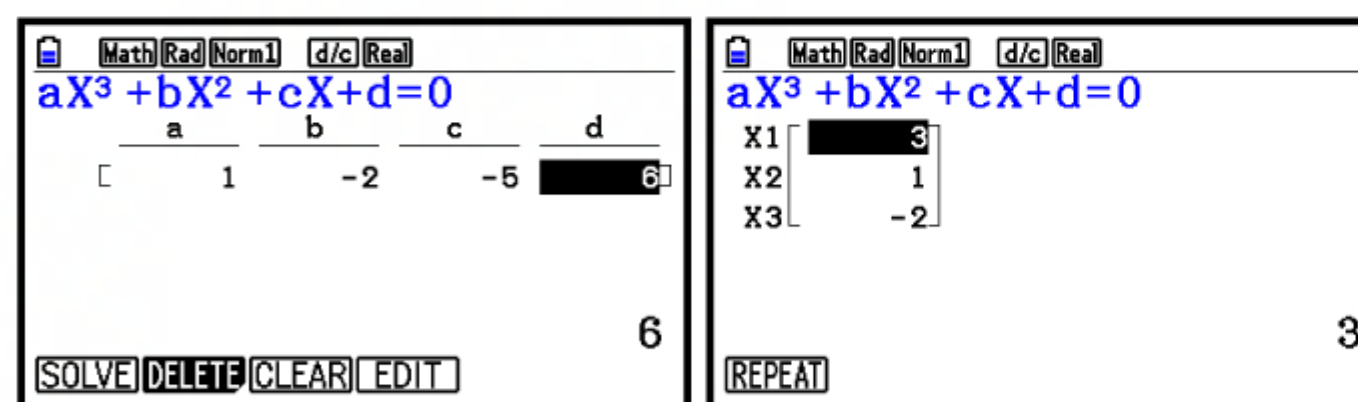
$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore x = -2, 1, \text{ or } 3$$

Since $2x^3 - 5x + 4 \geq x^3 + 2x^2 - 2$ for $-2 \leq x \leq 1$, and $x^3 + 2x^2 - 2 \geq 2x^3 - 5x + 4$ for $1 \leq x \leq 3$,

area

$$\begin{aligned} &= \int_{-2}^1 [(2x^3 - 5x + 4) - (x^3 + 2x^2 - 2)] dx \\ &\quad + \int_1^3 [(x^3 + 2x^2 - 2) - (2x^3 - 5x + 4)] dx \\ &= \int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx + \int_1^3 (-x^3 + 2x^2 + 5x - 6) dx \\ &= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-2}^1 + \left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{5x^2}{2} - 6x \right]_1^3 \\ &= \left(\frac{1^4}{4} - \frac{2(1)^3}{3} - \frac{5(1)^2}{2} + 6(1) \right) - \left(\frac{(-2)^4}{4} - \frac{2(-2)^3}{3} - \frac{5(-2)^2}{2} + 6(-2) \right) \\ &\quad + \left(-\frac{3^4}{4} + \frac{2(3)^3}{3} + \frac{5(3)^2}{2} - 6(3) \right) - \left(-\frac{1^4}{4} + \frac{2(1)^3}{3} + \frac{5(1)^2}{2} - 6(1) \right) \\ &= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left(4 + \frac{16}{3} - 10 - 12 \right) + \left(-\frac{81}{4} + 18 + \frac{45}{2} - 18 \right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{5}{2} - 6 \right) \\ &= -\frac{79}{4} - \frac{20}{3} + \frac{35}{2} + 30 \\ &= -\frac{107}{12} + 30 \\ &= \frac{253}{12} \\ &= 21\frac{1}{12} \text{ units}^2 \end{aligned}$$



89 $G(t) = \frac{2.5}{t+1}$ metres per year

a $G(t) > 0$ for all $t \geq 0$.

\therefore the rate at which the tree grows is always positive.

\therefore the tree is always growing taller.

b i
$$\begin{aligned} \int_0^5 G(t) dt &= \int_0^5 \frac{2.5}{t+1} dt \\ &= [2.5 \ln|t+1|]_0^5 \\ &= 2.5 \ln 6 - 2.5 \ln 1 \\ &= 2.5 \ln 6 \approx 4.48 \text{ metres} \end{aligned}$$

The tree grew about 4.48 metres in the first 5 years of its lifetime.

ii
$$\begin{aligned} \int_5^{10} G(t) dt &= \int_5^{10} \frac{2.5}{t+1} dt \\ &= [2.5 \ln|t+1|]_5^{10} \\ &= 2.5 \ln 11 - 2.5 \ln 6 \\ &= 2.5 \ln\left(\frac{11}{6}\right) \approx 1.52 \text{ metres} \end{aligned}$$

The tree grew about 1.52 metres between the 5th and 10th years of its lifetime.

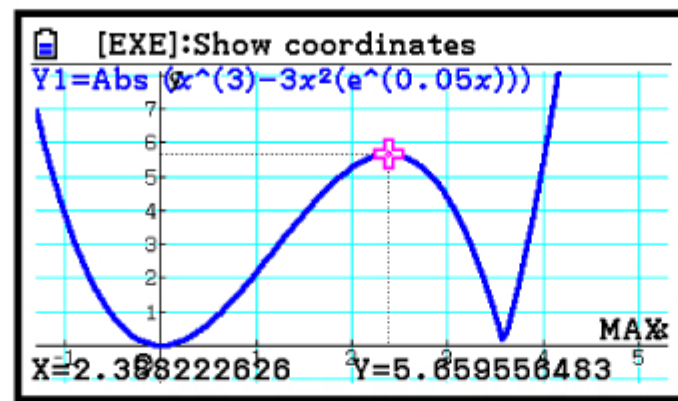
c
$$\begin{aligned} \int_0^{15} G(t) dt &= \int_0^{15} \frac{2.5}{t+1} dt \\ &= [2.5 \ln|t+1|]_0^{15} \\ &= 2.5 \ln 16 - 2.5 \ln 1 \\ &= 2.5 \ln 16 \approx 6.93 \text{ metres} \end{aligned}$$

The tree grew about 6.93 metres over its entire lifetime.

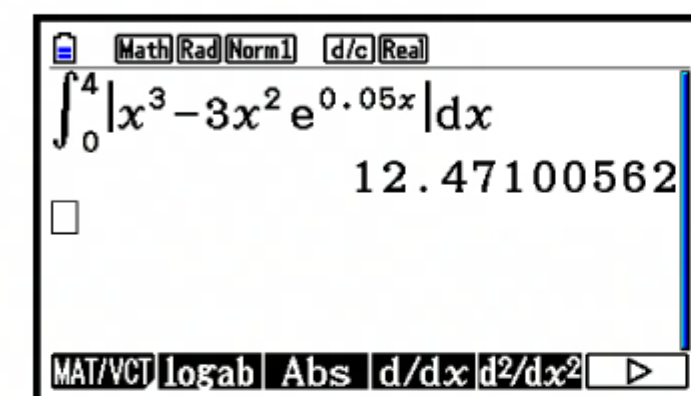
90 $v(t) = t^3 - 3t^2 e^{0.05t}$, $t \geq 0$ seconds

a speed = $|v(t)|$

Using technology, the maximum of $|v(t)|$ over $0 \leq t \leq 4$ is $\approx 5.66 \text{ m s}^{-1}$ when $t \approx 2.39$.



b Total distance =
$$\begin{aligned} \int_0^4 |v(t)| dt &= \int_0^4 |t^3 - 3t^2 e^{0.05t}| dt \\ &\approx 12.5 \text{ m} \end{aligned}$$



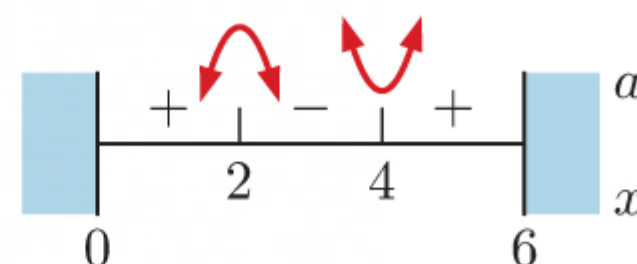
91 $v = t^3 - 9t^2 + 24t \text{ m s}^{-1}$, $0 \leq t \leq 6$

a
$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 3t^2 - 18t + 24 \text{ m s}^{-2} \end{aligned}$$

b The greatest velocity of the particle occurs when $\frac{dv}{dt} = a = 0$

$$\begin{aligned} \therefore 3t^2 - 18t + 24 &= 0 \\ \therefore t^2 - 6t + 8 &= 0 \\ \therefore (t-2)(t-4) &= 0 \\ \therefore t &= 2 \text{ or } 4 \end{aligned}$$

The sign diagram of a is



\therefore there is a local maximum at $t = 2$.

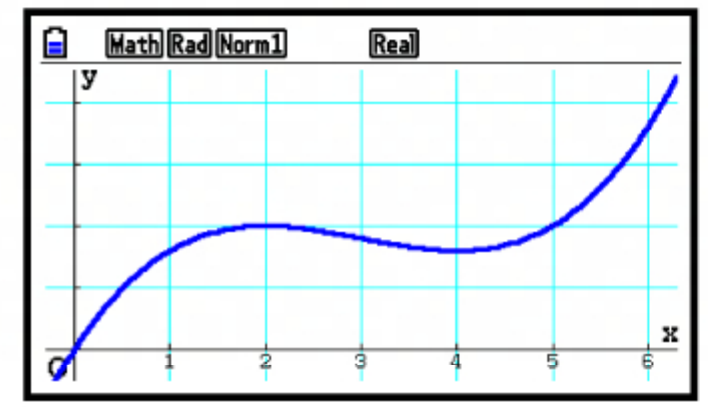
Critical value (t)	$v \text{ (m s}^{-1}\text{)}$
0 (end point)	0
2 (local maximum)	20
6 (end point)	36

\therefore the greatest velocity of the particle is 36 m s^{-1} which occurs at $t = 6$ seconds.

- c** The speed of the particle is decreasing when v and a have opposite signs.

Now $v \geq 0$ for all $0 \leq t \leq 6$.

\therefore the speed of the particle is decreasing when $a < 0$, which is when $2 < t < 4$.



92 a $a(t) = 2 - 3t \text{ m s}^{-1}$, $v(1) = 0 \text{ m s}^{-1}$

$$\begin{aligned}\therefore v(t) &= \int a(t) dt \\ &= \int (2 - 3t) dt \\ &= 2t - \frac{3}{2}t^2 + c\end{aligned}$$

Now $v(1) = 0 \quad \therefore 2(1) - \frac{3}{2}(1)^2 + c = 0$

$$\therefore 2 - \frac{3}{2} + c = 0$$

$$\therefore \frac{1}{2} + c = 0$$

$$\therefore c = -\frac{1}{2}$$

$$\therefore v(t) = 2t - \frac{3}{2}t^2 - \frac{1}{2}$$

c $v(t) = 2t - \frac{3}{2}t^2 - \frac{1}{2}$, $s(0) = 3 \text{ m}$

$$\begin{aligned}\therefore s(t) &= \int v(t) dt \\ &= \int \left(2t - \frac{3}{2}t^2 - \frac{1}{2}\right) dt \\ &= t^2 - \frac{1}{2}t^3 - \frac{1}{2}t + c\end{aligned}$$

Now $s(0) = 3 \quad \therefore 0^2 - \frac{1}{2}(0)^3 - \frac{1}{2}(0) + c = 3$

$$\therefore c = 3$$

$$\therefore s(t) = t^2 - \frac{1}{2}t^3 - \frac{1}{2}t + 3$$

93 $s(t) = 12t - 3t^3 + 1 \text{ cm}$, $t \geq 0$ seconds

a $v(t) = s'(t)$
 $= 12 - 9t^2 \text{ cm s}^{-1}$

$a(t) = v'(t)$
 $= -18t \text{ cm s}^{-2}$

b i $v(1) = 12 - 9(1)^2$
 $= 3 \text{ m s}^{-1}$

speed $= |v(1)| = 3 \text{ m s}^{-1}$

ii $v(2) = 12 - 9(2)^2$
 $= 12 - 9(4)$
 $= 12 - 36$
 $= -24 \text{ m s}^{-1}$

\therefore speed $= |v(2)| = 24 \text{ m s}^{-1}$

- c i** The sign diagram of $a(t)$ is



\therefore the velocity of the particle is always decreasing.

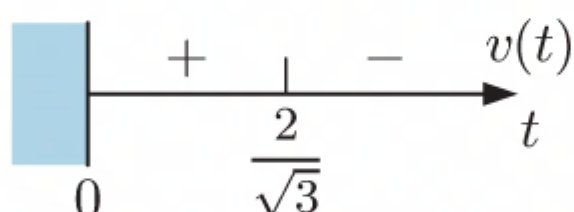
ii $v(t) = 0$ when $12 - 9t^2 = 0$

$$\therefore 9t^2 = 12$$

$$\therefore t^2 = \frac{4}{3}$$

$$\therefore t = \frac{2}{\sqrt{3}} \quad \{t \geq 0\}$$

The sign diagram of $v(t)$ is



The speed of the particle is decreasing when $v(t)$ and $a(t)$ have opposite sign, that is when $0 \leq t \leq \frac{2}{\sqrt{3}}$.

94 $v = 2\sqrt{t} - t \text{ m s}^{-1}, \quad t \geq 0$

a When $t = 5$, $v = 2\sqrt{5} - 5 \approx -0.528 \text{ m s}^{-1}$
 $\therefore \text{speed} = |v| \approx 0.528 \text{ m s}^{-1}$

b $a = \frac{dv}{dt}$
 $= \frac{2}{2\sqrt{t}} - 1$
 $= \frac{1}{\sqrt{t}} - 1 \text{ m s}^{-2}$

c The direction of motion changes when $v = 0$

$$\therefore 2\sqrt{t} - t = 0$$

$$\therefore 2\sqrt{t} = t$$

$$\therefore 4t = t^2$$

$$\therefore t^2 - 4t = 0$$

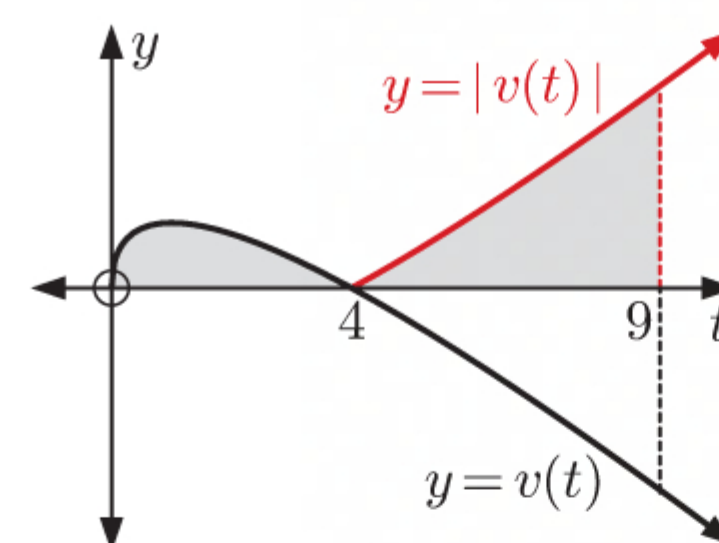
$$\therefore t(t - 4) = 0$$

$$\therefore t = 0 \text{ or } 4$$

Now $v = 0$ when $t = 0$ means that the particle was initially stationary. So, it does not make sense to talk about the direction of motion changing when there is nothing to compare its current direction to.

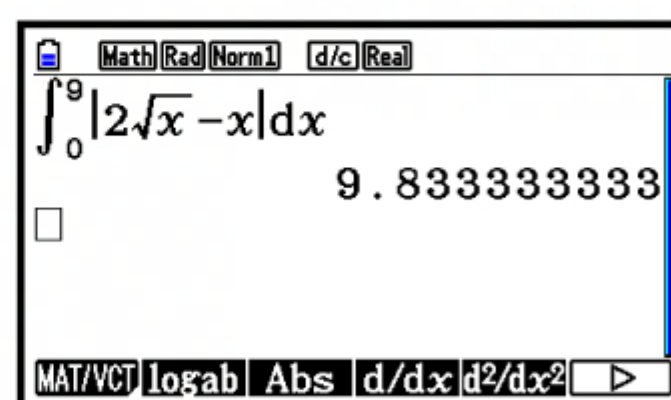
\therefore the direction of motion changes at $t = 4$ seconds.

d Total distance $= \int_0^9 |v| dt$
 $= \int_0^4 v dt + \int_4^9 -v dt$
 $= \int_0^4 v dt - \int_4^9 v dt$
 $= \int_0^4 (2\sqrt{t} - t) dt - \int_4^9 (2\sqrt{t} - t) dt$
 $= \left[\frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_0^4 - \left[\frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_4^9$
 $= \left(\frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right) - 0 - \left(\frac{4}{3}(9)^{\frac{3}{2}} - \frac{1}{2}(9)^2 \right) + \left(\frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right)$
 $= \frac{32}{3} - 8 - 36 + \frac{81}{2} + \frac{32}{3} - 8$
 $= \frac{64}{3} + \frac{81}{2} - 52$
 $= \frac{59}{6}$
 $= 9\frac{5}{6} \text{ m}$




Alternatively, using technology:

$$\int_0^9 |v| dt \approx 9.833 \approx 9\frac{5}{6} \text{ m}$$



MIXED QUESTIONS

MIXED QUESTIONS SET 1

1 $y = 2x^2 - 9x + 3$ has $a = 2$, $b = -9$, and $c = 3$. Since $a > 0$, the shape is .

a $\frac{-b}{2a} = \frac{-(-9)}{2(2)} = \frac{9}{4}$

The axis of symmetry is $x = \frac{9}{4}$.

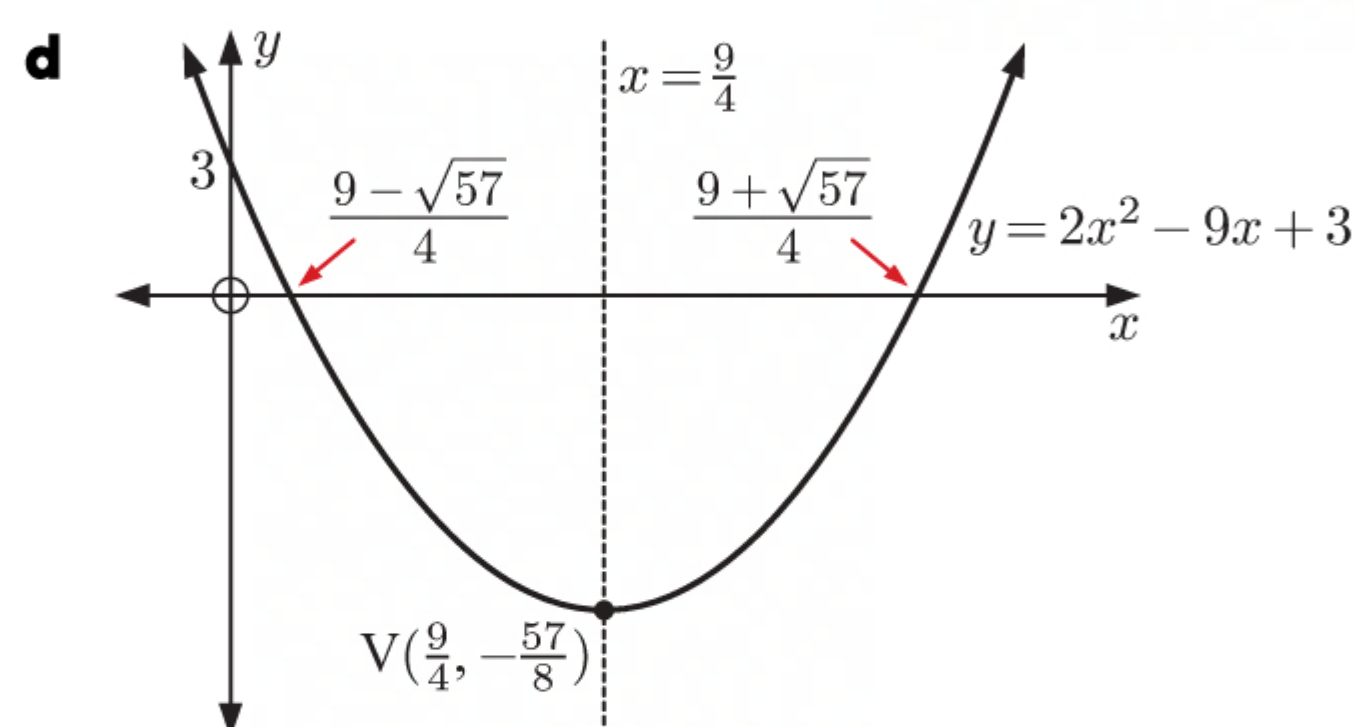
c The y -intercept is 3.

$$\begin{aligned} \text{When } y = 0, \\ 2x^2 - 9x + 3 &= 0 \\ \therefore x^2 - \frac{9}{2}x + \frac{3}{2} &= 0 \\ \therefore x^2 - \frac{9}{2}x &= -\frac{3}{2} \\ \therefore x^2 - \frac{9}{2}x + \left(-\frac{9}{4}\right)^2 &= -\frac{3}{2} + \left(-\frac{9}{4}\right)^2 \\ \therefore \left(x - \frac{9}{4}\right)^2 &= \frac{57}{16} \\ \therefore x - \frac{9}{4} &= \pm \frac{\sqrt{57}}{4} \\ \therefore x &= \frac{9 \pm \sqrt{57}}{4} \end{aligned}$$

\therefore the x -intercepts are $\frac{9 \pm \sqrt{57}}{4}$.

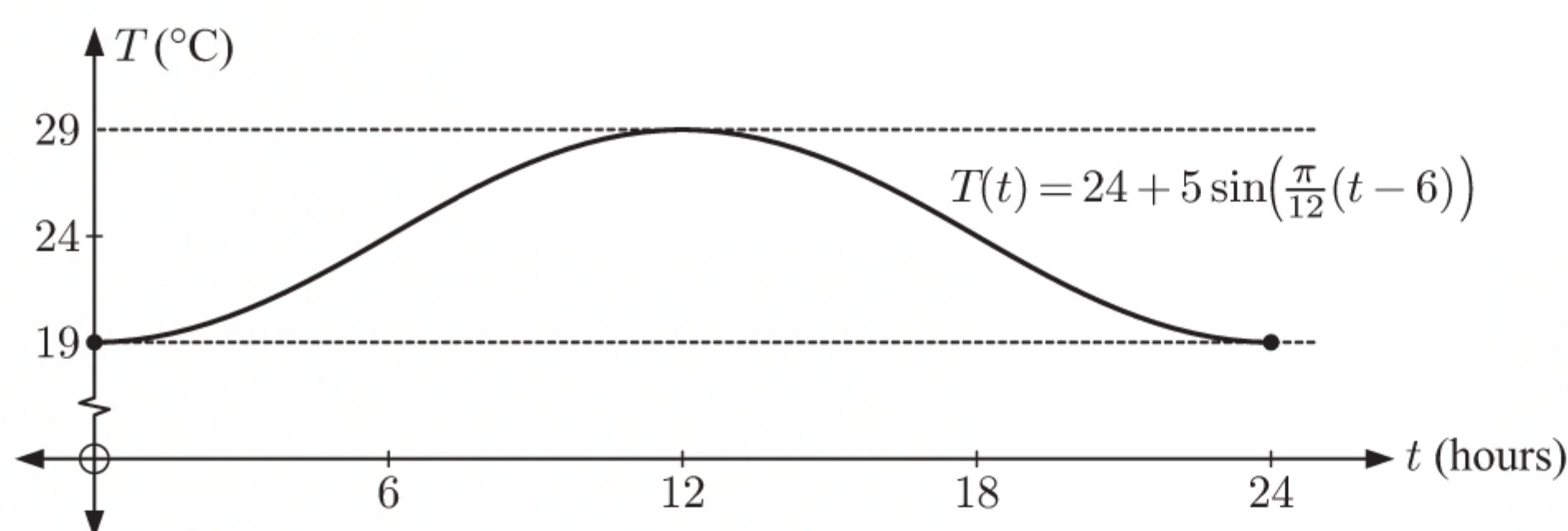
b When $x = \frac{9}{4}$, $y = 2\left(\frac{9}{4}\right)^2 - 9\left(\frac{9}{4}\right) + 3$
 $= \frac{162}{16} - \frac{81}{4} + 3$
 $= -\frac{57}{8}$

\therefore the vertex is $\left(\frac{9}{4}, -\frac{57}{8}\right)$.



2 a For $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right)$:

- the amplitude is 5
- the period is $\frac{2\pi}{(\frac{\pi}{12})} = 24$ hours
- the horizontal translation is 6 hours to the right
- the principal axis is $T = 24$.



b i 2 pm is 8 hours after 6 am.

$$\begin{aligned} T(8) &= 24 + 5 \sin\left(\frac{\pi}{12}(8 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 2\right) \\ &= 24 + 5 \sin \frac{\pi}{6} \\ &= 24 + 5 \times \frac{1}{2} \\ &= 26.5 \end{aligned}$$

\therefore at 2 pm, the temperature inside Pam's caravan is 26.5°C .

ii 9 pm is 15 hours after 6 am.

$$\begin{aligned} T(15) &= 24 + 5 \sin\left(\frac{\pi}{12}(15 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 9\right) \\ &= 24 + 5 \sin \frac{3\pi}{4} \\ &= 24 + 5 \times \frac{1}{\sqrt{2}} \\ &\approx 27.5 \end{aligned}$$

\therefore at 9 pm, the temperature inside Pam's caravan is about 27.5°C .

c The maximum temperature inside Pam's caravan is $24 + 5 = 29^\circ\text{C}$, which occurs when $t = 12$. 12 hours after 6 am is 6 pm.

So, the maximum temperature inside Pam's caravan occurs at 6 pm.

3 $f(x) = \ln(x\sqrt{1-2x})$

a $\ln(x\sqrt{1-2x})$ is defined when

$$x\sqrt{1-2x} > 0$$

$$\therefore x > 0 \quad \text{and} \quad 1-2x > 0$$

$$\therefore x > 0 \quad \text{and} \quad 2x < 1$$

$$\therefore x > 0 \quad \text{and} \quad x < \frac{1}{2}$$

So, the domain is $\{x \mid 0 < x < \frac{1}{2}\}$.

c At the point(s) where the normal has gradient $-\frac{6}{5}$, the tangent has gradient $\frac{5}{6}$.

Now $f'(x) = \frac{5}{6}$ where $\frac{1-3x}{x(1-2x)} = \frac{5}{6}$ {from **b**}

$$\therefore 6(1-3x) = 5x(1-2x)$$

$$\therefore 6-18x = 5x-10x^2$$

$$\therefore 10x^2 - 23x + 6 = 0$$

$$\therefore (10x-3)(x-2) = 0$$

$$\therefore x = \frac{3}{10} \quad \{0 < x < \frac{1}{2}\}$$

$$f\left(\frac{3}{10}\right) = \ln\left(\frac{3}{10}\sqrt{1-2\left(\frac{3}{10}\right)}\right)$$

$$= \ln\left(\frac{3}{10}\sqrt{1-\frac{3}{5}}\right)$$

$$= \ln\left(\frac{3}{10}\sqrt{\frac{2}{5}}\right)$$

$$\approx -1.66$$

\therefore the normal to $y = f(x)$ has gradient $-\frac{6}{5}$ at $\left(\frac{3}{10}, -1.66\right)$.

4 $f(x) = 5^x$, $g(x) = 2x + 1$

a $(f \circ g)(x) = f(g(x))$
 $= f(2x + 1)$
 $= 5^{2x+1}$

b $f \circ g$ is $y = 5^{2x+1}$
 $\therefore (f \circ g)^{-1}$ is $x = 5^{2y+1}$
 $\therefore \log_5 x = 2y + 1$
 $\therefore 2y = \log_5 x - 1$
 $\therefore y = \frac{1}{2} \log_5 x - \frac{1}{2}$
 $\therefore (f \circ g)^{-1}(x) = \frac{1}{2} \log_5 x - \frac{1}{2}$
 $\therefore (f \circ g)^{-1}(0.2) = \frac{1}{2} \log_5(0.2) - \frac{1}{2}$
 $= \frac{1}{2} \log_5\left(\frac{1}{5}\right) - \frac{1}{2}$
 $= \frac{1}{2} \log_5(5^{-1}) - \frac{1}{2}$
 $= -\frac{1}{2} - \frac{1}{2}$
 $= -1$

5 a Let u_0 be the original value of the car.

Since the value of the car depreciates by 10% each year, the value of the car after 3 years is

$$u_0 \times (1 - 0.1)^3 = u_0 \times (0.9)^3.$$

$$\therefore u_0 \times (0.9)^3 = 26\,244$$

$$\therefore u_0 = \frac{26\,244}{(0.9)^3}$$

$$= 36\,000$$

\therefore the original value of the car is \$36 000.

b $f(x) = \ln(x(1-2x)^{\frac{1}{2}})$

$$\therefore f'(x) = \frac{(1-2x)^{\frac{1}{2}} + \frac{1}{2}x(1-2x)^{-\frac{1}{2}}(-2)}{x(1-2x)^{\frac{1}{2}}} \times \frac{(1-2x)^{\frac{1}{2}}}{(1-2x)^{\frac{1}{2}}}$$

$$= \frac{(1-2x) - x}{x(1-2x)}$$

$$= \frac{1-3x}{x(1-2x)}$$

c Under a horizontal stretch with scale factor k , $f(x)$ becomes $f\left(\frac{1}{k}x\right)$.

$$\therefore y = 5^x \quad \text{becomes} \quad y = 5^{\frac{x}{k}}$$

The resulting graph passes through $\left(\frac{1}{6}, \sqrt{5}\right)$.

$$\therefore \sqrt{5} = 5^{\frac{1}{k}\left(\frac{1}{6}\right)}$$

$$\therefore 5^{\frac{1}{2}} = 5^{\frac{1}{6k}}$$

$$\therefore \frac{1}{2} = \frac{1}{6k} \quad \{\text{equating indices}\}$$

$$\therefore 6k = 2$$

$$\therefore k = \frac{1}{3}$$

b Let u_n be the value of the car after n years.

$$\therefore u_n = u_0 \times (1-d)^n$$

$$= 36\,000 \times (0.9)^n$$

which describes a geometric sequence with

$$u_0 = 36\,000 \quad \text{and} \quad r = 0.9.$$

- c** For the value of the car to fall below \$10 000, we need to find when

$$36\,000 \times (0.9)^n = 10\,000$$

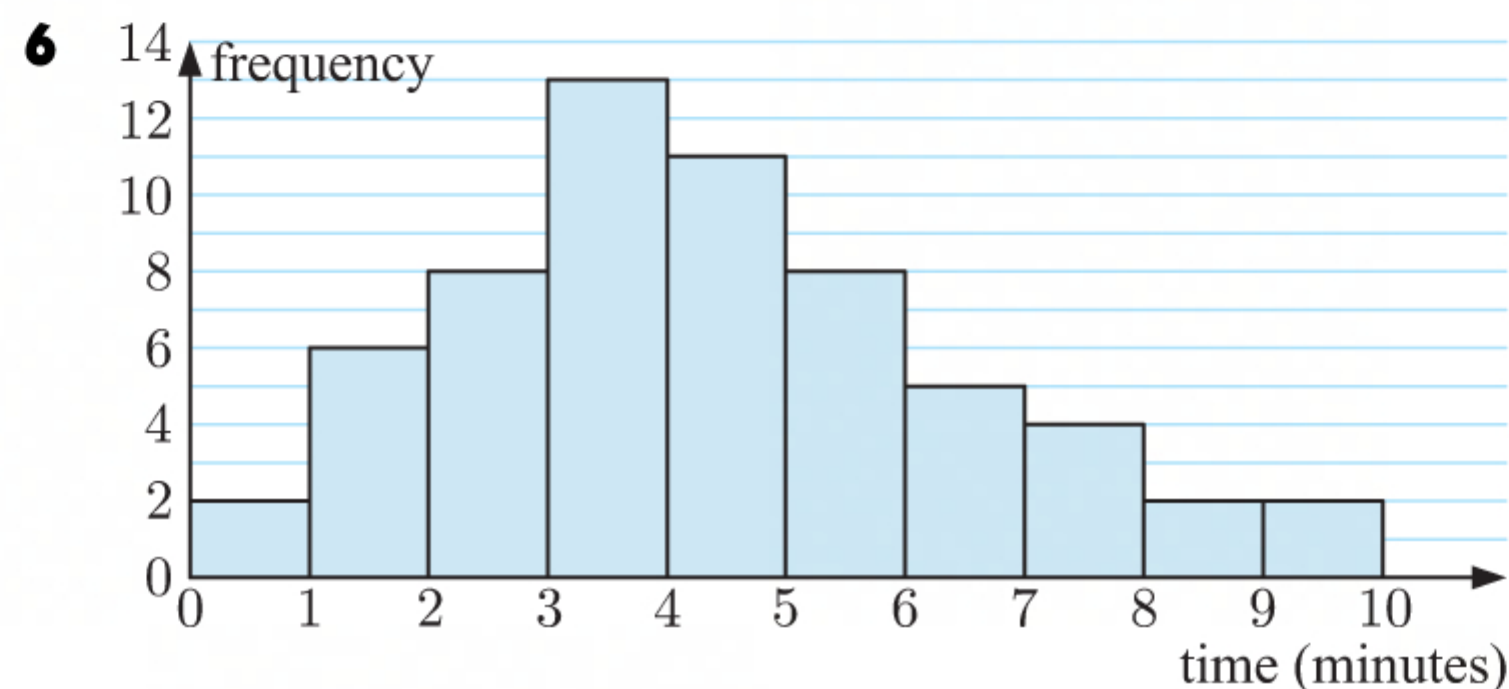
$$\therefore (0.9)^n = \frac{10\,000}{36\,000}$$

$$\therefore (0.9)^n = \frac{5}{18}$$

$$\therefore n \log(0.9) = \log\left(\frac{5}{18}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{18}\right)}{\log(0.9)} \approx 12.2$$

So, in the 13th year the value of the car falls below \$10 000.



- a** The modal class is $3 \leq t < 4$ where t is the time in minutes.

b

Duration of call (t min)	Frequency (f)	Midpoint (x)	Product (xf)
$0 \leq t < 1$	2	0.5	1
$1 \leq t < 2$	6	1.5	9
$2 \leq t < 3$	8	2.5	20
$3 \leq t < 4$	13	3.5	45.5
$4 \leq t < 5$	11	4.5	49.5
$5 \leq t < 6$	8	5.5	44
$6 \leq t < 7$	5	6.5	32.5
$7 \leq t < 8$	4	7.5	30
$8 \leq t < 9$	2	8.5	17
$9 \leq t < 10$	2	9.5	19
Total	$\sum f = 61$		$\sum xf = 267.5$

c
$$\bar{x} = \frac{\sum xf}{\sum f}$$

$$= \frac{267.5}{61}$$

$$\approx 4.39$$

d
$$P(\geq 6 \text{ minutes}) \approx \frac{5 + 4 + 2 + 2}{61}$$

$$\approx \frac{13}{61}$$

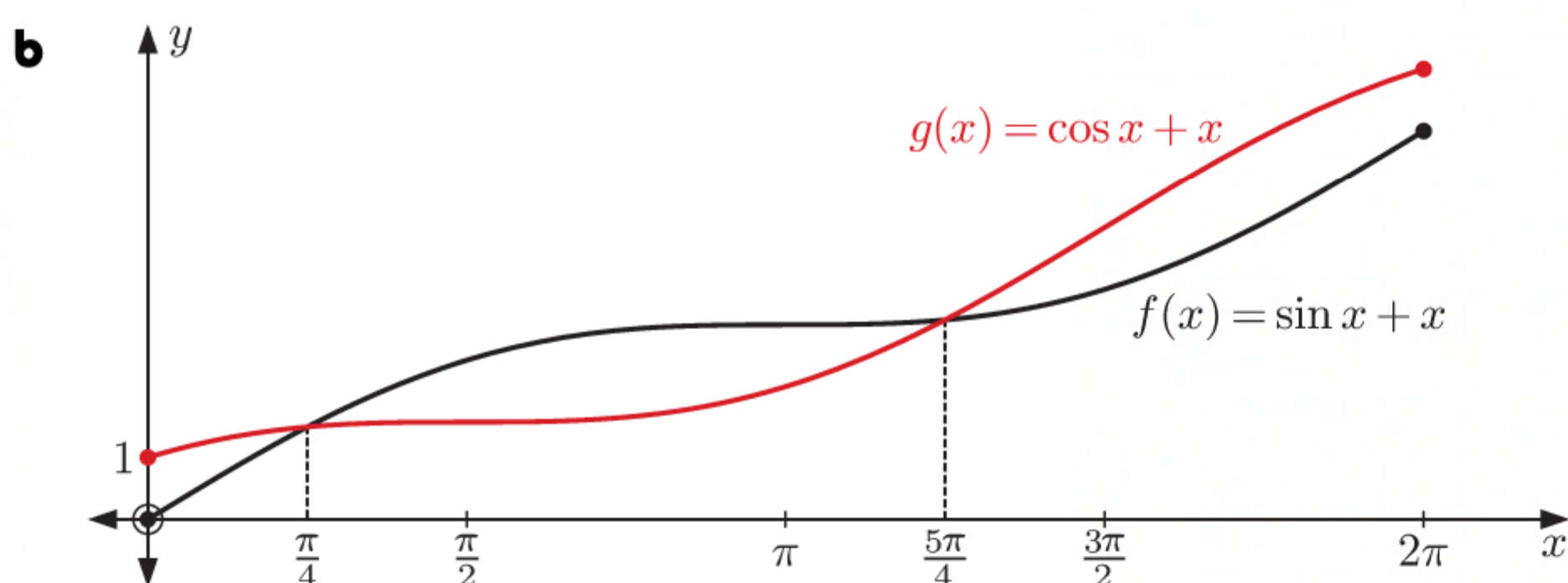
$$\approx 0.213$$

\therefore the mean length of a phone call is about 4.39 minutes.

7 a
$$\int h(x) dx = \int (\sin x - \cos x) dx$$

$$= -\cos x - \sin x + c$$

\therefore the antiderivative of $h(x)$ is $H(x) = -\cos x - \sin x$.



$$\begin{aligned}
 \text{c } y = f(x) \text{ and } y = g(x) \text{ meet where } \sin x + x &= \cos x + x \\
 \therefore \sin x &= \cos x \\
 \therefore \tan x &= 1 \\
 \therefore x &= \frac{\pi}{4}, \frac{5\pi}{4} \quad \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

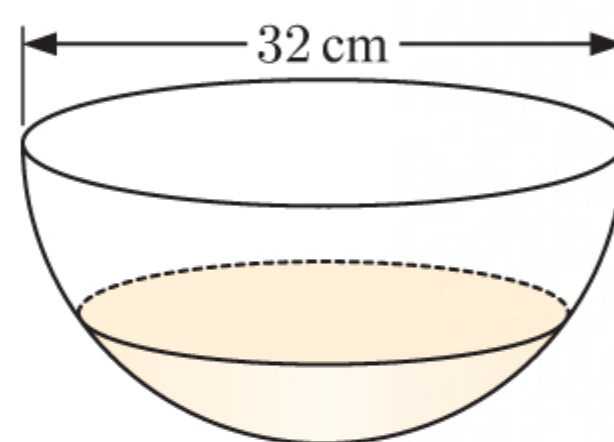
Since $\sin x + x \geq \cos x + x$ on the interval $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$,

$$\begin{aligned}
 \text{Area enclosed} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} ((\sin x + x) - (\cos x + x)) dx \\
 &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) dx \\
 &= \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \quad \{\text{using a}\} \\
 &= \left(-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right) \\
 &= \left(-\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) \right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\
 &= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\
 &= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= 2\sqrt{2} \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a } \log_4(x^2 - x + 3) \\
 &= \frac{\log_2(x^2 - x + 3)}{\log_2 4} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 &= \frac{\log_2(x^2 - x + 3)}{\log_2(2^2)} \\
 &= \frac{\log_2(x^2 - x + 3)}{2} \\
 &= \frac{1}{2} \log_2(x^2 - x + 3) \\
 &= \log_2((x^2 - x + 3)^{\frac{1}{2}}) \\
 &= \log_2 \sqrt{x^2 - x + 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_2(x + 2) &= \log_4(x^2 - x + 3) \\
 \therefore \log_2(x + 2) &= \log_2 \sqrt{x^2 - x + 3} \quad \{\text{using a}\} \\
 \therefore x + 2 &= \sqrt{x^2 - x + 3} \\
 \therefore (x + 2)^2 &= x^2 - x + 3, \quad x \geq -2 \\
 \therefore x^2 + 4x + 4 &= x^2 - x + 3, \quad x \geq -2 \\
 \therefore 5x &= -1, \quad x \geq -2 \\
 \therefore x &= -\frac{1}{5}
 \end{aligned}$$

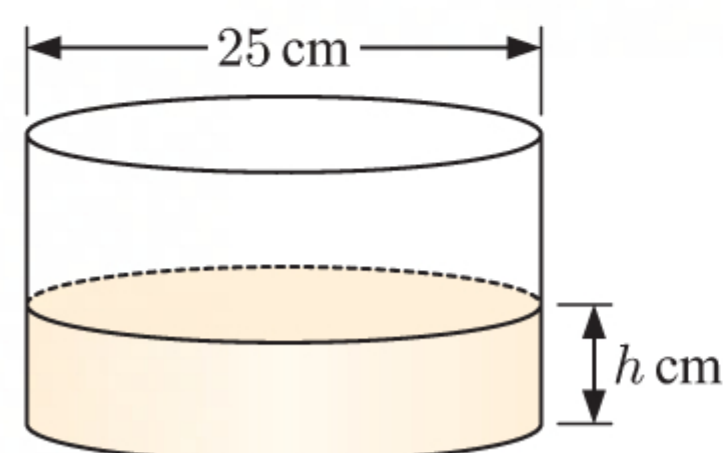
$$\begin{aligned}
 \text{9 a } V &= \frac{1}{2} \times \text{volume of sphere} \\
 &= \frac{1}{2} \times \frac{4}{3} \pi r^3 \\
 &= \frac{2}{3} \times \pi \times \left(\frac{32}{2}\right)^3 \text{ cm}^3 \\
 &= \frac{8192}{3} \pi \text{ cm}^3 \\
 &\approx 8580 \text{ cm}^3
 \end{aligned}$$



The capacity of the bowl is about 8580 mL or 8.58 L.

b i When 20% full, the bowl contains $\frac{8192}{3} \pi \times 0.2 \approx 1720$ mL or about 1.72 L of cake batter.

$$\begin{aligned}
 \text{ii } V &= \frac{8192}{3} \pi \times 0.2 \text{ cm}^3 \\
 \therefore \pi \times \left(\frac{25}{2}\right)^2 \times h &= \frac{8192}{3} \pi \times 0.2 \\
 \therefore h &= \frac{8192 \times 0.2}{3 \times (12.5)^2} \\
 &\approx 3.50 \text{ cm}
 \end{aligned}$$



The cake batter will reach about 3.50 cm up the tin.

10 Let W denote Wollongong, P denote Picton, and C denote Canberra.

a North-west is in the direction 315° . South-west is in the direction 225° .

$$\widehat{PWN} = 360^\circ - 315^\circ = 45^\circ$$

$$\widehat{WPN} = 180^\circ - 45^\circ = 135^\circ \quad \{\text{co-interior angles}\}$$

$$\widehat{CPW} = 225^\circ - 135^\circ = 90^\circ$$

$\therefore \triangle CPW$ is right angled at P.

$$CW^2 = 36^2 + 210^2 \quad \{\text{Pythagoras}\}$$

$$\therefore CW = \sqrt{36^2 + 210^2} \quad \{\text{as } CW > 0\}$$

$$\approx 213 \text{ km}$$

So, Canberra is about 213 km from Wollongong.

b $\widehat{CPN} = 360^\circ - 225^\circ = 135^\circ$

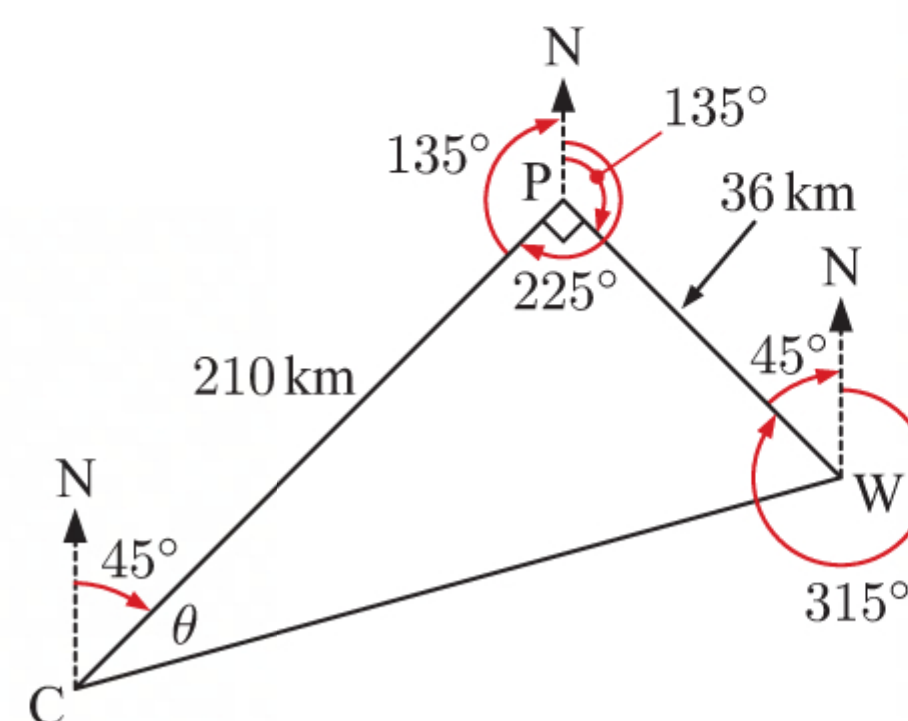
$$\widehat{NCP} = 180^\circ - 135^\circ = 45^\circ \quad \{\text{co-interior angles}\}$$

$$\tan \theta = \frac{36}{210}$$

$$\therefore \theta = \tan^{-1}\left(\frac{36}{210}\right) \approx 9.73^\circ$$

$$\therefore \text{the bearing of Wollongong from Canberra} \approx 45^\circ + 9.73^\circ$$

$$\approx 054.7^\circ$$



11 $v(t) = 30 - 20e^{-0.2t} \text{ m s}^{-1}$

a i $v(0) = 30 - 20e^0$
 $= 10$

\therefore the initial velocity of the boat is 10 m s^{-1} .

ii $v(2) = 30 - 20e^{-0.2 \times 2}$
 ≈ 16.6

\therefore the velocity of the boat after 2 seconds is about 16.6 m s^{-1} .

b $v(t) = 20$ when $30 - 20e^{-0.2t} = 20$

$$\therefore 20e^{-0.2t} = 10$$

$$\therefore e^{-0.2t} = \frac{1}{2}$$

$$\therefore e^{0.2t} = 2$$

$$\therefore 0.2t = \ln 2$$

$$\therefore t = 5 \ln 2 \approx 3.47$$

It will take about 3.47 seconds for the boat's velocity to reach 20 m s^{-1} .

c As $t \rightarrow \infty$, $e^{-0.2t} \rightarrow 0$

$$\therefore v(t) \rightarrow 30 - 20(0) = 30$$

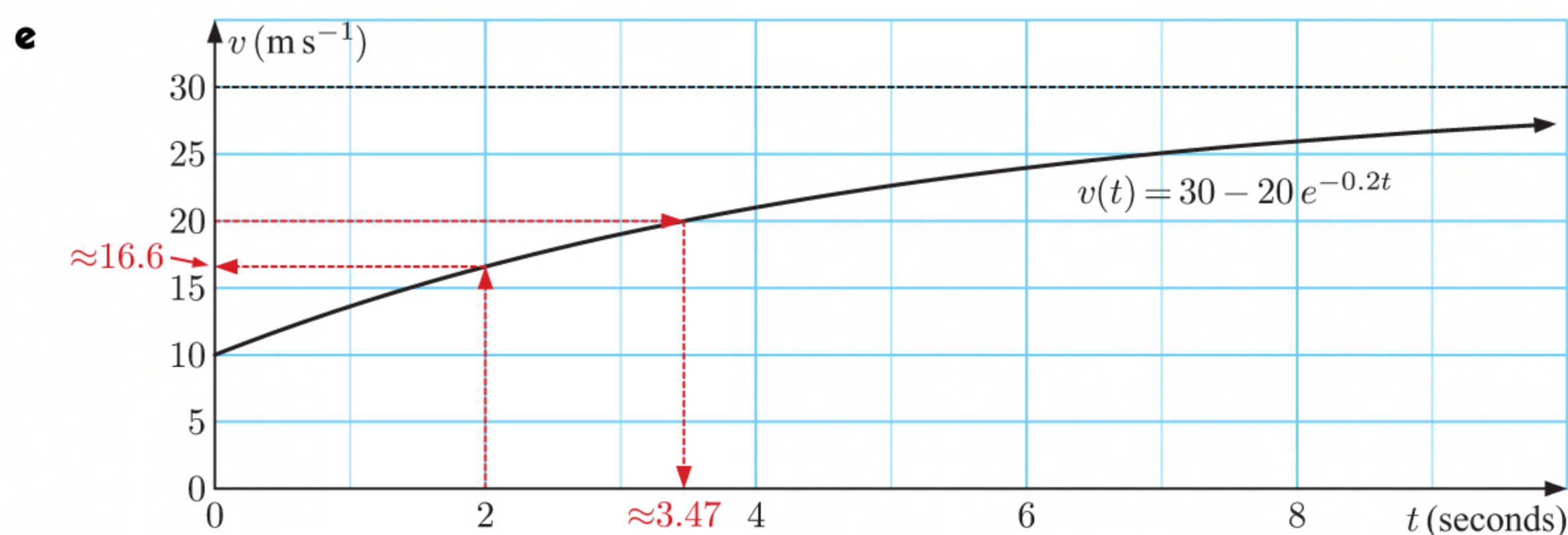
d $v(t) = 30 - 20e^{-0.2t}$

$$\therefore v'(t) = -20e^{-0.2t}(-0.2) \quad \{\text{chain rule}\}$$

$$\therefore v'(t) = 4e^{-0.2t}$$

$$\therefore v'(t) > 0 \quad \{\text{as } e^{-0.2t} > 0 \text{ for all } t\}$$

So, the acceleration $v'(t)$ is always positive.



- f** The boat's velocity reached 20 m s^{-1} after $5 \ln 2$ seconds. {from **b**}

$$\begin{aligned}
 \text{Distance travelled in first } 5 \ln 2 \text{ seconds} &= \int_0^{5 \ln 2} v(t) dt \\
 &= \int_0^{5 \ln 2} (30 - 20e^{-0.2t}) dt \\
 &= [30t + 100e^{-0.2t}]_0^{5 \ln 2} \\
 &= (150 \ln 2 + 100e^{-\ln 2}) - (0 + 100) \\
 &= 150 \ln 2 + 100 \times \frac{1}{2} - 100 \\
 &= 150 \ln 2 - 50 \\
 &\approx 54.0 \text{ m}
 \end{aligned}$$

\therefore the boat travelled about 54.0 m before its velocity reached 20 m s^{-1} .

12

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

- a** X can only take the values -2, 0, 3, or 5.

$\therefore X$ is a discrete random variable.

- b** Since this is a probability distribution, $P(X = -2) + P(X = 0) + P(X = 3) + P(X = 5) = 1$

$$\therefore \frac{1}{3} + \frac{1}{6} + k + \frac{1}{12} = 1$$

$$\therefore k + \frac{7}{12} = 1$$

$$\therefore k = \frac{5}{12}$$

- c** Since $P(X = 3)$ is the greatest probability, 3 is the mode of the distribution.

$$P(X = -2) = \frac{1}{3} \approx 0.333$$

$$P(X = -2) + P(X = 0) = \frac{1}{3} + \frac{1}{6} = 0.5$$

Since $P(X = -2) + P(X = 0) \geq 0.5$, the median is 0.

- d** $E(X) = -2(\frac{1}{3}) + 0(\frac{1}{6}) + 3(\frac{5}{12}) + 5(\frac{1}{12})$
 $= -\frac{2}{3} + 0 + \frac{5}{4} + \frac{5}{12}$
 $= 1$

MIXED QUESTIONS SET 2

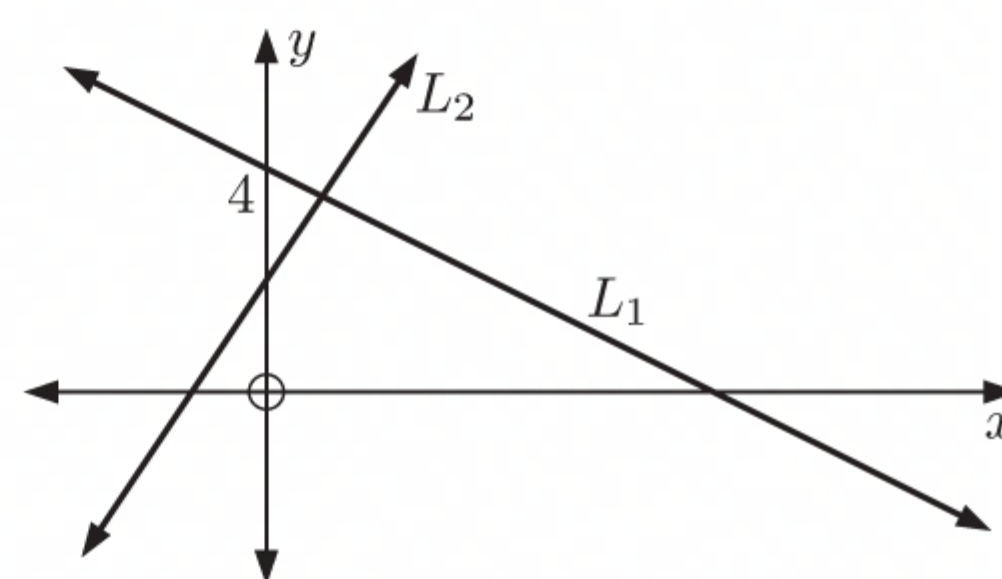
- 1 a** L_1 has gradient $-\frac{1}{2}$ and passes through $(0, 4)$.

$$\therefore L_1 \text{ has equation } y - 4 = -\frac{1}{2}(x - 0)$$

$$\therefore y - 4 = -\frac{1}{2}x$$

$$\therefore \frac{1}{2}x + y - 4 = 0$$

$$\therefore x + 2y - 8 = 0 \quad \dots (1)$$



- b** L_2 has gradient $\frac{8 - (-1)}{4 - (-2)} = \frac{9}{6} = \frac{3}{2}$, and passes through $(-2, -1)$.

$$\therefore L_2 \text{ has equation } y - (-1) = \frac{3}{2}(x - (-2))$$

$$\therefore y + 1 = \frac{3}{2}(x + 2)$$

$$\therefore y + 1 = \frac{3}{2}x + 3$$

$$\therefore \frac{3}{2}x - y + 2 = 0$$

$$\therefore 3x - 2y + 4 = 0 \quad \dots (2)$$

Adding (1) and (2) gives

$$x + 2y - 8 = 0$$

$$3x - 2y + 4 = 0$$

$$\hline 4x - 4 = 0$$

$$\therefore 4x = 4$$

$$\therefore x = 1$$

Substituting $x = 1$ into (1) gives $1 + 2y - 8 = 0$

$$\therefore 2y = 7$$

$$\therefore y = \frac{7}{2}$$

\therefore the point of intersection of L_1 and L_2 is $(1, \frac{7}{2})$.

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad u_4 &= u_1 r^3 \\
 \therefore 8 &= 27r^3 \\
 \therefore r^3 &= \frac{8}{27} \\
 \therefore r &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad u_n &= u_1 r^{n-1} \\
 \therefore u_n &= 27\left(\frac{2}{3}\right)^{n-1} \\
 \text{So, } S &= \sum_{n=1}^{\infty} u_n \\
 &= \sum_{n=1}^{\infty} 27\left(\frac{2}{3}\right)^{n-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad u_6 &= u_1 r^5 \\
 &= 27\left(\frac{2}{3}\right)^5 \\
 &= \frac{32}{9}
 \end{aligned}$$

\mathbf{d} Since $|r| = \left|\frac{2}{3}\right|$ is < 1 , the sum of the infinite series converges.

$$\begin{aligned}
 \therefore S &= \frac{u_1}{1-r} \\
 &= \frac{27}{1-\frac{2}{3}} \\
 &= \frac{27}{\frac{1}{3}} \\
 &= 81
 \end{aligned}$$

$$\mathbf{3}$$

Event	Time (seconds)	μ (seconds)	σ (seconds)
100 m	9.99	10.20	0.113
200 m	17.30	18.50	0.706

$$\mathbf{a} \quad \text{For the 100 m event, } z\text{-score} = \frac{9.99 - 10.20}{0.113} \approx -1.86$$

$$\text{For the 200 m event, } z\text{-score} = \frac{17.30 - 18.50}{0.706} \approx -1.70$$

\mathbf{b} A lower z -score is better as it indicates that the time is lower, and hence that Carl ran faster.
 \therefore Carl performed better in the 100 m event.

$\mathbf{4} \quad \mathbf{a}$ The tangent to the curve $y = ax^3 - bx^2$ at the point where $x = 3$ is $y = x - 6$.
 \therefore the tangent has gradient 1, and the point of contact is $(3, 3 - 6)$ which is $(3, -3)$.

$$\text{Now } f(x) = ax^3 - bx^2$$

$$\therefore f'(x) = 3ax^2 - 2bx$$

$$\text{So, } f'(3) = 1 \quad \text{and} \quad f(3) = -3$$

$$\therefore 3a(3)^2 - 2b(3) = 1 \quad \therefore a(3)^3 - b(3)^2 = -3$$

$$\therefore 27a - 6b = 1 \quad \dots (1) \quad \therefore 27a - 9b = -3 \quad \dots (2)$$

$$\begin{array}{rcl}
 \text{Subtracting (2) from (1) gives} & 27a - 6b = 1 & \\
 & -(27a - 9b = -3) & \\
 \hline
 & 3b = 4 & \\
 \therefore b = \frac{4}{3} & &
 \end{array}$$

$$\begin{array}{rcl}
 \text{Substituting } b = \frac{4}{3} \text{ into (1) gives} & 27a - 6\left(\frac{4}{3}\right) = 1 & \\
 \therefore 27a - 8 = 1 & & \\
 \therefore 27a = 9 & & \\
 \therefore a = \frac{1}{3} & &
 \end{array}$$

$$\mathbf{b} \quad f(x) = \frac{1}{3}x^3 - \frac{4}{3}x^2 = \frac{1}{3}(x^3 - 4x^2) \quad \{\text{from a}\}$$

$$\text{The tangent meets the curve where } x - 6 = \frac{1}{3}(x^3 - 4x^2)$$

$$\therefore 3x - 18 = x^3 - 4x^2$$

$$\therefore x^3 - 4x^2 - 3x + 18 = 0$$

Since the tangent touches the curve at $x = 3$, there must be a repeated solution at this point.

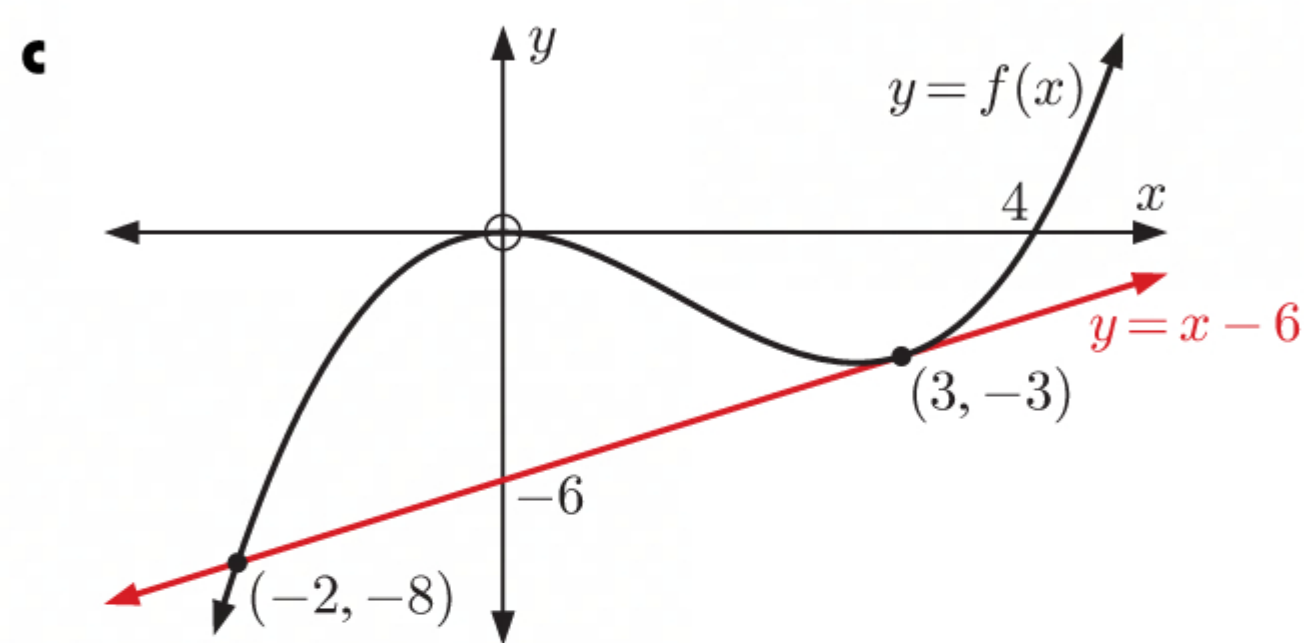
$\therefore (x - 3)^2$ must be a factor of this cubic.

$$\therefore (x - 3)^2(x + 2) = 0$$

\therefore the tangent meets the curve again when $x = -2$.

$$\text{When } x = -2, \quad y = -2 - 6 = -8$$

\therefore the tangent meets the curve again at $(-2, -8)$.



5 a
$$\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1 - \cos^2 \theta}{1 + \cos \theta}$$

$$= \frac{(1 + \cos \theta)(1 - \cos \theta)}{1 + \cos \theta}$$

$$= 1 - \cos \theta \quad \text{for all } \theta \text{ such that } \cos \theta \neq -1.$$

b
$$\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1}{2}$$

$$\therefore 1 - \cos \theta = \frac{1}{2} \quad \{\text{from a as } \cos \theta \neq -1 \text{ for all } -\pi < \theta < \pi\}$$

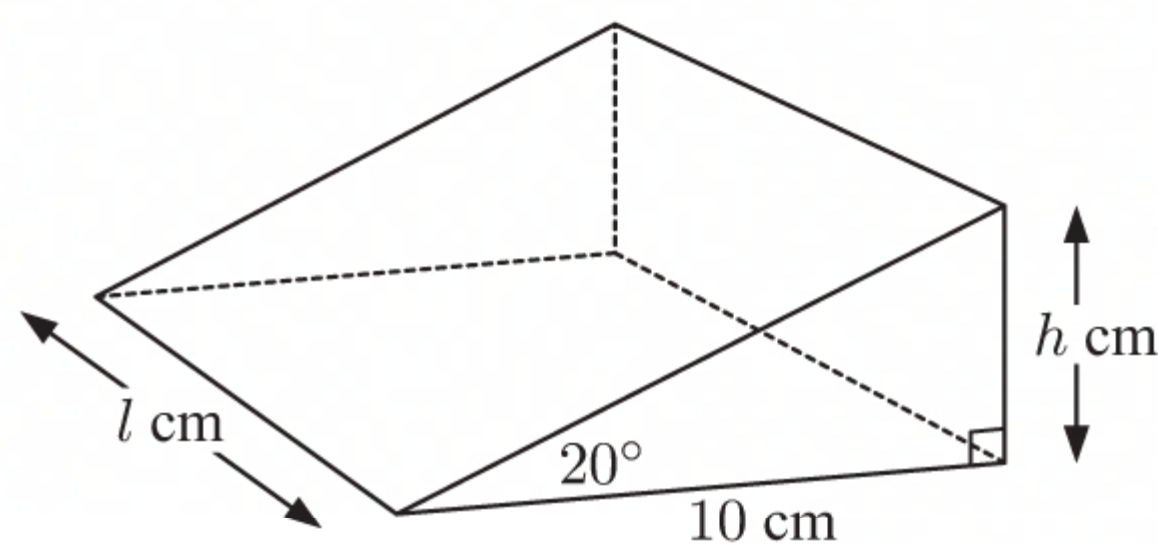
$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = -\frac{\pi}{3} \text{ or } \frac{\pi}{3} \quad \{-\pi < \theta < \pi\}$$

6 a
$$\tan 20^\circ = \frac{h}{10}$$

$$\therefore h = 10 \tan 20^\circ$$

$$\approx 3.640$$



b Area of triangular end $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times 10 \times h$$

$$= 5 \times 10 \tan 20^\circ \quad \{\text{from a}\}$$

$$= 50 \tan 20^\circ \text{ cm}^2$$

$$\approx 18.2 \text{ cm}^2$$

c Volume of door-stop $= 60 \text{ cm}^3$

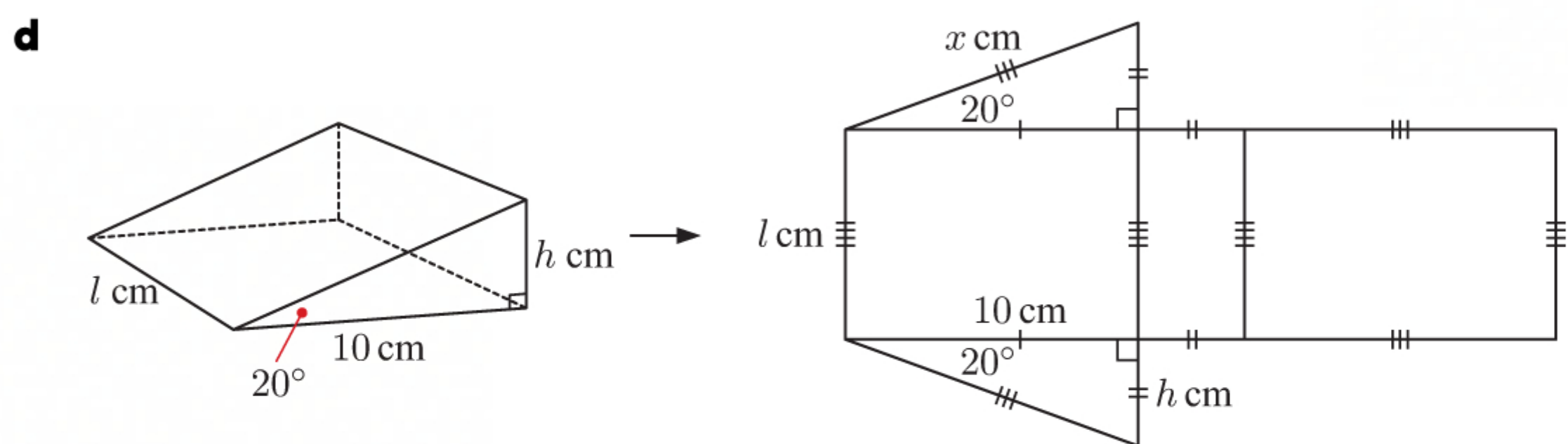
$$\therefore \text{area of triangular end} \times \text{length} = 60$$

$$\therefore 50 \tan 20^\circ \times l = 60 \quad \{\text{from b}\}$$

$$\therefore l = \frac{60}{50 \tan 20^\circ}$$

$$\therefore l = \frac{6}{5 \tan 20^\circ}$$

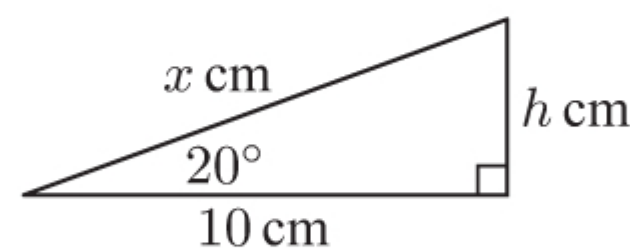
$$\therefore l \approx 3.30$$



Let the hypotenuse of the triangular end be x cm.

$$\cos 20^\circ = \frac{10}{x}$$

$$\therefore x = 10 \cos 20^\circ$$



Surface area $= (10 \times l) + (h \times l) + (x \times l) + 2 \times (\frac{1}{2} \times 10 \times h)$

$$= (10 + h + x) \times l + 10h$$

$$= (10 + 10 \tan 20^\circ + 10 \cos 20^\circ) \times \frac{6}{5 \tan 20^\circ} + 10 \times 10 \tan 20^\circ \quad \{\text{from a and c}\}$$

$$\approx 112 \text{ cm}^2$$

- 7 a**
- The survey is likely to under-represent full-time weekday workers.
 - The survey was taken at a suburban shopping centre, so the people surveyed are likely to prefer suburban shopping. Therefore the sample is likely to be biased toward suburban shopping.
- b** The conclusion is unreasonable since the survey is likely to contain a coverage error, as in **a**, and so the results may not accurately represent the opinions of the whole population.

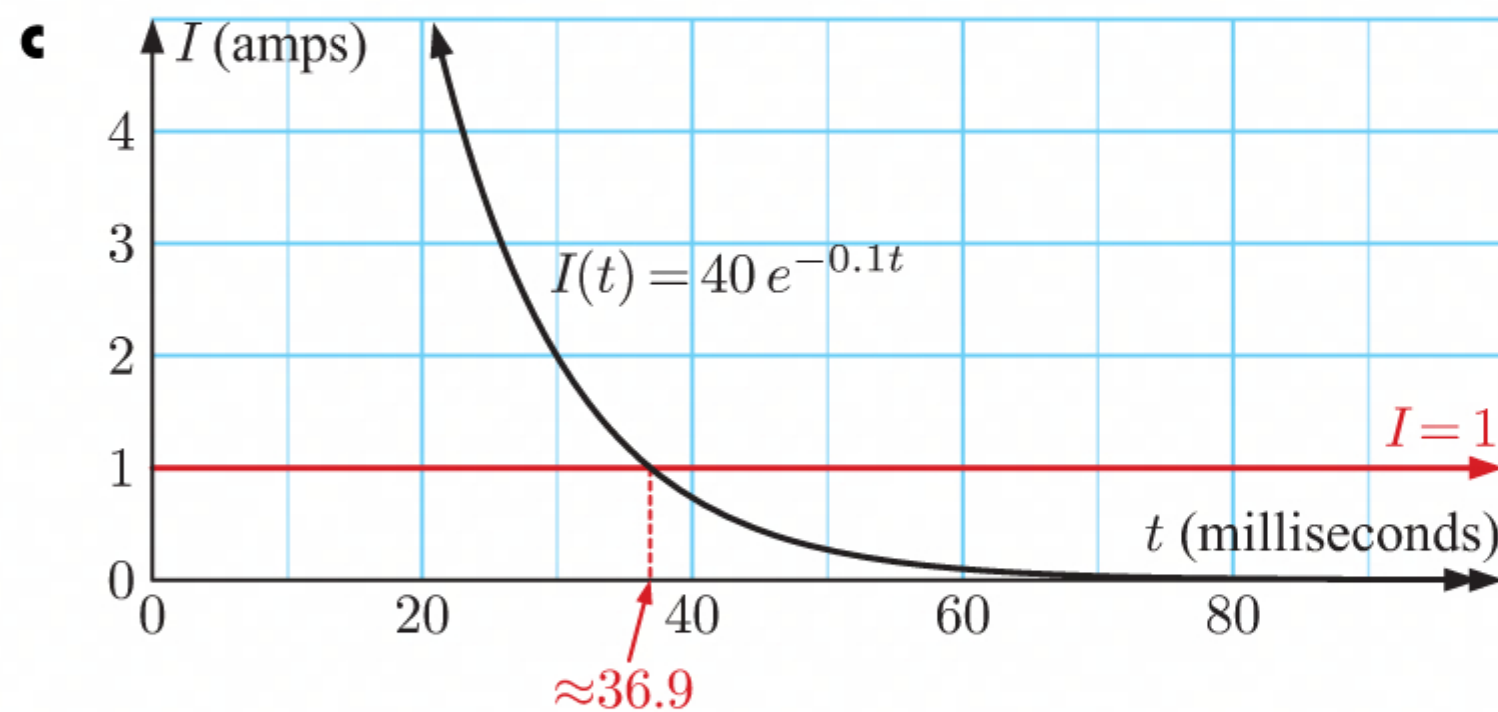
8 $I(t) = 40e^{-0.1t}$ amps

a $I(0) = 40e^0$
 $= 40$

\therefore there was 40 amps of current flowing through the circuit initially.

b $I(100) = 40e^{-0.1 \times 100}$
 ≈ 0.00182

\therefore after 100 milliseconds, there was about 0.00182 amps flowing through the circuit.



d The graphs meet where $40e^{-0.1t} = 1$
 $\therefore e^{-0.1t} = \frac{1}{40}$
 $\therefore e^{0.1t} = 40$
 $\therefore 0.1t = \ln 40$
 $\therefore t = 10 \ln 40$
 $\therefore t \approx 36.9$

\therefore it took about 36.9 milliseconds for the current to fall to 1 amp.

9 a $QR^2 = x^2 + 8^2$ {Pythagoras}
 $\therefore QR = \sqrt{x^2 + 64}$ {as $QR > 0$ }

Also $QS = PS - PQ$
 $= 11 - x$

So, the length of pipeline under the sea is $\sqrt{x^2 + 64}$ km,
and the length of pipeline overland is $(11 - x)$ km.

\therefore the cost $C(x) = 5\sqrt{x^2 + 64} + 3(11 - x)$ million dollars
 $= 5\sqrt{x^2 + 64} + 33 - 3x$ million dollars.

b $C(x) = 5(x^2 + 64)^{\frac{1}{2}} + 33 - 3x$
 $\therefore C'(x) = 5 \times \frac{1}{2}(x^2 + 64)^{-\frac{1}{2}}(2x) - 3 = \frac{5x}{\sqrt{x^2 + 64}} - 3$

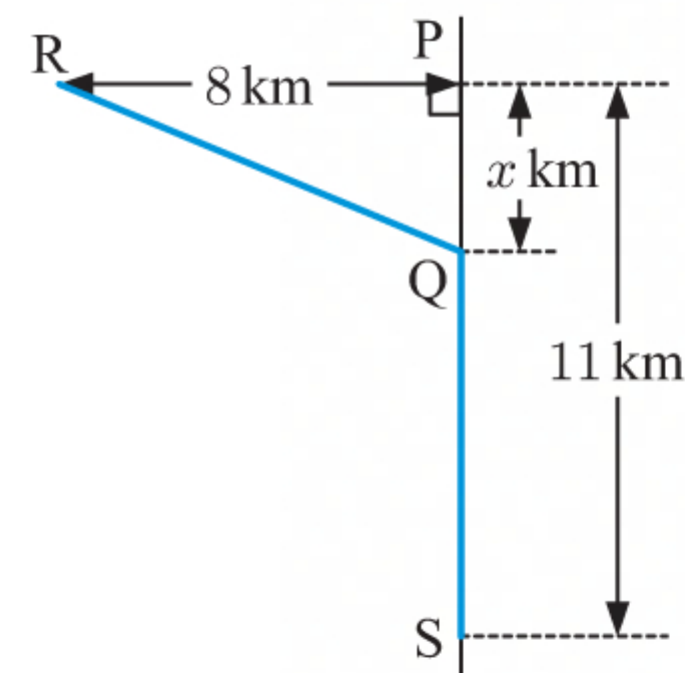
Now $C'(x) = 0$ where $\frac{5x}{\sqrt{x^2 + 64}} - 3 = 0$
 $\therefore \frac{5x}{\sqrt{x^2 + 64}} = 3$
 $\therefore 5x = 3\sqrt{x^2 + 64}$
 $\therefore (5x)^2 = 9(x^2 + 64)$
 $\therefore 25x^2 = 9x^2 + 576$
 $\therefore 16x^2 = 576$
 $\therefore x^2 = 36$
 $\therefore x = 6$ $\{0 \leq x \leq 11\}$

$C'(x)$ has sign diagram:

The minimum cost occurs when $x = 6$.

$C(6) = 5\sqrt{6^2 + 64} + 33 - 3(6)$
 $= 5\sqrt{100} + 33 - 18$
 $= 65$

\therefore the minimum cost of the pipeline is 65 million dollars.



10 $a > b > c > 0$

a i $a > b > 0$
 $\therefore a^2 > b^2$
 $\therefore a^2 - b^2 > 0$

ii $b > c > 0$
 $\therefore b^2 > c^2$
 $\therefore b^2 - c^2 > 0$

b $(a^2 - b^2)(b^2 - c^2) > 0$ {using **a i** and **a ii**}
 $\therefore a^2b^2 - a^2c^2 - b^4 + b^2c^2 > 0$
 $\therefore (ab)^2 + (bc)^2 - (ac)^2 > b^4$

11 We extend the table to include totals for each row and column.

	Defective	Not defective	Total
Corn	37	581	618
Pineapple	24	617	641
Total	61	1198	1259

a There were 1259 tins included in the sample.

b i 1198 of the 1259 tins were not defective.

$$\therefore P(\text{is not defective}) \approx \frac{1198}{1259} \approx 0.952$$

iii 37 of the 618 tins of corn were defective.

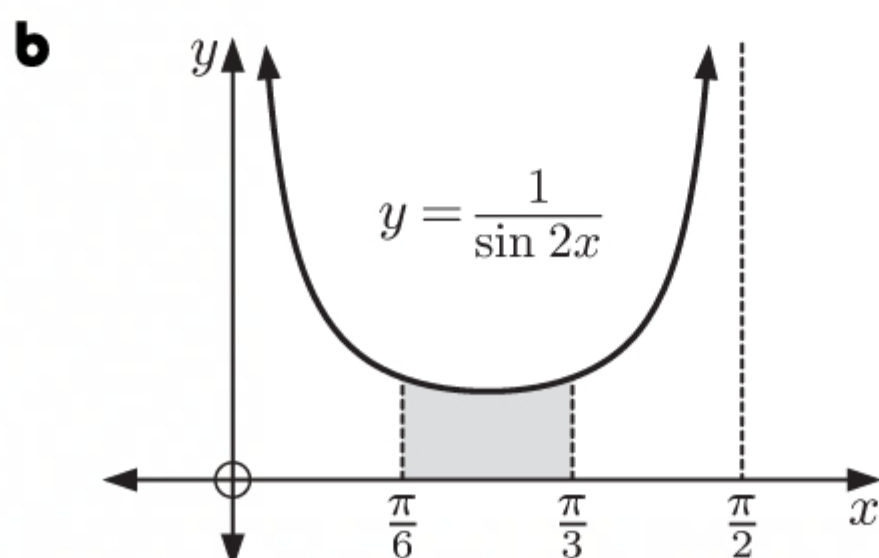
$$\therefore P(\text{is defective, given it is a tin of corn}) \approx \frac{37}{618} \approx 0.0599$$

ii 24 of the 1259 tins were defective tins of pineapple.

$$\therefore P(\text{is a defective tin of pineapple}) \\ \approx \frac{24}{1259} \approx 0.0191$$

12 a $y = \ln(\tan x), \quad 0 < x < \frac{\pi}{2}$

$$\begin{aligned} &= \ln\left(\frac{\sin x}{\cos x}\right) \\ &= \ln(\sin x) - \ln(\cos x) \\ \therefore \frac{dy}{dx} &= \frac{\cos x}{\sin x} - \frac{-\sin x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \\ &= \frac{1}{\frac{1}{2} \sin 2x} \quad \{\cos^2 x + \sin^2 x = 1, \quad \sin 2x = 2 \sin x \cos x\} \\ &= \frac{2}{\sin 2x} \end{aligned}$$



$$\begin{aligned} \text{Shaded area} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin 2x} dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2}{\sin 2x} dx \\ &= \frac{1}{2} \left[\ln(\tan x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \quad \{\text{using a}\} \\ &= \frac{1}{2} (\ln(\tan \frac{\pi}{3}) - \ln(\tan \frac{\pi}{6})) \\ &= \frac{1}{2} (\ln \sqrt{3} - \ln(\frac{1}{\sqrt{3}})) \\ &= \frac{1}{2} (\ln \sqrt{3} - \ln 1 + \ln \sqrt{3}) \\ &= \frac{1}{2} (2 \ln \sqrt{3}) \\ &= \frac{1}{2} \ln 3 \text{ units}^2 \end{aligned}$$

MIXED QUESTIONS SET 3

1 a $x^2 + 8x + k = 0$ has $a = 1$, $b = 8$, and $c = k$ $\therefore \Delta = b^2 - 4ac$

$$\begin{aligned} &= 8^2 - 4(1)(k) \\ &= 64 - 4k \end{aligned}$$

b i For no real roots $\Delta < 0$

$$\therefore 64 - 4k < 0$$

$$\therefore 4k > 64$$

$$\therefore k > 16$$

ii For two distinct real roots $\Delta > 0$

$$\therefore 64 - 4k > 0$$

$$\therefore 4k < 64$$

$$\therefore k < 16$$

$$2 \quad f(x) = \frac{x-3}{2-x} = -\frac{x-3}{x-2} = -\left(\frac{x-2-1}{x-2}\right) = \frac{1}{x-2} - 1$$

a The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq -1\}$.

c $f(0) = \frac{-3}{2} = -\frac{3}{2}$, so the y -intercept is $-\frac{3}{2}$.

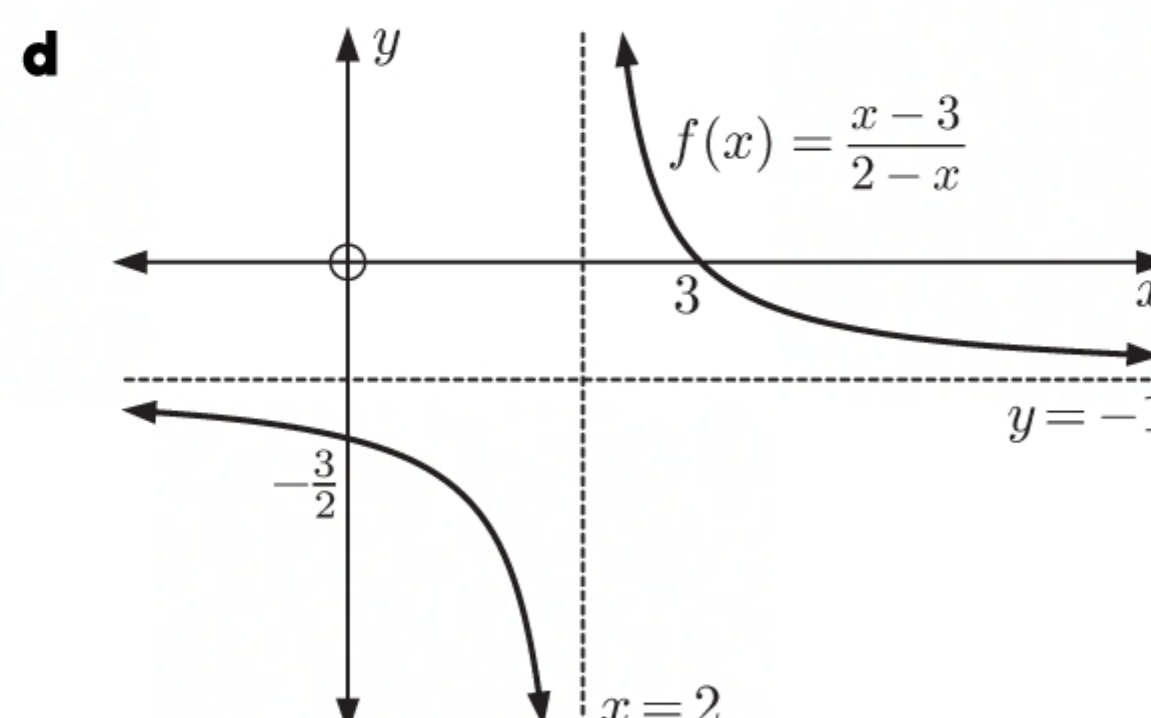
$$f(x) = 0 \quad \text{when} \quad x-3=0$$

$$\therefore x=3$$

\therefore the x -intercept is 3.

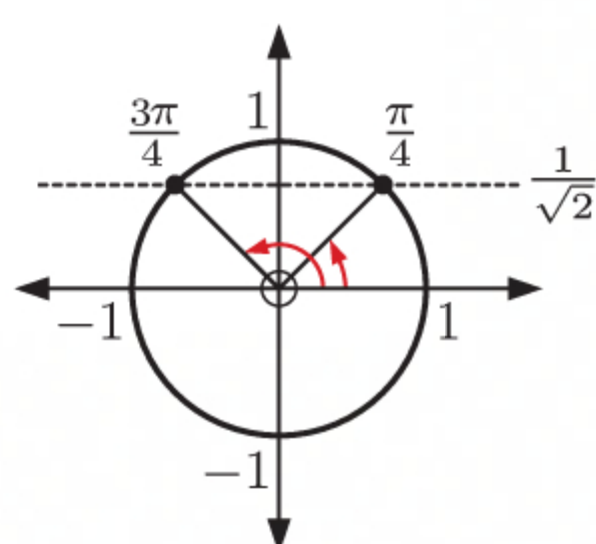
b The vertical asymptote is $x=2$.

The horizontal asymptote is $y=-1$.



3 $\sqrt{2} \sin(2(x - \frac{\pi}{6})) = 1, \quad -\pi \leq x \leq 2\pi$

$$\therefore \sin(2(x - \frac{\pi}{6})) = \frac{1}{\sqrt{2}}$$



There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.

They correspond to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Since $-\pi \leq x \leq 2\pi$

$$\therefore -\frac{7\pi}{6} \leq x - \frac{\pi}{6} \leq \frac{11\pi}{6}$$

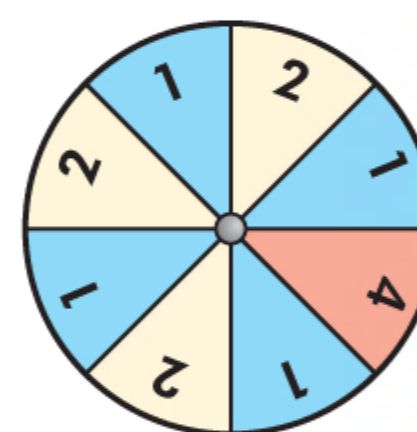
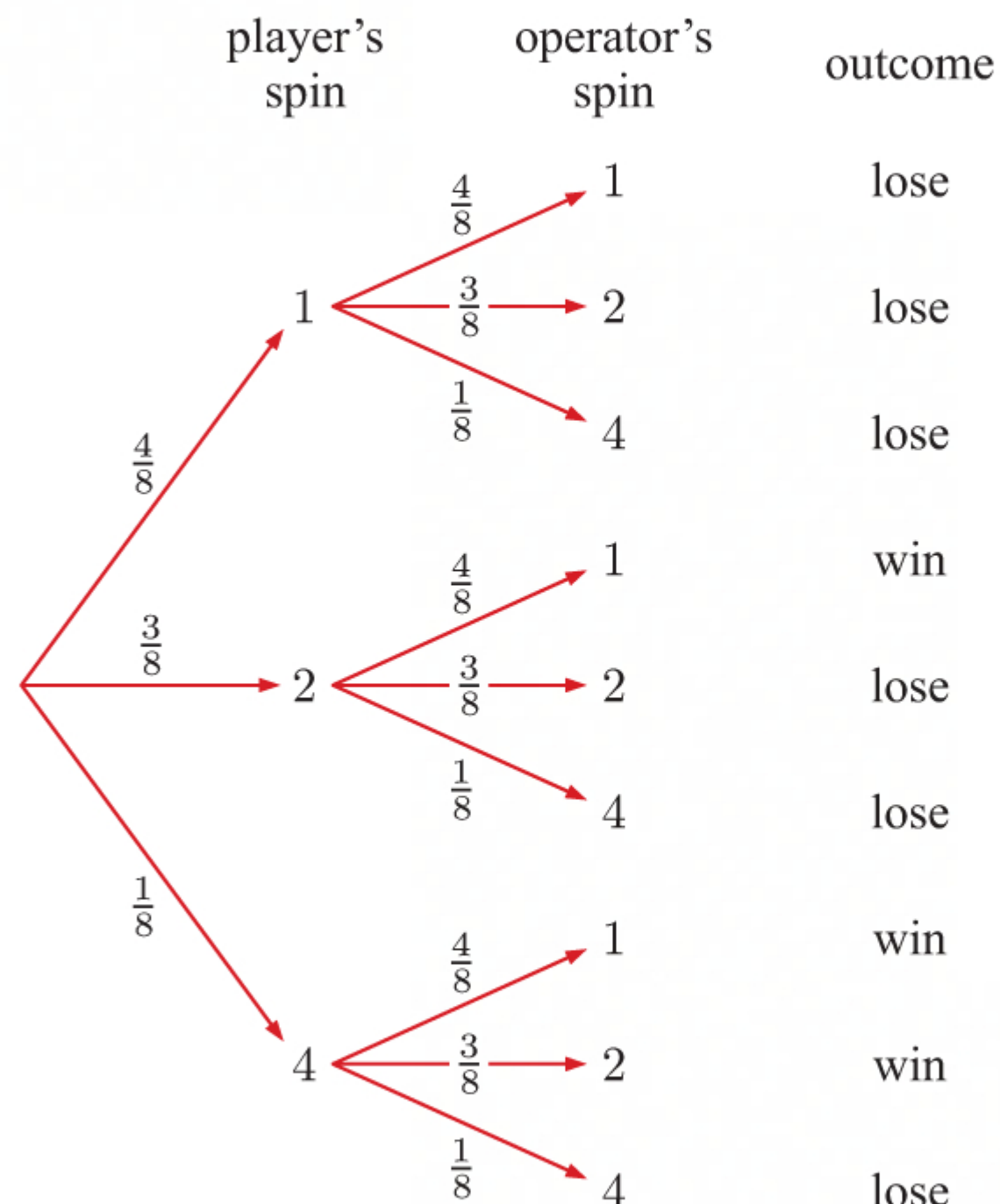
$$\therefore -\frac{7\pi}{3} \leq 2(x - \frac{\pi}{6}) \leq \frac{11\pi}{3}$$

So, $2(x - \frac{\pi}{6}) = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{11\pi}{4}$

$$\therefore x - \frac{\pi}{6} = -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{11\pi}{8}$$

$$\therefore x = -\frac{17\pi}{24}, -\frac{11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{31\pi}{24}, \text{ or } \frac{37\pi}{24}$$

4 We first construct a tree diagram of the possible outcomes.



The player wins if their spin is higher than the operator.

$$\therefore P(\text{win}) = \left(\frac{3}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{3}{8}\right)$$

$$= \frac{12}{64} + \frac{4}{64} + \frac{3}{64}$$

$$= \frac{19}{64}$$

Outcome	Win	Lose
Winnings	\$a	\$0
Probability	$\frac{19}{64}$	$\frac{41}{64}$

Let X denote the return from one game.

$$E(X) = (a \times \frac{19}{64}) + (0 \times \frac{41}{64})$$

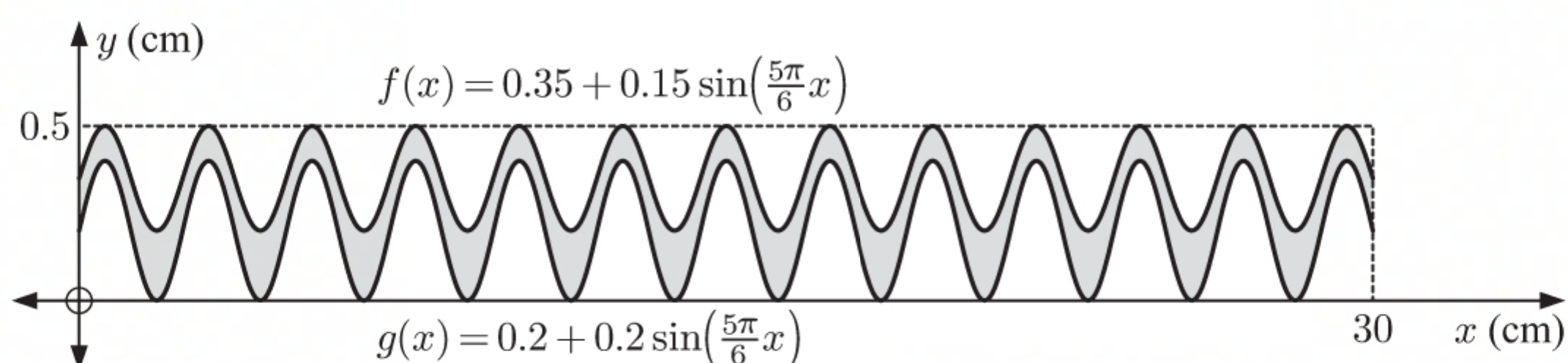
$$= \frac{19a}{64} \text{ dollars}$$

It costs $\$k$ to play a game, so the expected gain $= \frac{19a}{64} - k$ dollars.

The game is fair when the expected gain is 0.

$$\therefore \frac{19a}{64} - k = 0 \quad \text{or} \quad 19a = 64k$$

5

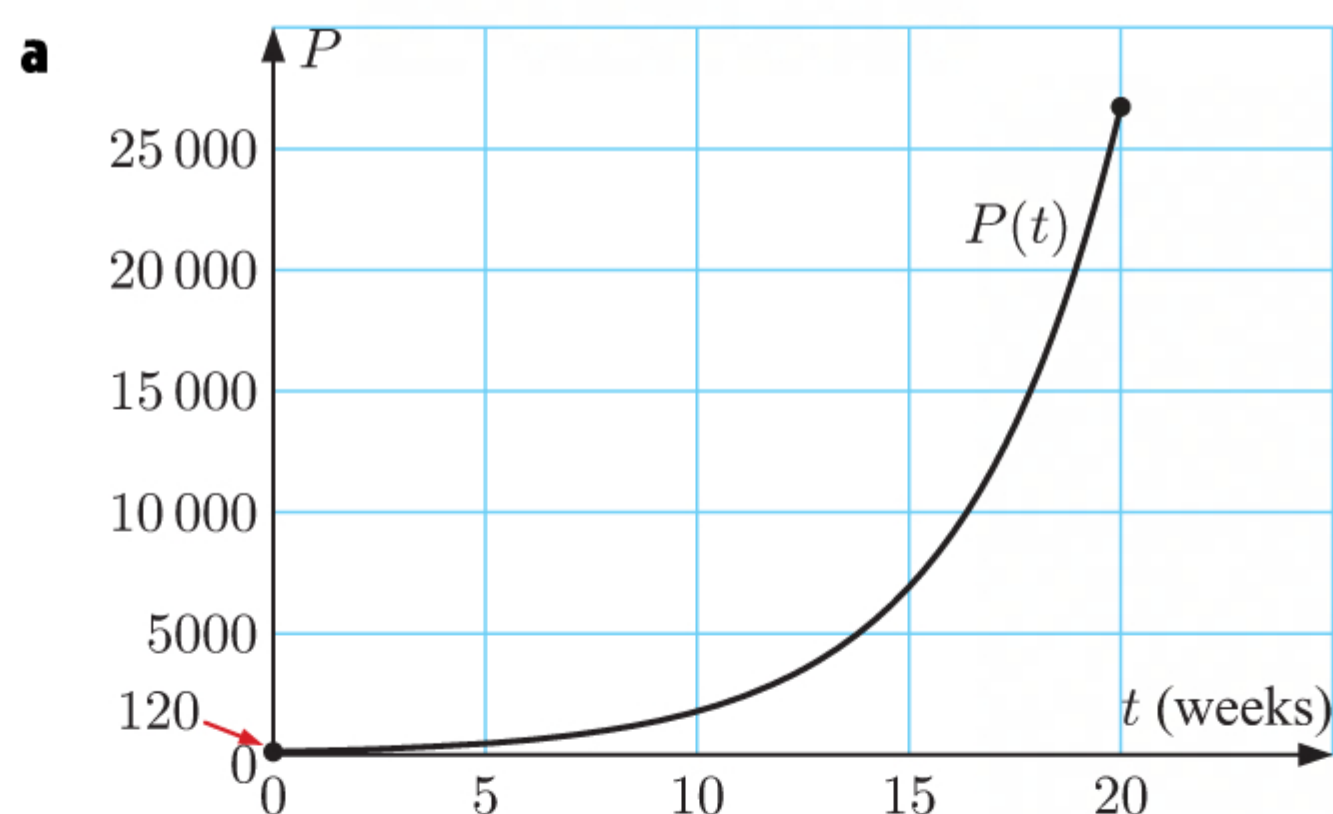


$$\begin{aligned}
 \text{Cross-sectional area} &= \int_0^{30} (f(x) - g(x)) dx \\
 &= \int_0^{30} \left((0.35 + 0.15 \sin(\frac{5\pi}{6}x)) - (0.2 + 0.2 \sin(\frac{5\pi}{6}x)) \right) dx \\
 &= \int_0^{30} (0.15 - 0.05 \sin(\frac{5\pi}{6}x)) dx \\
 &= \left[0.15x + 0.05\left(\frac{6}{5\pi}\right) \cos(\frac{5\pi}{6}x) \right]_0^{30} \\
 &= (0.15 \times 30 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 30)) - (0.15 \times 0 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 0)) \\
 &= 4.5 + \frac{3}{50\pi} \cos(25\pi) - \frac{3}{50\pi} \\
 &= 4.5 + \frac{3}{50\pi} \cos \pi - \frac{3}{50\pi} \quad \{\cos \theta = \cos(\theta + 2\pi)\} \\
 &= 4.5 + \frac{3}{50\pi}(-1) - \frac{3}{50\pi} \\
 &= 4.5 - \frac{3}{25\pi} \text{ cm}^2
 \end{aligned}$$

So, volume = cross-sectional area \times length

$$\begin{aligned}
 &= (4.5 - \frac{3}{25\pi}) \times 100 \quad \{1 \text{ m} \equiv 100 \text{ cm}\} \\
 &\approx 446 \text{ cm}^3
 \end{aligned}$$

6 $P(t) = 120 \times (2.25)^{\frac{t}{3}}$



b $P(10) = 120 \times (2.25)^{\frac{10}{3}}$
 ≈ 1790

\therefore the population of bees in the hive is about 1790 after 10 weeks.

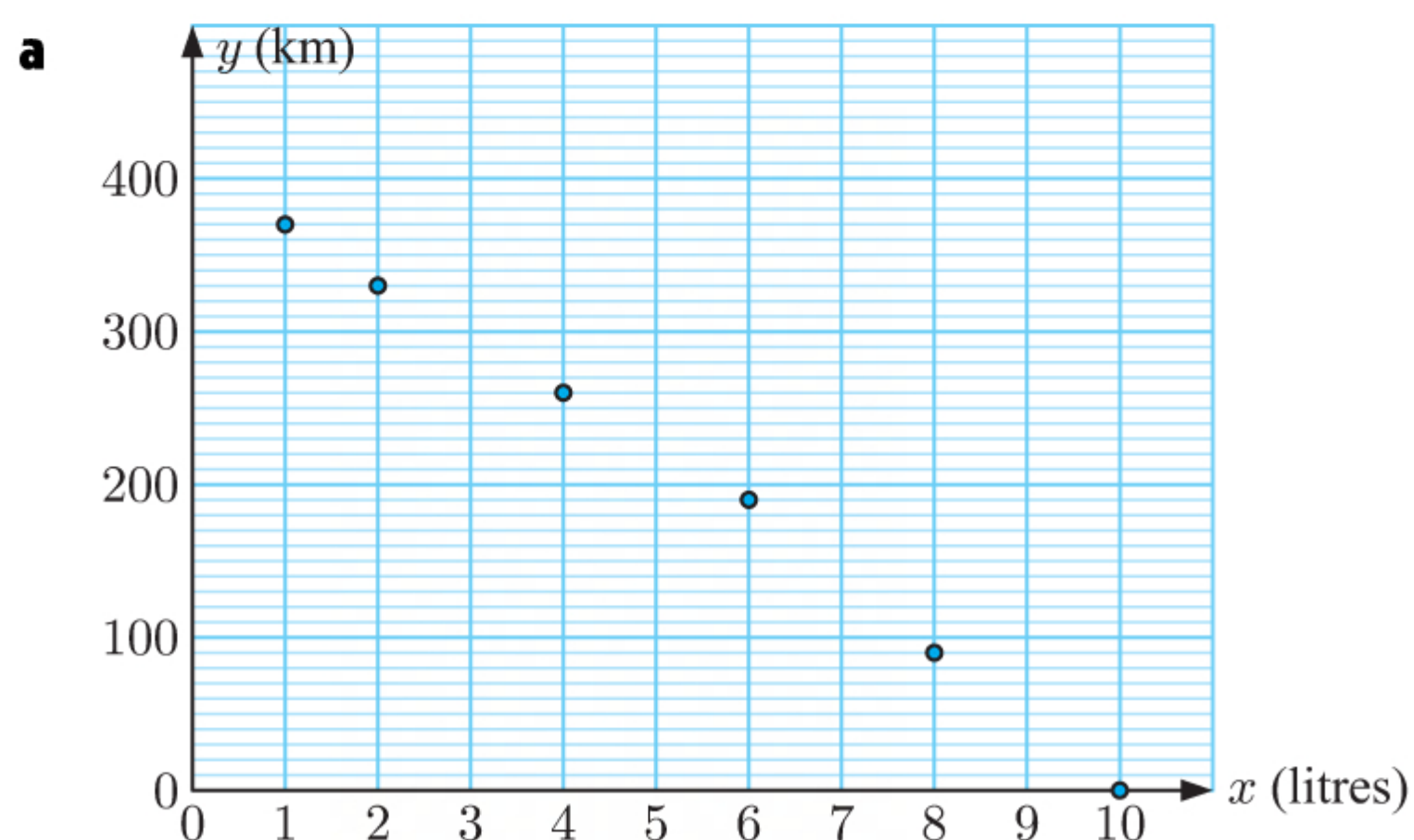
c $P = 120 \times (2.25)^{\frac{t}{3}}$
 $\therefore \frac{P}{120} = (2.25)^{\frac{t}{3}}$
 $\therefore \ln\left(\frac{P}{120}\right) = \frac{t}{3} \ln(2.25)$
 $\therefore t = \frac{3 \ln\left(\frac{P}{120}\right)}{\ln(2.25)}$

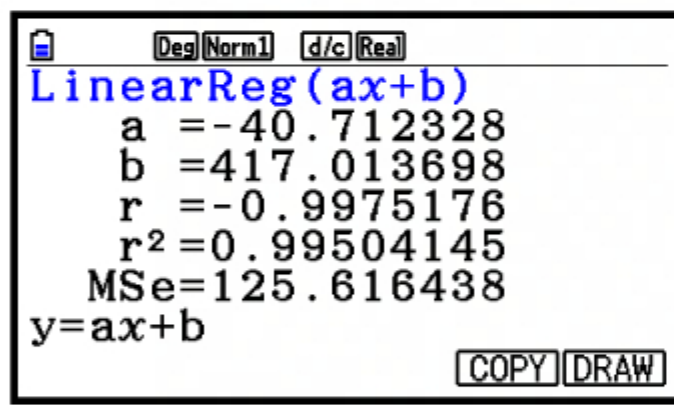
d When $P = 5000$, $t = \frac{3 \ln\left(\frac{5000}{120}\right)}{\ln(2.25)}$
 $= \frac{3 \ln\left(\frac{125}{3}\right)}{\ln(2.25)}$
 ≈ 13.8

\therefore it will take about 13.8 weeks for the population to reach 5000.

7

Remaining fuel (x litres)	10	8	6	4	2	1
Distance (y km)	0	90	190	260	330	370



bUsing technology, the regression line is $y \approx -40.7x + 417$.**c** The y -intercept of the regression line ≈ 417 . This indicates that the motorbike can travel about 417 km on a full tank of petrol.**d i** When $y = 220$, $220 \approx -40.7x + 417$

$$\therefore -197 \approx -40.7x$$

$$\therefore x \approx 4.84$$

 \therefore there is about 4.84 litres of fuel left in the tank after the motorbike has travelled 220 km.**ii** Average distance travelled per litre $\approx \frac{220}{10 - 4.84} \approx 42.6$ km per litre.**8** $a(t) = 2 - 6t \text{ m s}^{-2}$, $t \geq 0$

$$\begin{aligned} \text{a } v(t) &= \int a(t) dt \\ &= \int (2 - 6t) dt \\ &= 2t - 3t^2 + c \end{aligned}$$

The particle is initially at rest, so $v(0) = 0$

$$\therefore 2(0) - 3(0)^2 + c = 0$$

$$\therefore c = 0$$

$$\therefore v(t) = 2t - 3t^2, \quad t \geq 0$$

c The particle changes direction where $v(t) = 0$

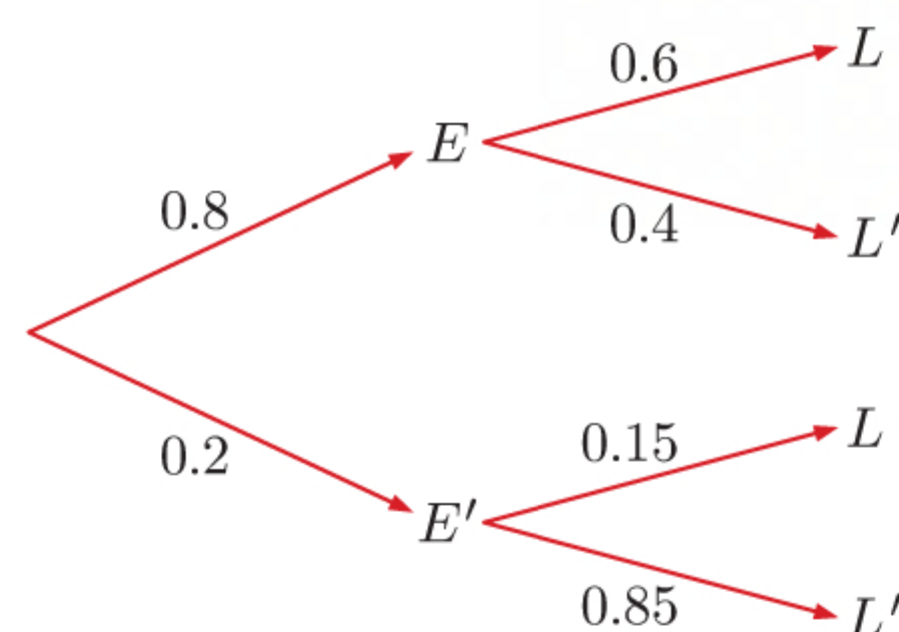
$$\therefore 2t - 3t^2 = 0$$

$$\therefore t(2 - 3t) = 0$$

$$\therefore t = 0 \text{ or } t = \frac{2}{3}$$

So, total distance travelled in first second

$$\begin{aligned} &= \int_0^1 |v(t)| dt \\ &= \int_0^{\frac{2}{3}} |2t - 3t^2| dt + \int_{\frac{2}{3}}^1 |2t - 3t^2| dt \\ &= \int_0^{\frac{2}{3}} (2t - 3t^2) dt + \int_{\frac{2}{3}}^1 (3t^2 - 2t) dt \\ &= \left[t^2 - t^3 \right]_0^{\frac{2}{3}} + \left[t^3 - t^2 \right]_{\frac{2}{3}}^1 \\ &= \left[\left(\left(\frac{2}{3} \right)^2 - \left(\frac{2}{3} \right)^3 \right) - 0 \right] + \left[(1 - 1) - \left(\left(\frac{2}{3} \right)^3 - \left(\frac{2}{3} \right)^2 \right) \right] \\ &= \frac{4}{9} - \frac{8}{27} - \frac{8}{27} + \frac{4}{9} \\ &= \frac{8}{27} \approx 0.296 \text{ m} \end{aligned}$$

9 a Let E be the event that Mark wakes up early, and L be the event that Mark packs his lunch.

$$\begin{aligned} \text{b } P(L) &= P(E \cap L) + P(E' \cap L) \\ &= 0.8 \times 0.6 + 0.2 \times 0.15 \\ &= 0.51 \end{aligned}$$

- 10 a** In $\triangle ABC$, by the cosine rule:

$$\begin{aligned} BC^2 &= 65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ \\ \therefore BC &= \sqrt{65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ} \quad \{\text{as } BC > 0\} \\ \therefore BC &= 91 \text{ m} \end{aligned}$$

b Area of $\triangle ABC = \frac{1}{2} \times AB \times AC \times \sin \widehat{BAC}$

$$\begin{aligned} &= \frac{1}{2} \times 65 \times 104 \times \sin 60^\circ \\ &= 65 \times 52 \times \frac{\sqrt{3}}{2} \\ &= 1690\sqrt{3} \text{ m}^2 \\ &\approx 2930 \text{ m}^2 \end{aligned}$$

\therefore the total area of the field is about 2930 m^2 .

c Area of $A_1 = \frac{1}{2} \times AB \times AD \times \sin \widehat{BAD}$

$$\begin{aligned} &= \frac{1}{2} \times 65 \times x \times \sin 30^\circ \\ &= \frac{65x}{2} \times \frac{1}{2} \\ &= \frac{65x}{4} \text{ m}^2 \end{aligned}$$

Area of $A_2 = \frac{1}{2} \times AC \times AD \times \sin \widehat{CAD}$

$$\begin{aligned} &= \frac{1}{2} \times 104 \times x \times \sin 30^\circ \\ &= 52x \times \frac{1}{2} \\ &= 26x \text{ m}^2 \end{aligned}$$

Now, the total area of the field $= A_1 + A_2$

$$\therefore 1690\sqrt{3} = \frac{65x}{4} + 26x \quad \{\text{from a}\}$$

$$\therefore 1690\sqrt{3} = x\left(\frac{65}{4} + 26\right)$$

$$\therefore x = \frac{1690\sqrt{3}}{\frac{65}{4} + 26}$$

$$\therefore x \approx 69.3$$

- 11** Since p , q , and r are consecutive odd integers with $p < q < r$, $p = q - 2$ and $r = q + 2$.

So, $2q(p + r) = 2q((q - 2) + (q + 2))$

$$\begin{aligned} &= 2q(2q) \\ &= (2q)^2 \quad \text{which is a perfect square} \end{aligned}$$

- 12** $y = xe^{2x}$

a $\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

$$= e^{2x}(1 + 2x)$$

The tangent to the curve is horizontal where $\frac{dy}{dx} = 0$

$$\therefore e^{2x}(1 + 2x) = 0$$

$$\therefore x = -\frac{1}{2} \quad \{e^{2x} > 0\}$$

When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})e^{2(-\frac{1}{2})} = -\frac{1}{2e}$.

$\therefore y = k$ is a horizontal tangent to the curve when $k = -\frac{1}{2e}$.

b When $y = 0$, $xe^{2x} = 0$

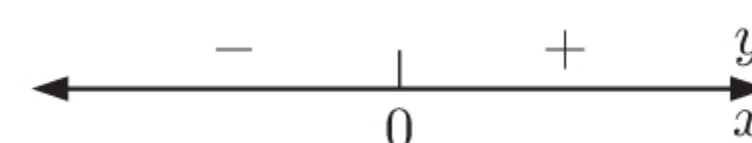
$$\therefore x = 0 \quad \{e^{2x} > 0\}$$

When $\frac{dy}{dx} = 0$, $x = -\frac{1}{2}$ {from a}

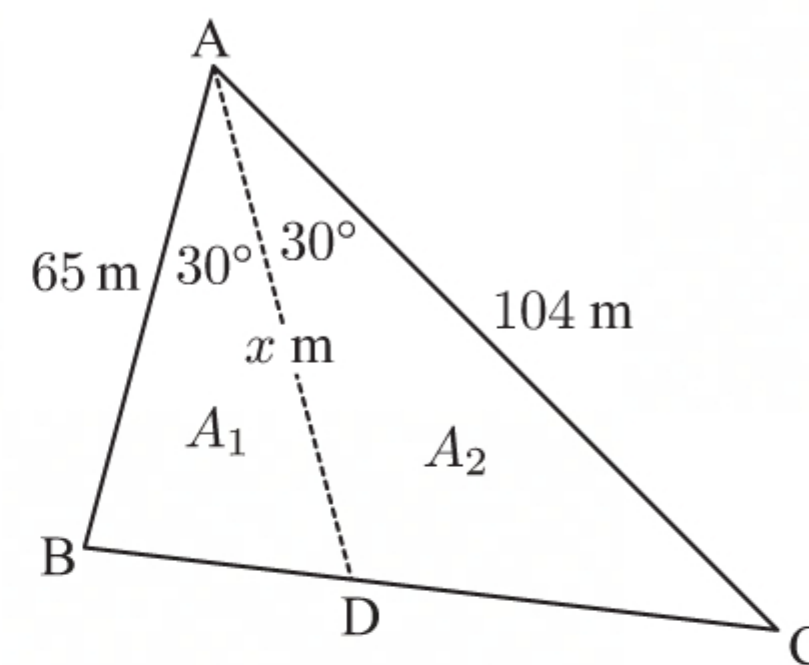
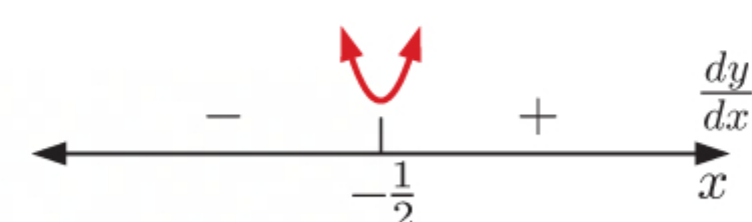
Now $\frac{d^2y}{dx^2} = 2e^{2x}(1 + 2x) + 2e^{2x}$

$$\begin{aligned} &= 2e^{2x}(2 + 2x) \\ &= 4e^{2x}(1 + x) \end{aligned}$$

So, y has sign diagram:



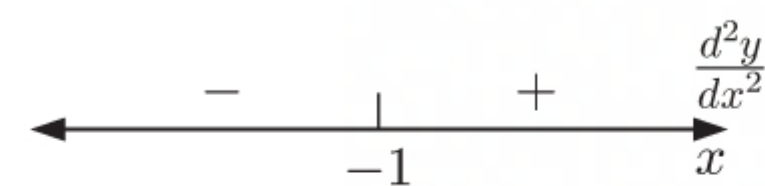
So, $\frac{dy}{dx}$ has sign diagram:



When $\frac{d^2y}{dx^2} = 0$, $4e^{2x}(1+x) = 0$

$$\therefore x = -1 \quad \{e^{2x} > 0\}$$

So, $\frac{d^2y}{dx^2}$ has sign diagram:

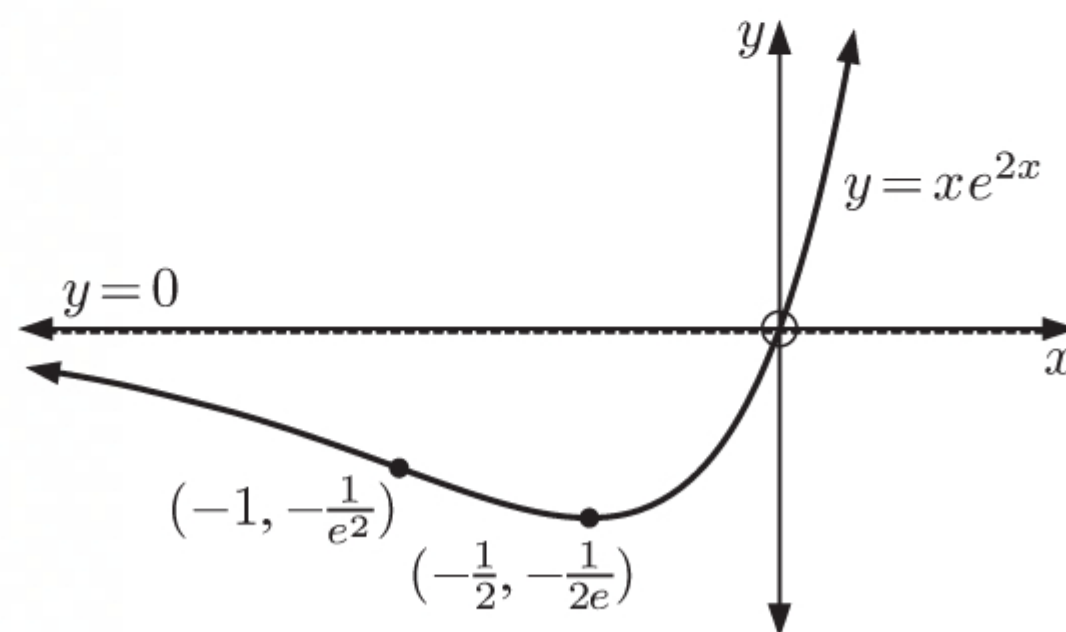


When $x = -1$, $y = (-1)e^{2(-1)} = -\frac{1}{e^2}$.

\therefore there is a non-stationary point of inflection at $\left(-1, -\frac{1}{e^2}\right)$.

So, as $x \rightarrow \infty$, $y = xe^{2x} \rightarrow \infty$

and as $x \rightarrow -\infty$, $y = xe^{2x} \rightarrow 0^-$.



i $y = k$ meets the curve at exactly one point for $k = -\frac{1}{2e}$ or $k \geq 0$.

ii $y = k$ meets the curve at two distinct points for $-\frac{1}{2e} < k < 0$.

iii $y = k$ meets the curve at no points for $k < -\frac{1}{2e}$.

c $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$

i $y = x$ meets the curve $y = xe^{ax}$ where $xe^{ax} = x$

$$\therefore xe^{ax} - x = 0$$

$$\therefore x(e^{ax} - 1) = 0$$

$$\therefore x = 0 \quad \text{or} \quad e^{ax} = 1$$

$$\therefore x = 0 \quad \text{or} \quad ax = 0$$

$$\therefore x = 0 \quad \{a > 0\}$$

Now $y = xe^{ax}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= e^{ax} + axe^{ax} \\ &= e^{ax}(1 + ax) \end{aligned}$$

When $x = 0$, $y = 0e^0 = 0$

and $\frac{dy}{dx} = e^0(1 + 0) = 1$

\therefore the tangent to the curve at $x = 0$ has gradient 1, and the point of contact is $(0, 0)$.

\therefore the equation of the tangent is $y - 0 = 1(x - 0)$

$\therefore y = x$ as required.

ii From **i**, the tangent to the curve at $y = xe^{ax}$ when $x = 0$ is $y = x$.

\therefore the normal to the curve at $x = 0$ has gradient -1 , and the point of contact is $(0, 0)$.

\therefore the equation of the normal is $y - 0 = -1(x - 0)$

$\therefore y = -x$.

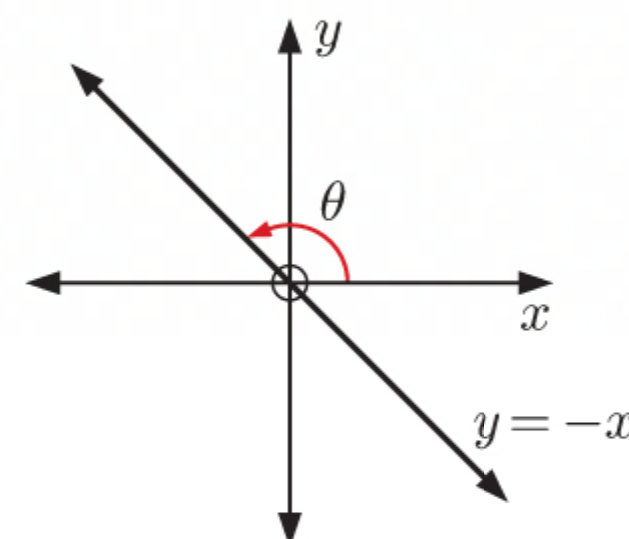
Let θ be the angle the normal makes with the positive x -axis.

The normal has gradient -1 .

$$\therefore \tan \theta = -1$$

$$\therefore \theta = \frac{3\pi}{4} \quad \{0 < \theta < \pi\}$$

So, the acute angle the normal makes with the x -axis is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.



MIXED QUESTIONS SET 4

$$\begin{aligned}
 \mathbf{1} \quad & f \text{ is } y = 4x - 3 & g \text{ is } y = x + 2 \\
 & \therefore f^{-1} \text{ is } x = 4y - 3 & \therefore g^{-1} \text{ is } x = y + 2 \\
 & \therefore 4y = x + 3 & \therefore y = x - 2 \\
 & \therefore y = \frac{1}{4}x + \frac{3}{4} & \therefore g^{-1}(x) = x - 2 \\
 & \therefore f^{-1}(x) = \frac{1}{4}x + \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } (f \circ g^{-1})(x) &= f^{-1}(x) \text{ where } f(g^{-1}(x)) = \frac{1}{4}x + \frac{3}{4} \\
 &\therefore f(x - 2) = \frac{1}{4}x + \frac{3}{4} \\
 &\therefore 4(x - 2) - 3 = \frac{1}{4}x + \frac{3}{4} \\
 &\therefore 4x - 11 = \frac{1}{4}x + \frac{3}{4} \\
 &\therefore 16x - 44 = x + 3 \\
 &\therefore 15x = 47 \\
 &\therefore x = \frac{47}{15}
 \end{aligned}$$

$$\mathbf{2} \quad f'(x) = a\sqrt{x} + bx, \text{ where } a \text{ and } b \text{ are constants}$$

$$\begin{aligned}
 \therefore f(x) &= \int (ax^{\frac{1}{2}} + bx) dx \\
 &= \frac{2}{3}ax^{\frac{3}{2}} + \frac{1}{2}bx^2 + c \\
 f(0) &= -4, \text{ so } c = -4
 \end{aligned}$$

$$\text{Thus } f(x) = \frac{2}{3}ax^{\frac{3}{2}} + \frac{1}{2}bx^2 - 4 \text{ where } f(1) = -1 \text{ and } f(2) = 4\sqrt{2}.$$

$$\begin{aligned}
 \text{So, } \frac{2}{3}a(1)^{\frac{3}{2}} + \frac{1}{2}b(1)^2 - 4 &= -1 & \text{and } \frac{2}{3}a(2)^{\frac{3}{2}} + \frac{1}{2}b(2)^2 - 4 &= 4\sqrt{2} \\
 \therefore \frac{2}{3}a + \frac{1}{2}b &= 3 & \therefore \frac{4\sqrt{2}}{3}a + 2b &= 4\sqrt{2} + 4 \\
 \therefore \frac{1}{2}b &= 3 - \frac{2}{3}a & \therefore \frac{4\sqrt{2}}{3}a + 2(6 - \frac{4}{3}a) &= 4\sqrt{2} + 4 \quad \{\text{using } (*)\} \\
 \therefore b &= 6 - \frac{4}{3}a \quad \dots (*) & \therefore \frac{4\sqrt{2}}{3}a + 12 - \frac{8}{3}a &= 4\sqrt{2} + 4 \\
 & & \therefore \frac{4\sqrt{2} - 8}{3}a &= 4\sqrt{2} - 8 \\
 & & \therefore a &= 3
 \end{aligned}$$

$$\text{Substituting } a = 3 \text{ into } (*) \text{ gives } b = 6 - \frac{4}{3}(3) = 2.$$

$$\begin{aligned}
 \text{So, } f(x) &= \frac{2}{3}(3)x^{\frac{3}{2}} + \frac{1}{2}(2)x^2 - 4 \\
 \therefore f(x) &= 2x^{\frac{3}{2}} + x^2 - 4
 \end{aligned}$$

$$\begin{array}{l}
 \mathbf{3} \quad \text{Neighbourhood A:} \quad \begin{array}{cccccccccc} 275 & 281 & 320 & 265 & 305 & 258 & 310 & 430 & 285 \\ & 290 & 297 & 345 & 195 & 230 & 269 & 300 & 258 & 273 \end{array} \\
 \text{Neighbourhood B:} \quad \begin{array}{cccccccccc} 325 & 300 & 412 & 370 & 297 & 505 & 340 & 333 & 290 \\ & 428 & 305 & 520 & 360 & 410 & 275 & 320 & 431 & 410 \end{array}
 \end{array}$$

a The sale price of a house can be counted, so it is a discrete variable.

b Neighbourhood A:

	Des	Norm1	d/c	Real
1-Variable				
n	=	18		
minX	=	195		
Q1	=	265		
Med	=	283		
Q3	=	305		
maxX	=	430		

minimum = \$195 000
 Q_1 = \$265 000
 median = \$283 000
 Q_3 = \$305 000
 maximum = \$430 000

Neighbourhood B:

	Des	Norm1	d/c	Real
1-Variable				
n	=	18		
minX	=	275		
Q1	=	305		
Med	=	350		
Q3	=	412		
maxX	=	520		

minimum = \$275 000
 Q_1 = \$305 000
 median = \$350 000
 Q_3 = \$412 000
 maximum = \$520 000

- c** For Neighbourhood A, $IQR = 305\,000 - 265\,000 = 40\,000$

$$\begin{array}{ll}
 \text{Test for outliers:} & \text{upper boundary} \qquad \qquad \text{and} \qquad \text{lower boundary} \\
 & = \text{upper quartile} + 1.5 \times IQR \qquad \qquad = \text{lower quartile} - 1.5 \times IQR \\
 & = 305\,000 + 1.5 \times 40\,000 \qquad \qquad = 265\,000 - 1.5 \times 40\,000 \\
 & = 365\,000 \qquad \qquad \qquad \qquad \qquad = 205\,000
 \end{array}$$

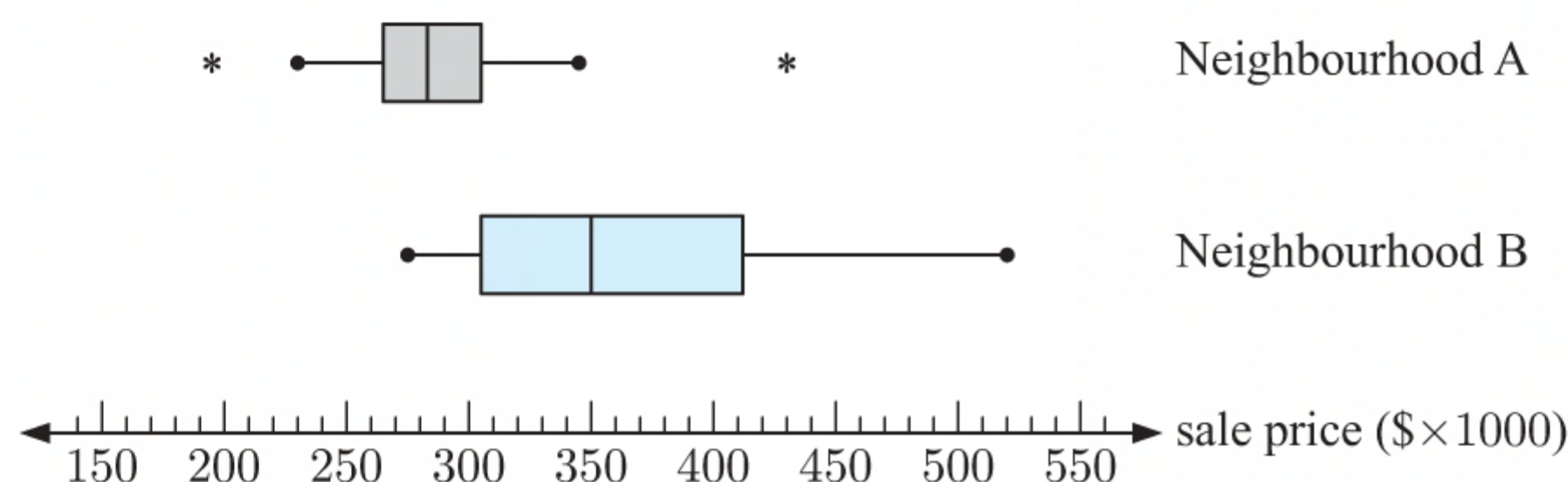
\$430 000 is above the upper boundary, so it is an outlier.

\$195 000 is below the lower boundary, so it is an outlier.

For Neighbourhood B, $IQR = 412\,000 - 305\,000 = 107\,000$

$$\begin{array}{ll}
 \text{Test for outliers:} & \text{upper boundary} \qquad \qquad \text{and} \qquad \text{lower boundary} \\
 & = \text{upper quartile} + 1.5 \times IQR \qquad \qquad = \text{lower quartile} - 1.5 \times IQR \\
 & = 412\,000 + 1.5 \times 107\,000 \qquad \qquad = 305\,000 - 1.5 \times 107\,000 \\
 & = 572\,500 \qquad \qquad \qquad \qquad \qquad = 144\,500
 \end{array}$$

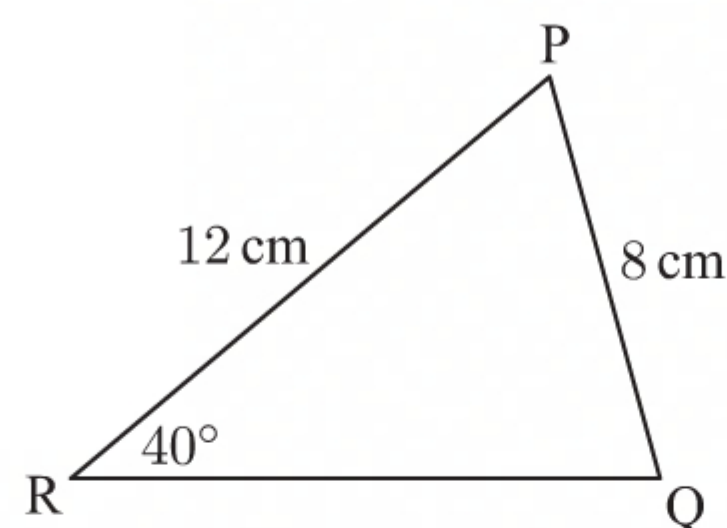
\therefore there are no outliers.



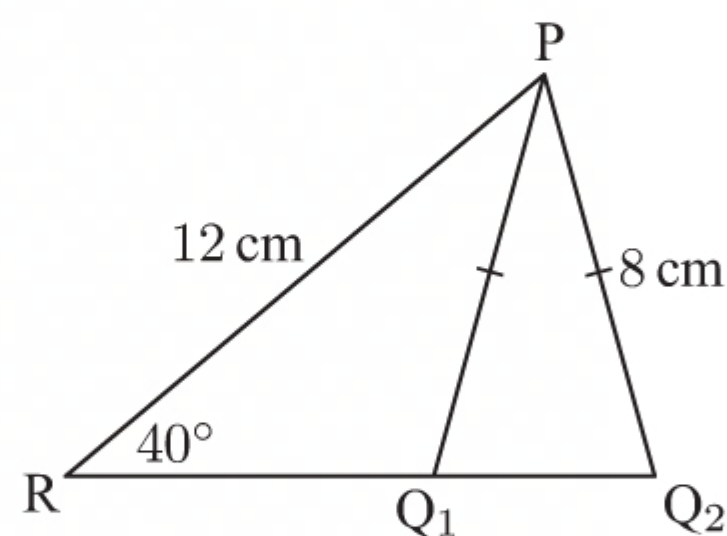
- d** Both sets of data are positively skewed. The sale price of houses in Neighbourhood B are generally higher than those in Neighbourhood A. With the outliers removed, there is more variation in the sale price of houses in Neighbourhood B compared to Neighbourhood A.

- 4 a** Using the sine rule,

$$\begin{aligned}
 \frac{\sin \widehat{PQR}}{12} &= \frac{\sin 40^\circ}{8} \\
 \therefore \sin \widehat{PQR} &= \frac{12 \times \sin 40^\circ}{8} \\
 \therefore \widehat{PQR} &= \sin^{-1} \left(\frac{12 \times \sin 40^\circ}{8} \right) \text{ or its supplement} \\
 \therefore \widehat{PQR} &\approx 74.6^\circ \text{ or } 180^\circ - 74.6^\circ \\
 \therefore \widehat{PQR} &\approx 74.6^\circ \text{ or } 105.4^\circ
 \end{aligned}$$



b



- c** For the case in which $\widehat{PQR} \approx 74.6^\circ$:

i $\widehat{QPR} \approx 180^\circ - 40^\circ - 74.6^\circ$ {angles in a triangle}

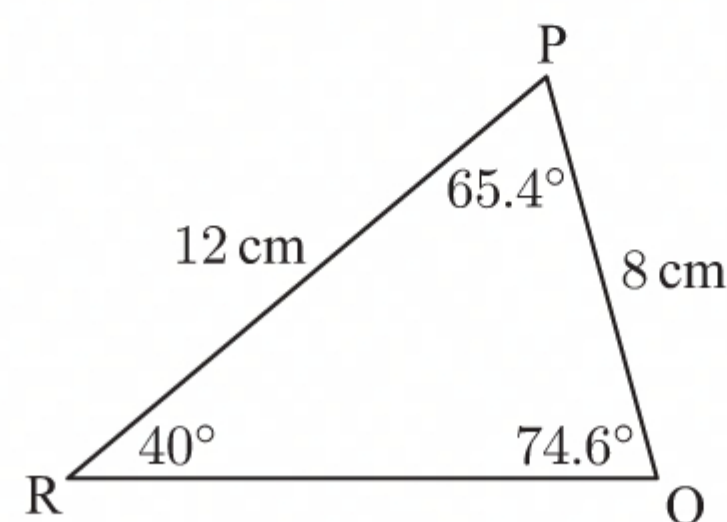
$\therefore \widehat{QPR} \approx 65.4^\circ$

ii $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$ {sine rule}

$\therefore \frac{QR}{\sin 65.4^\circ} \approx \frac{8}{\sin 40^\circ}$

$\therefore QR \approx \frac{8 \times \sin 65.4^\circ}{\sin 40^\circ}$

$\therefore QR \approx 11.3 \text{ cm}$



So, perimeter of $\triangle PQR \approx (12 + 8 + 11.3) \text{ cm}$
 $\approx 31.3 \text{ cm}$

For the case in which $\widehat{PQR} \approx 105.4^\circ$:

i $\widehat{QPR} \approx 180^\circ - 40^\circ - 105.4^\circ$ {angles in a triangle}

$\therefore \widehat{QPR} \approx 34.6^\circ$

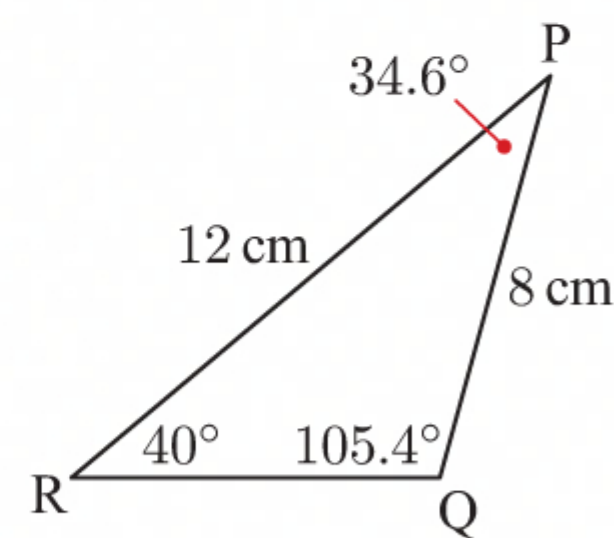
ii $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$ {sine rule}

$\therefore \frac{QR}{\sin 34.6^\circ} \approx \frac{8}{\sin 40^\circ}$

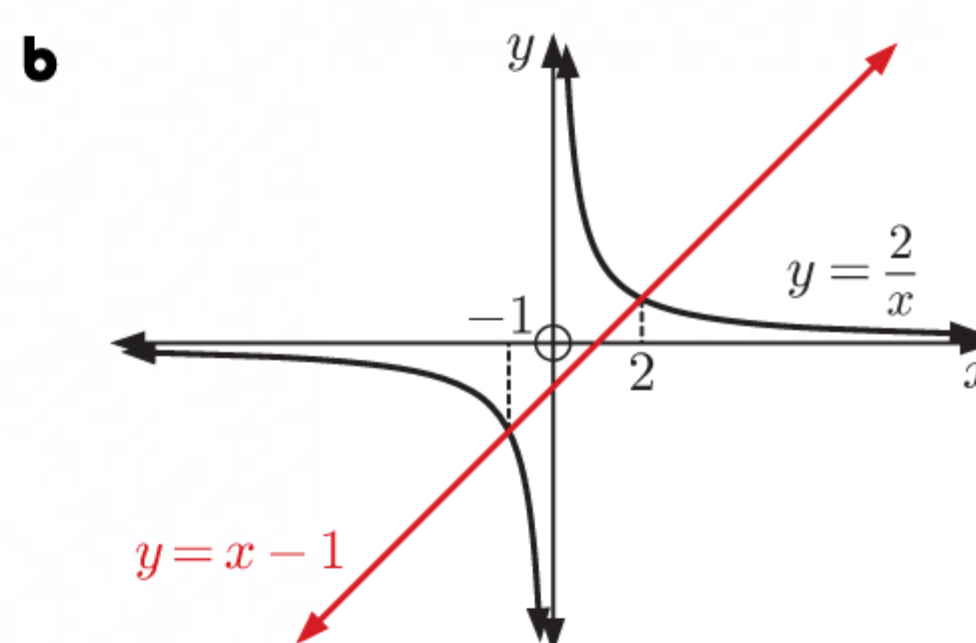
$\therefore QR \approx \frac{8 \times \sin 34.6^\circ}{\sin 40^\circ}$

$\therefore QR \approx 7.07 \text{ cm}$

So, perimeter of $\triangle PQR \approx (12 + 8 + 7.07) \text{ cm}$
 $\approx 27.1 \text{ cm}$



5 a $\frac{2}{x} = x - 1$
 $\therefore 2 = x^2 - x$
 $\therefore x^2 - x - 2 = 0$
 $\therefore (x - 2)(x + 1) = 0$
 $\therefore x = 2 \text{ or } -1$



c If $\frac{2}{x} < x - 1$, the graph of $y = \frac{2}{x}$ is below the graph of $y = x - 1$. This occurs when $-1 < x < 0$ or $x > 2$.

6 $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

So, $1, 3a, 3a^2$ are consecutive terms in an arithmetic sequence.

Since the terms are consecutive, $3a - 1 = 3a^2 - 3a$ {equating differences}

$\therefore 3a^2 - 6a + 1 = 0$

$\therefore a = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 1}}{2 \times 3}$

$\therefore a = \frac{6 \pm \sqrt{36 - 12}}{6}$

$\therefore a = \frac{6 \pm \sqrt{24}}{6}$

$\therefore a = \frac{3 \pm \sqrt{6}}{3}$

7 a i $X \sim N(\mu, (6.8)^2)$

$P(X < 45) = 0.75$

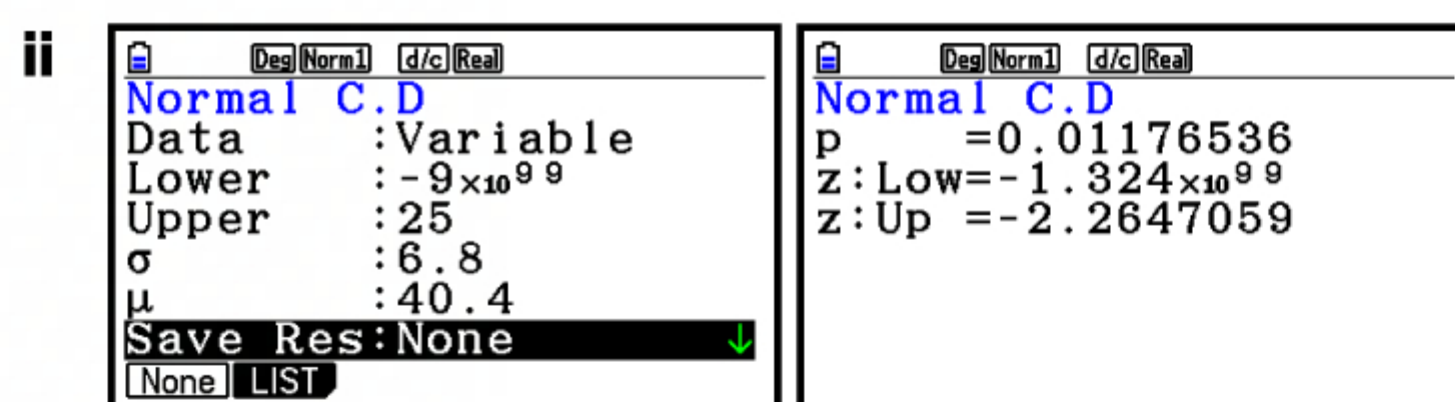
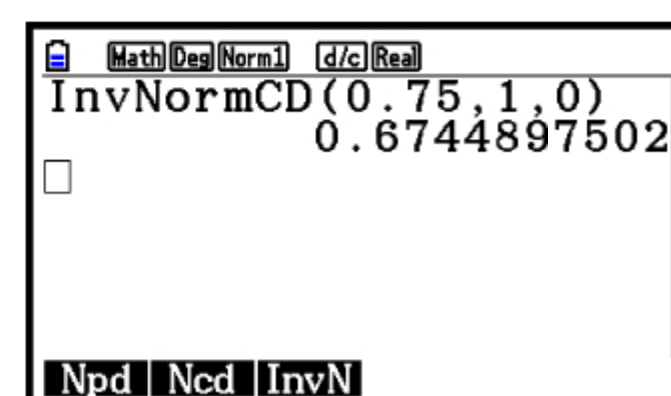
$\therefore P\left(\frac{X - \mu}{6.8} < \frac{45 - \mu}{6.8}\right) = 0.75$

$\therefore P\left(Z < \frac{45 - \mu}{6.8}\right) = 0.75$ $\left\{Z = \frac{X - \mu}{6.8}\right\}$

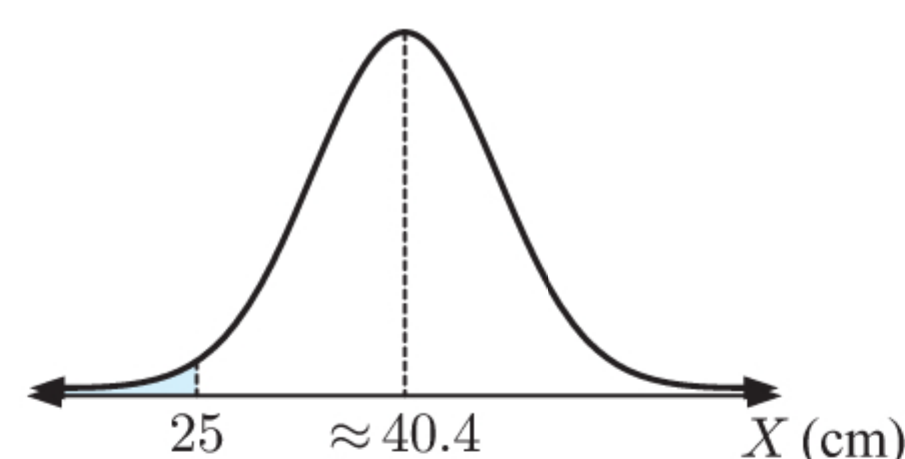
$\therefore \frac{45 - \mu}{6.8} \approx 0.674$ $\{Z \sim N(0, 1^2)\}$

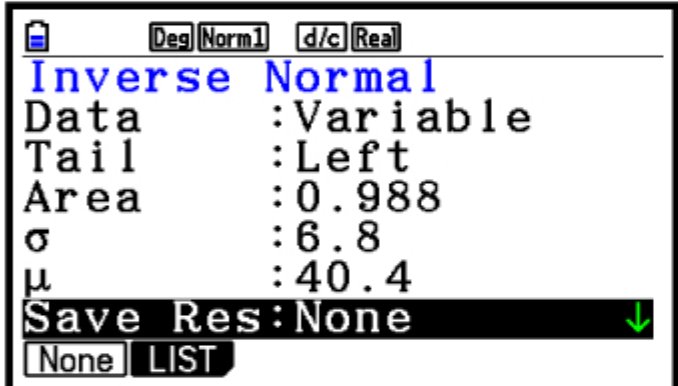
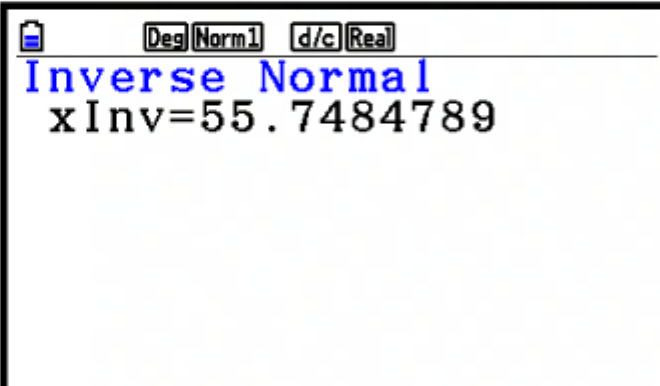
$\therefore 45 - \mu \approx 4.58$

$\therefore \mu \approx 40.4$

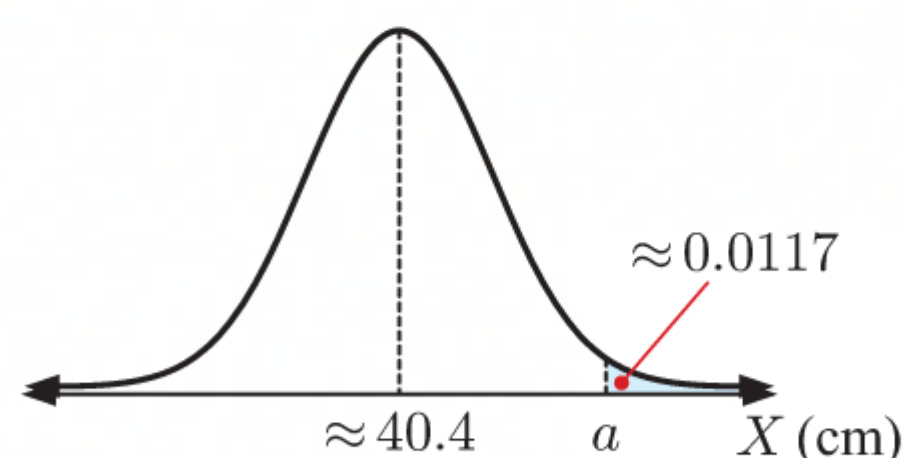


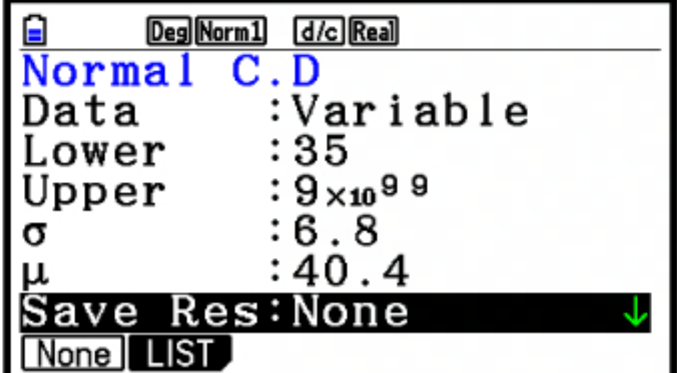
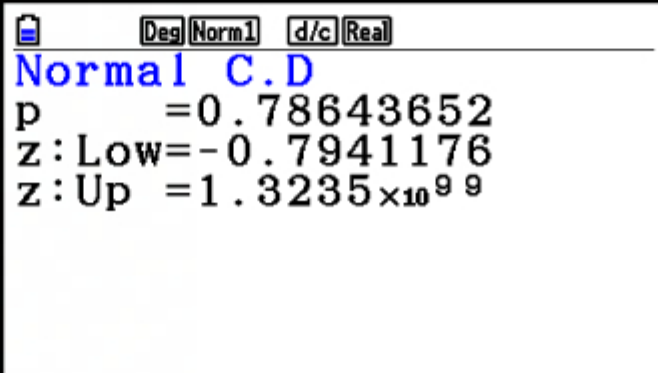
$P(X < 25) \approx 0.0118$



iii		
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If $P(X < 25) = P(X > a)$
 then $P(X > a) \approx 0.0117$ {from ii}
 $\therefore P(X < a) \approx 0.988$
 $\therefore a \approx 55.7$



b		
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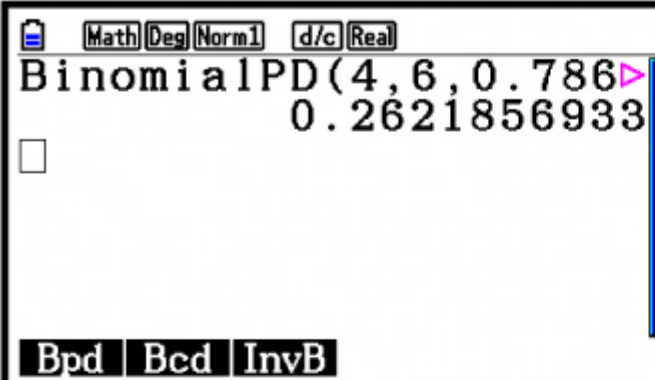
$$P(X > 35) \approx 0.786$$

Let Y be the number of maize plants more than 35 cm high.

$n = 6$, so $Y = 0, 1, 2, 3, 4, 5$, or 6 and $p \approx 0.786$.

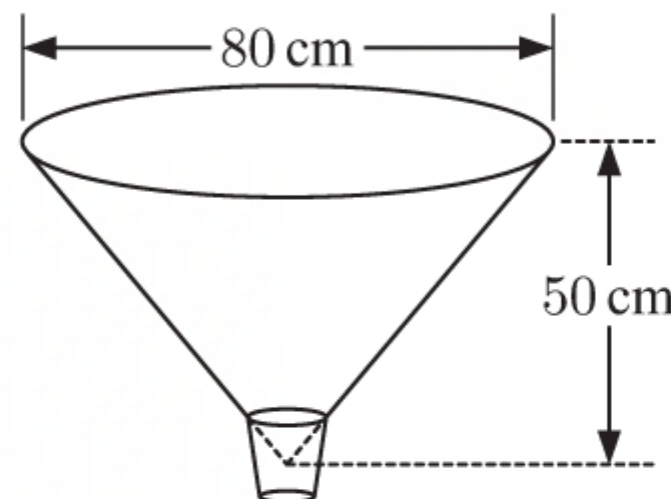
$$Y \sim B(6, 0.786)$$

$$\text{So, } P(Y = 4) \approx \binom{6}{4} (0.786)^4 (1 - 0.786)^2 \approx 0.262$$



8 a $V \approx$ volume of cone

$$\begin{aligned} &\approx \frac{1}{3} \pi r^2 h \\ &\approx \frac{1}{3} \times \pi \times \left(\frac{80}{2}\right)^2 \times 50 \text{ cm}^3 \\ &\approx \frac{80\,000}{3} \pi \text{ cm}^3 \\ &\approx 83\,800 \text{ cm}^3 \end{aligned}$$

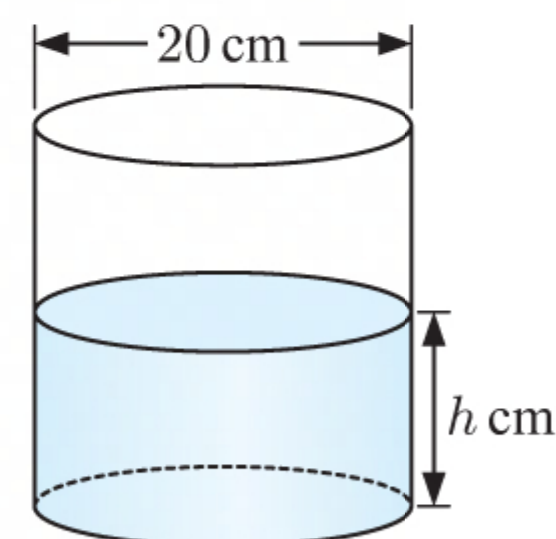


The capacity of the funnel is about 83 800 mL or 8.38×10^4 mL.

b When half full, the funnel contains about $\frac{80\,000}{3} \pi \times 0.5 \approx \frac{40\,000}{3} \pi$ mL of liquid.

$$\begin{aligned} V &\approx \frac{40\,000}{3} \pi \text{ cm}^3 \\ \therefore \pi \times \left(\frac{20}{2}\right)^2 \times h &\approx \frac{40\,000}{3} \pi \\ \therefore h &\approx \frac{40\,000}{3 \times 10^2} \\ \therefore h &\approx 133 \text{ cm} \end{aligned}$$

The liquid will reach about 133 cm up the tube.



9 $a(t) = 6 \cos 2t \text{ cm s}^{-2}$

$$\begin{aligned} \text{a } v(t) &= \int 6 \cos 2t \, dt \\ &= 3 \sin 2t + c \end{aligned}$$

But $v(0) = 0$, so $0 = 3 \sin 0 + c$
 $\therefore c = 0$

Thus, $v(t) = 3 \sin 2t \text{ cm s}^{-1}$.

Now $v(4) = 3 \sin(2 \times 4)$
 ≈ 2.97

\therefore the speed of the tip of the pendulum after 4 seconds is about 2.97 cm s^{-1} .

- b** Total distance travelled in first 5 seconds

$$\begin{aligned}
 &= \int_0^5 |v(t)| dt \\
 &= \int_0^5 |3 \sin 2t| dt \\
 &\approx 9.24 \text{ cm} \quad \{\text{using technology}\}
 \end{aligned}$$

Math Rad Norm1 d/c Real
 $\int_0^5 3 \sin(2x) dx$
 9.241392706
 MAT/VCT logab Abs d/dx d²/dx² ▶

- 10 a** Let the cosine model be $H(t) = a \cos(b(t - c)) + d$.

$$\text{Low tide} = 4.7 - 2.4 = 2.3 \text{ m}$$

$$\therefore \text{the mean height} = \frac{2.3 + 4.7}{2} = 3.5 \text{ m, so } d = 3.5.$$

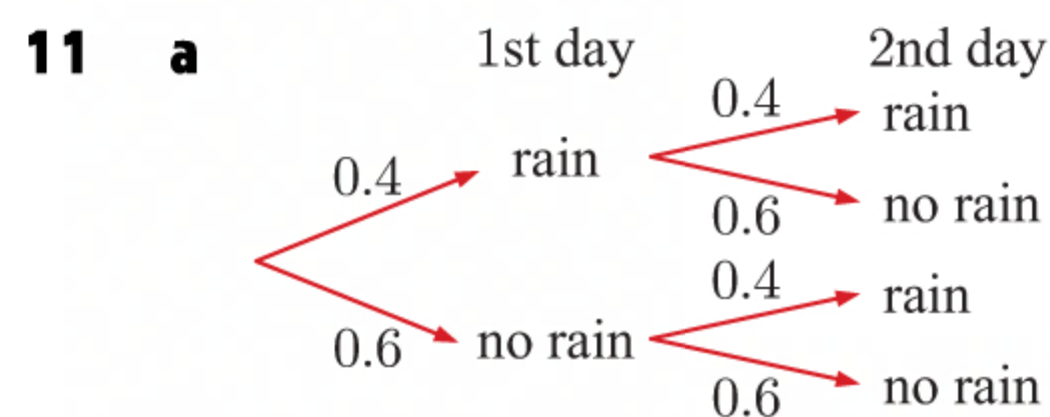
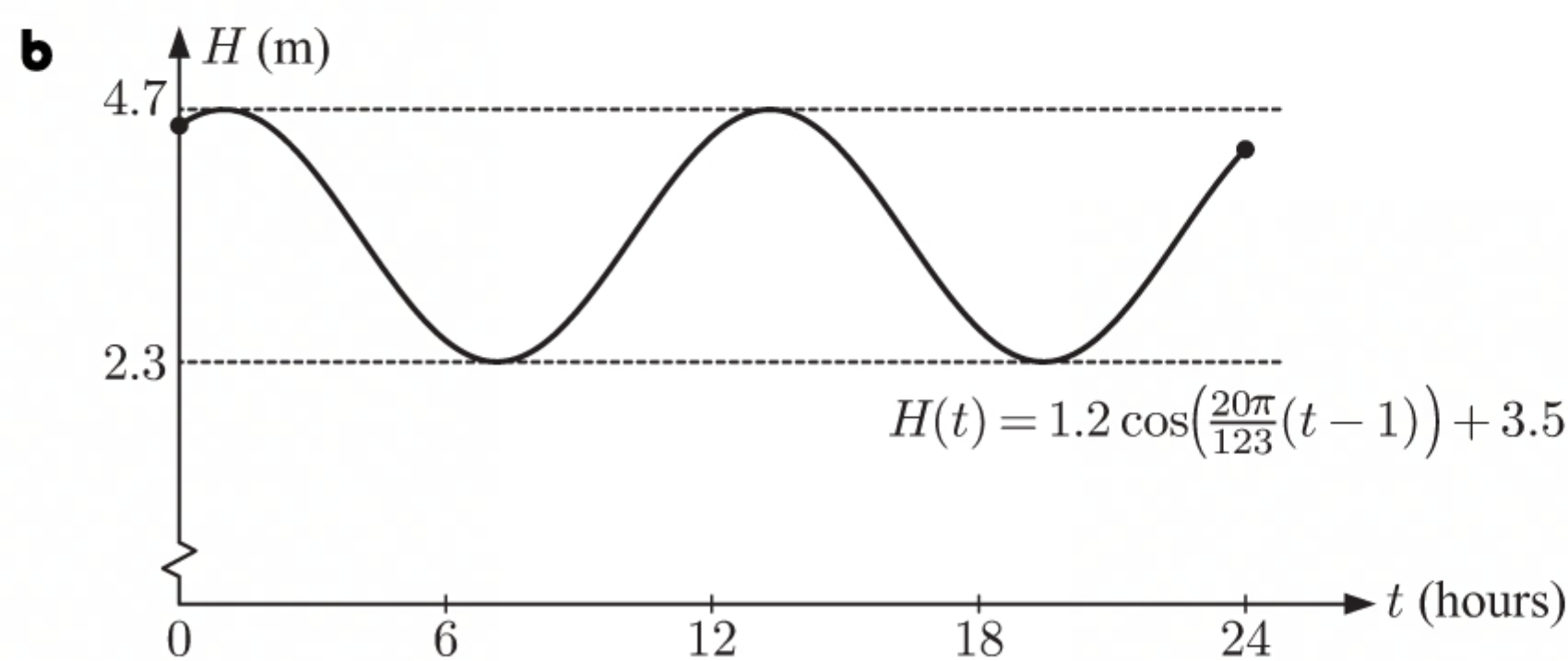
$$\text{The amplitude} = \frac{2.4}{2} = 1.2 \text{ m, so } a = 1.2.$$

$$\text{The period} = 12.3 \text{ hours, so } b = \frac{2\pi}{12.3} = \frac{20\pi}{123}.$$

High tide occurs at 1 am, so the function is shifted 1 hour to the right, thus $c = 1$.

If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 1.2 \cos\left(\frac{20\pi}{123}(t - 1)\right) + 3.5 \text{ m.}$$



b i $P(\text{rain on both days}) = P(\text{rain} \cap \text{rain})$
 $= 0.4 \times 0.4$
 $= 0.16$

ii $P(\text{no rain on one day}) = P(\text{rain} \cap \text{no rain}) + P(\text{no rain} \cap \text{rain})$
 $= 0.4 \times 0.6 + 0.6 \times 0.4$
 $= 0.48$

c $P(\text{no rain on both days}) = P(\text{no rain} \cap \text{no rain})$
 $= 0.6 \times 0.6$
 $= 0.36$

$$\therefore P(\text{rain on at least one day}) = 1 - 0.36$$

$$= 0.64$$

$$\begin{aligned}
 \text{So, } P(\text{rain on 2nd day} \mid \text{rain on at least one day}) &= \frac{P(\text{rain on 2nd day} \cap \text{rain on at least one day})}{P(\text{rain on at least one day})} \\
 &= \frac{P(\text{rain on 2nd day})}{P(\text{rain on at least one day})} \\
 &= \frac{0.4}{0.64} \\
 &= 0.625
 \end{aligned}$$

12 a $f(0) = (0)e^{-0} = 0$

The y -intercept is 0.

b $f(x) = xe^{-x}$
 $\therefore f'(x) = e^{-x} - xe^{-x}$

Let a be the x -coordinate of A, so A is $(a, f(a))$.

Since A is a stationary point, $f'(a) = 0$

$$\therefore 0 = e^{-a} - ae^{-a}$$

$$\therefore e^{-a}(1 - a) = 0$$

$$\therefore a = 1 \quad \{e^{-a} > 0\}$$

Now $f(1) = (1)e^{-1} = \frac{1}{e}$, so A is $(1, \frac{1}{e})$.

c $f'(x) = e^{-x} - xe^{-x}$ {from **b**}

$$\therefore f''(x) = -e^{-x} - e^{-x} + xe^{-x}$$

$$= xe^{-x} - 2e^{-x}$$

Let b be the x -coordinate of B.

Since B is a point of inflection, $f''(b) = 0$

$$\therefore 0 = be^{-b} - 2e^{-b}$$

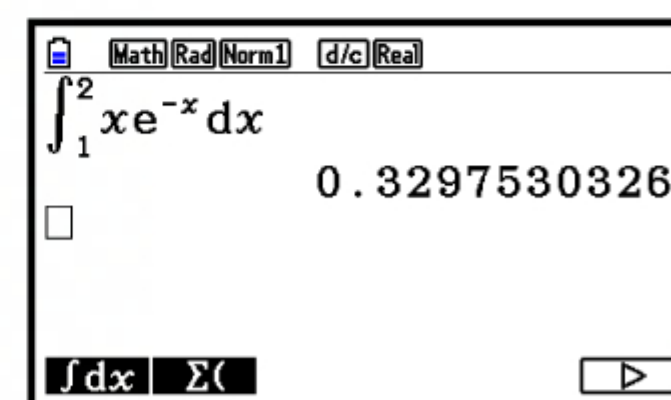
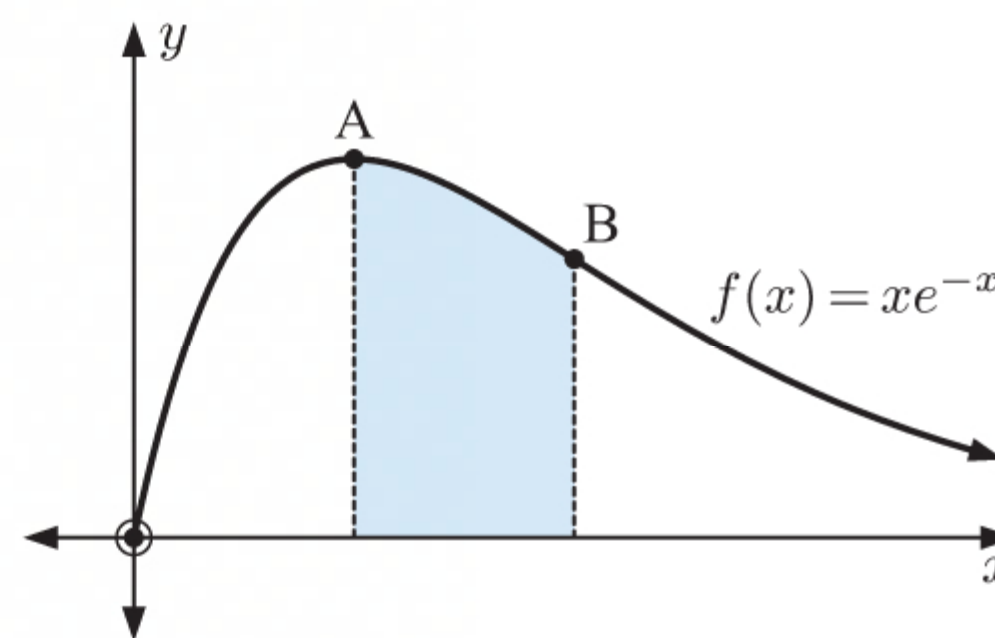
$$\therefore e^{-b}(b - 2) = 0$$

$$\therefore b = 2 \quad \{e^{-b} > 0\}$$

\therefore the x -coordinate of the point of inflection B is 2.

d Area of shaded region $= \int_1^2 xe^{-x} dx$ {using **b** and **c**}

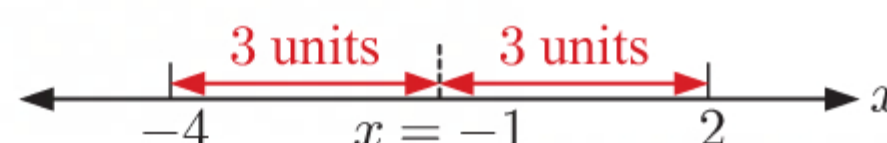
$$\approx 0.330 \text{ units}^2 \quad \{\text{using technology}\}$$



MIXED QUESTIONS SET 5

1 The axis of symmetry $x = -1$ lies midway between the x -intercepts.

\therefore the other x -intercept is 2.



Since the x -intercepts are -4 and 2 , the quadratic has the form $y = a(x + 4)(x - 2)$, $a \neq 0$.

When $x = 1$, $y = 5$

$$\therefore 5 = a(1 + 4)(1 - 2)$$

$$\therefore 5 = a(5)(-1)$$

$$\therefore a = -1$$

The quadratic has equation $y = -(x + 4)(x - 2)$

$$= -(x^2 + 2x - 8)$$

$$\therefore y = -x^2 - 2x + 8$$

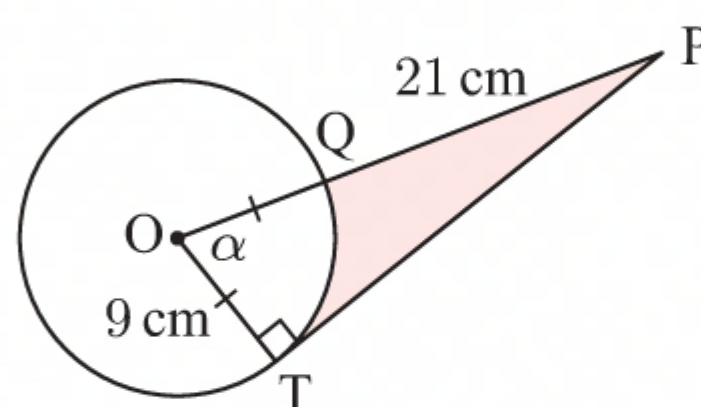
2 a $\widehat{OTP} = 90^\circ$ {radius-tangent}

$\therefore \triangle OPT$ is right angled at T.

$$\therefore \cos \alpha = \frac{9}{30}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{9}{30}\right)$$

$$\approx 72.5^\circ$$



$$\begin{aligned}\text{b Area of } \triangle OPT &= \frac{1}{2} \times 9 \times 30 \times \sin \alpha \\ &= 135 \sin \alpha \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of sector OQT} &= \frac{\theta}{360} \times \pi r^2 \\ &= \frac{\alpha}{360} \times \pi \times 9^2 \\ &= \frac{9\alpha\pi}{40} \text{ cm}^2\end{aligned}$$

So, shaded area = area of $\triangle OPT$ – area of sector OQT

$$\begin{aligned}&= 135 \sin \alpha - \frac{9\alpha\pi}{40} \\ &= 135 \sin(\cos^{-1}(\frac{9}{30})) - \frac{9\pi}{40} \cos^{-1}(\frac{9}{30}) \quad \{\text{using a}\} \\ &\approx 77.5 \text{ cm}^2\end{aligned}$$

- 3 a** Francesca adds \$0.50 in the first week, \$1 the next, \$1.50 the next, adding an additional \$0.50 each subsequent week.

\therefore in the n th week, Francesca adds $0.50n$ dollars to her money box.

Now the last week before her 11th birthday is the 51st week.

\therefore in the last week before her 11th birthday, Francesca added $\$0.50 \times 51 = \25.50 to her money box.

- b** Let $P(n)$ dollars be the amount Pierre had added to his money box after n weeks, and $F(n)$ dollars be the amount Francesca had added to her money box after n weeks.

Pierre adds \$10 each week, so after n weeks he has added $10n$ dollars.

So, $P(n) = 10n$

$$\therefore P(8) = 10 \times 8 = 80$$

After 8 weeks Pierre had added \$80 to his money box.

From **a**, Francesca adds $0.50n$ dollars in the n th week, so after n weeks she has added $0.50 + 1 + 1.50 + \dots + 0.50n$ dollars.

Now $0.50 + 1 + 1.50 + \dots + 0.50n$ is an arithmetic series with $u_1 = 0.5$ and $d = 0.5$.

$$\begin{aligned}\therefore 0.50 + 1 + 1.50 + \dots + 0.50n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 0.5 + (n-1) \times 0.5) \\ &= \frac{n}{2}(1 + 0.5n - 0.5) \\ &= \frac{n}{2}(0.5 + 0.5n) \\ &= 0.25n + 0.25n^2\end{aligned}$$

So, $F(n) = 0.25n + 0.25n^2$

$$\therefore F(8) = 0.25 \times 8 + 0.25 \times 8^2 = 18$$

After 8 weeks, Francesca added \$18 to her money box.

- c** There are 52 weeks in 1 year.

Now $P(52) = 10 \times 52 = 520$

$$\text{and } F(52) = 0.25 \times 52 + 0.25 \times 52^2 = 689$$

\therefore after 1 year, Pierre had $\$520 + \$100 = \$620$ in his money box, and Francesca had $\$689 + \$100 = \$789$ in her money box.

So, Francesca had more money in her money box after 1 year.

$$\mathbf{4} \quad 2 \sin^2 x = 3 \cos x + 2, \quad -\pi \leq x \leq \pi$$

$$\therefore 2(1 - \cos^2 x) = 3 \cos x + 2$$

$$\therefore 2 - 2 \cos^2 x = 3 \cos x + 2$$

$$\therefore 2 \cos^2 x + 3 \cos x = 0$$

$$\therefore \cos x(2 \cos x + 3) = 0$$

$$\therefore \cos x = 0 \quad \text{or} \quad \cos x = -\frac{3}{2}$$

$$\therefore \cos x = 0$$

$$\therefore x = -\frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2}$$

$$\{-1 \leq \cos x \leq 1\}$$

$$\{-\pi \leq x \leq \pi\}$$

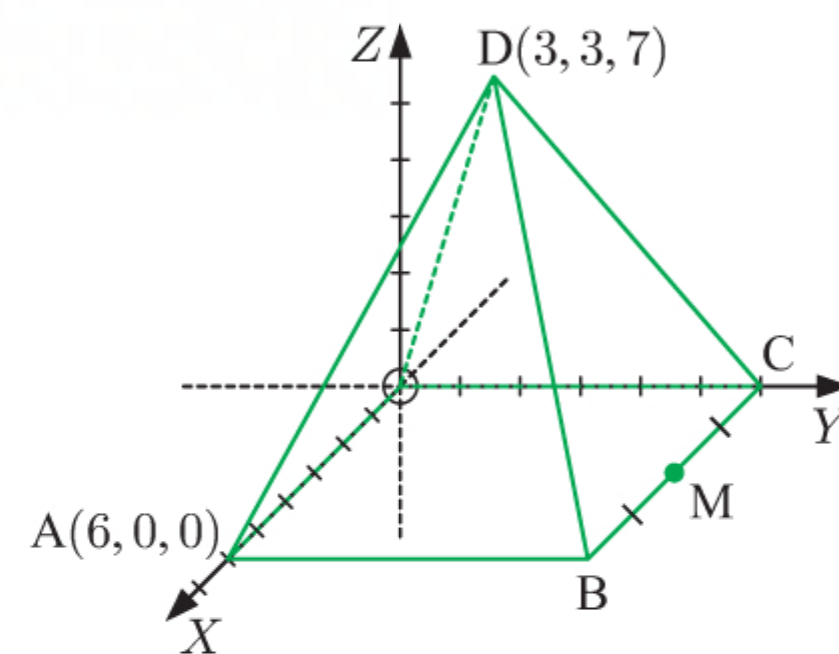
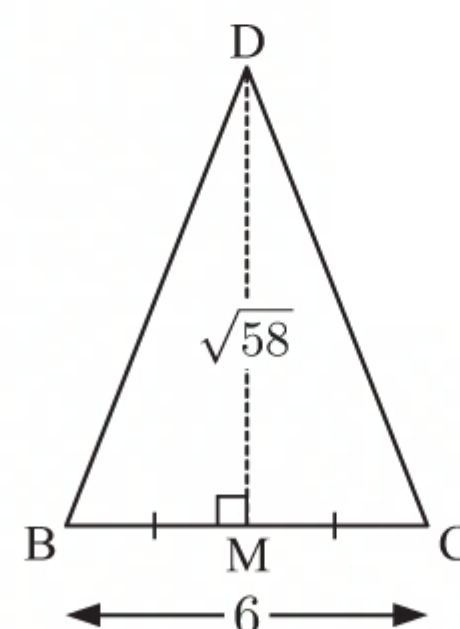
5 a B is (6, 6, 0) and C is (0, 6, 0).

b Volume = $\frac{1}{3} \times \text{area of base} \times \text{height}$
 $= \frac{1}{3} \times 6 \times 6 \times 7$
 $= 84 \text{ units}^3$

c The midpoint M of [BC] is $\left(\frac{6+0}{2}, \frac{6+6}{2}, \frac{0+0}{2}\right)$ which is (3, 6, 0).

d MD = $\sqrt{(3-3)^2 + (3-6)^2 + (7-0)^2}$
 $= \sqrt{0^2 + (-3)^2 + 7^2}$
 $= \sqrt{0 + 9 + 49}$
 $= \sqrt{58} \text{ units}$

Area of triangle BCD = $\frac{1}{2} \times 6 \times \sqrt{58}$
 $= 3\sqrt{58} \text{ units}^2$



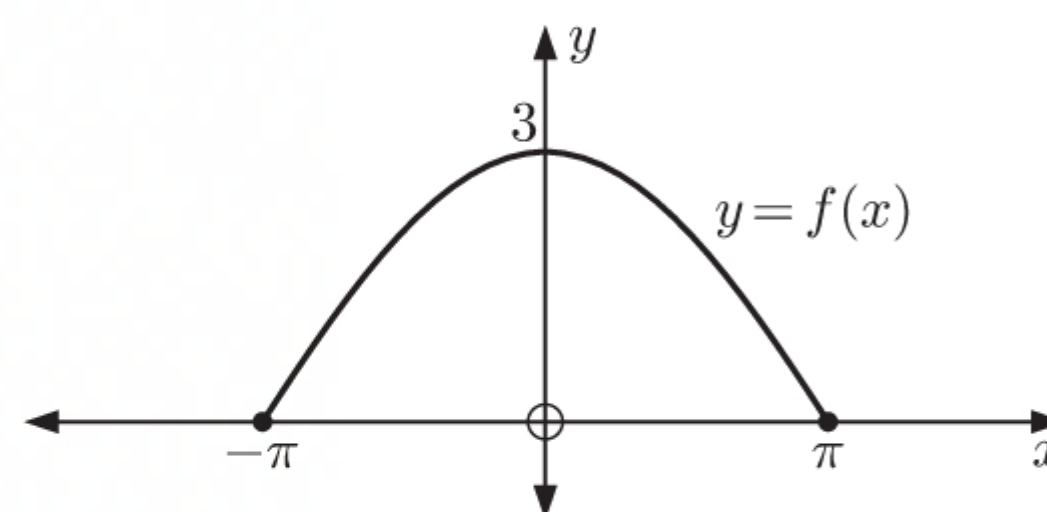
Surface area of pyramid = area of base + area of 4 triangular faces
 $= 6 \times 6 + 4 \times \text{area of } \triangle BCD$
 $= 36 + 4 \times 3\sqrt{58}$
 $= 36 + 12\sqrt{58} \text{ units}^2$
 $\approx 127 \text{ units}^2$

6 $4^x + 4 = 17(2^{x-1})$
 $\therefore (2^2)^x + 4 = 17\left(\frac{2^x}{2}\right)$
 $\therefore 2(2^x)^2 + 8 = 17(2^x)$
 $\therefore 2(2^x)^2 - 17(2^x) + 8 = 0$
 $\therefore (2(2^x) - 1)(2^x - 8) = 0 \quad \{2X^2 - 17X + 8 = (2X - 1)(X - 8)\}$
 $\therefore 2^x = \frac{1}{2} \quad \text{or} \quad 2^x = 8$
 $\therefore 2^x = 2^{-1} \quad \text{or} \quad 2^x = 2^3$
 $\therefore x = -1 \quad \text{or} \quad x = 3$

7 a $f(x) = a \cos bx, \quad -\pi \leq x \leq \pi$

From the graph:

- the amplitude is 3, so $a = 3$
- the period is $2 \times 2\pi, \quad \frac{2\pi}{b} = 4\pi$
 $\therefore b = \frac{1}{2}$



b $f(x) = 3 \cos\left(\frac{x}{2}\right), \quad -\pi \leq x \leq \pi$

i $f'(x) = -\frac{3}{2} \sin\left(\frac{x}{2}\right)$

Now $f(c) = 3 \cos\left(\frac{c}{2}\right)$ and $f'(c) = -\frac{3}{2} \sin\left(\frac{c}{2}\right)$.

\therefore the normal to $y = f(x)$ at the point where $x = 3$ has gradient $\frac{2}{3 \sin(\frac{c}{2})}$ and passes through $(c, 3 \cos(\frac{c}{2}))$.

\therefore the equation of the normal is $2x - 3 \sin\left(\frac{c}{2}\right)y = 2c - 3 \sin\left(\frac{c}{2}\right)(3 \cos(\frac{c}{2}))$

$\therefore 2x - 3 \sin\left(\frac{c}{2}\right)y = 2c - 9 \sin\left(\frac{c}{2}\right) \cos\left(\frac{c}{2}\right)$

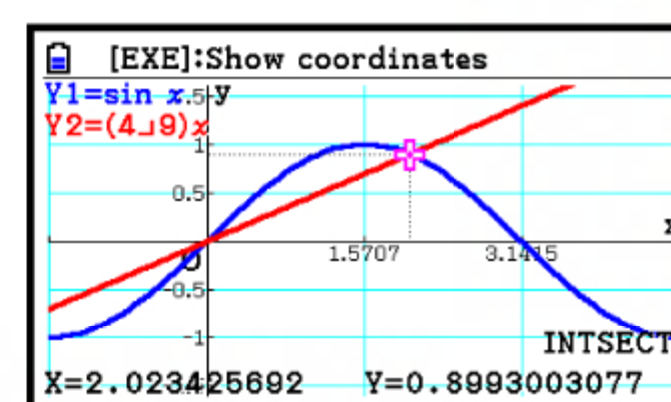
$\therefore 2x - 3 \sin\left(\frac{c}{2}\right)y = 2c - \frac{9}{2} \sin c \quad \{2 \sin \theta \cos \theta = \sin 2\theta\}$

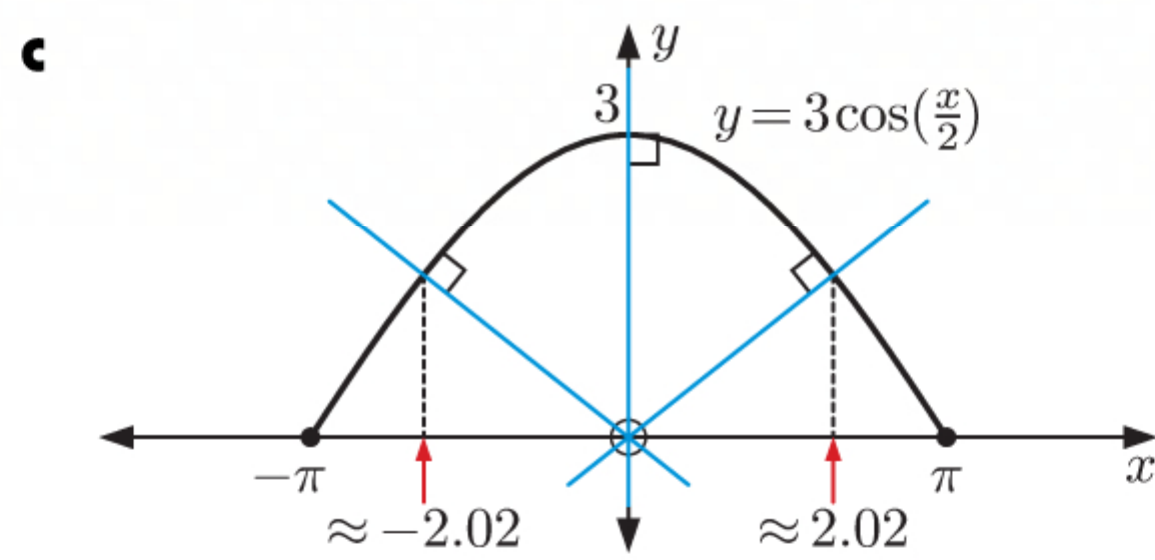
ii If the normal passes through the origin then

$2c - \frac{9}{2} \sin c = 0$

$\therefore \sin c = \frac{4}{9}c$

Using technology, $c = 0$ or $\approx \pm 2.02 \quad \{-\pi \leq c \leq \pi\}$

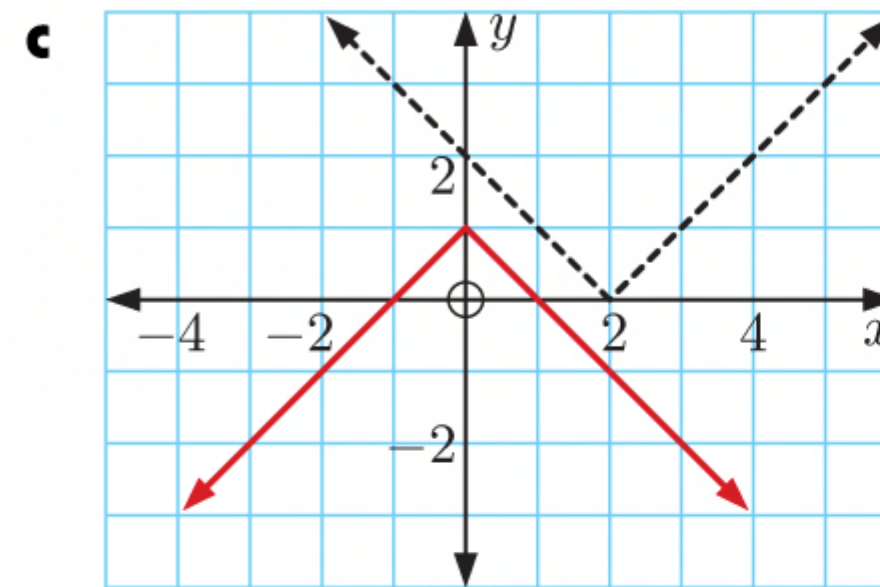




8 a Yes, as any vertical line cuts the graph no more than once.

b Domain = $\{x \mid x \in \mathbb{R}\}$

Range = $\{y \mid y \geq 0\}$



9 $R(t) = \frac{12}{\sqrt{t+1}} e^{-\sqrt{t+1}} \text{ L s}^{-1}$

a $R(10) = \frac{12}{\sqrt{10+1}} e^{-\sqrt{10+1}}$
 $= \frac{12}{\sqrt{11}} e^{-\sqrt{11}}$
 $\approx 0.131 \text{ L s}^{-1}$

After 10 seconds, the water is still overflowing at about 0.131 L s^{-1} .

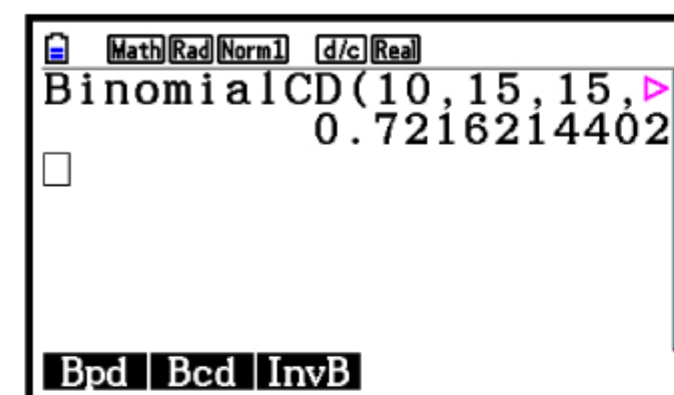
c $\int_0^{60} R(t) dt$
 $= \int_0^{60} \frac{12}{\sqrt{t+1}} e^{-\sqrt{t+1}} dt$
 $= \left[-24e^{-\sqrt{t+1}} \right]_0^{60} \quad \{\text{using b}\}$
 $= -24e^{-\sqrt{61}} + 24e^{-1}$
 $= 24(e^{-1} - e^{-\sqrt{61}})$
 ≈ 8.82

10 a Let X be the number of committee members who attend a randomly selected meeting.

$\therefore X \sim B(15, 0.7)$

Now $P(X \geq 10) \approx 0.722$ {using technology}

\therefore approximately 72.2% of meetings will go ahead.



b Suppose there are n committee members, and let Y be the number of committee members who attend a randomly selected meeting.

$\therefore Y \sim B(n, 0.7)$

We require $P(Y \geq 10) \geq 0.9$

If $n = 16$, $P(Y \geq 10) \approx 0.825$ ✗

If $n = 17$, $P(Y \geq 10) \approx 0.895$ ✗

If $n = 18$, $P(Y \geq 10) \approx 0.940$ ✓

So, 18 committee members are required to ensure that at least 90% of the meetings go ahead.

11 $s = \sin\left(\frac{\pi}{(t+1)^2}\right) \text{ cm}, \quad t \geq 0$

a When $t = 1$, $s = \sin\left(\frac{\pi}{(1+1)^2}\right)$
 $= \sin \frac{\pi}{4}$
 $= \frac{1}{\sqrt{2}} \text{ cm}$

The displacement of the object after 1 second is $\frac{1}{\sqrt{2}} \text{ cm}$.

b $s = 0.5 \text{ cm}$ when $\sin\left(\frac{\pi}{(t+1)^2}\right) = 0.5$
 $\therefore \frac{\pi}{(t+1)^2} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$
 $\therefore (t+1)^2 = 6 \text{ or } \frac{6}{5}$
 $\therefore t+1 = \sqrt{6} \text{ or } \sqrt{\frac{6}{5}} \quad \{t+1 > 0\}$
 $\therefore t = \sqrt{6} - 1 \text{ or } \sqrt{\frac{6}{5}} - 1$
 $\therefore t \approx 1.45 \text{ or } 0.0954 \text{ seconds}$

The object has displacement 0.5 cm for the first time after about 0.0954 seconds.

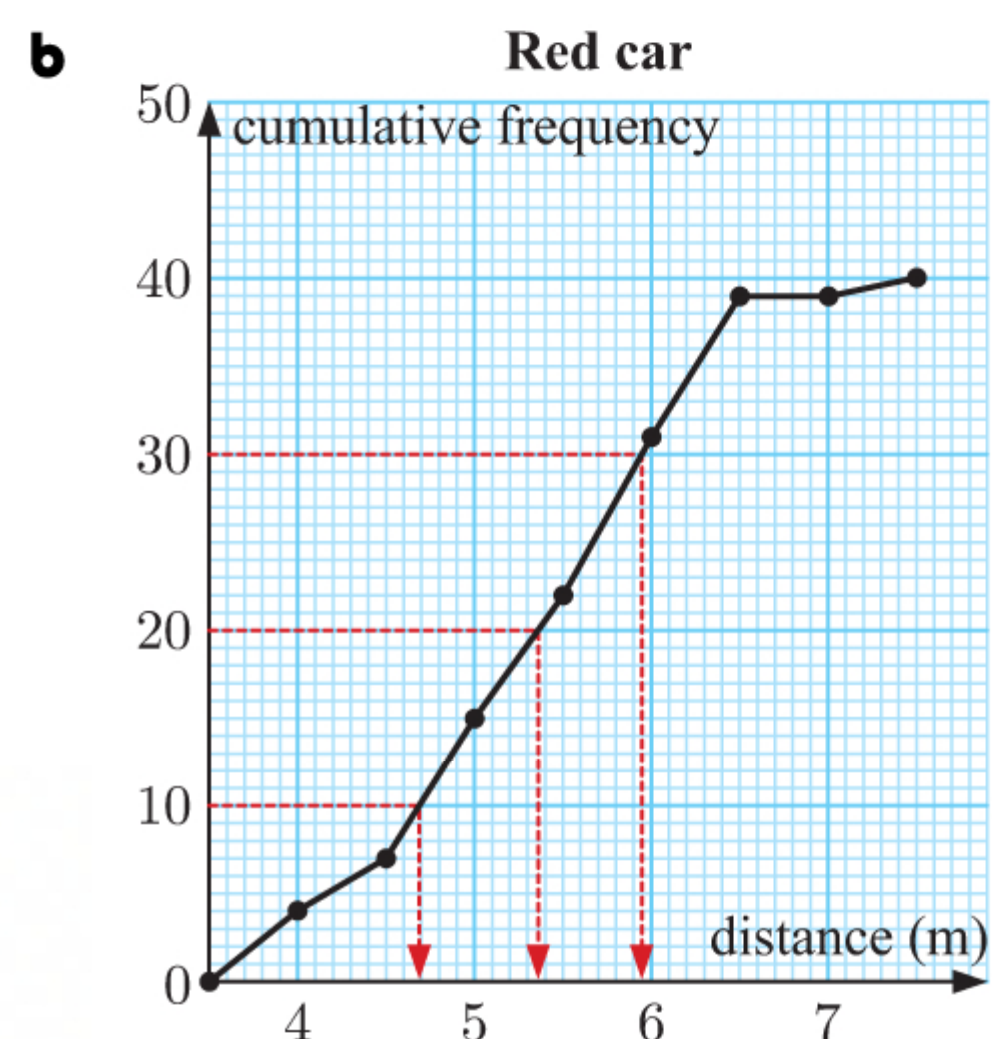
c $v = \frac{ds}{dt}$
 $= \cos\left(\frac{\pi}{(t+1)^2}\right) \times (-2\pi(t+1)^{-3}) \quad \{\text{chain rule}\}$
 $= \frac{-2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right) \text{ cm s}^{-1}$

d The object is stationary when $v = 0$
 $\therefore \frac{-2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right) = 0$
 $\therefore \cos\left(\frac{\pi}{(t+1)^2}\right) = 0$
 $\therefore \frac{\pi}{(t+1)^2} = \frac{\pi}{2}$
 $\therefore (t+1)^2 = 2$
 $\therefore t+1 = \sqrt{2} \quad \{t+1 > 0\}$
 $\therefore t = \sqrt{2} - 1 \approx 0.414 \text{ seconds}$

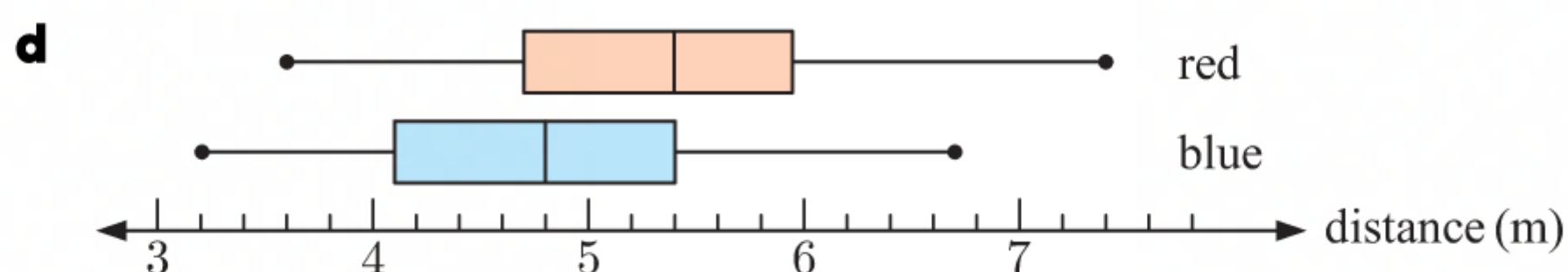
The object is stationary for the first time after about 0.414 seconds.

12 a

Distance (m)	Frequency	Cumulative frequency
$3.5 \leq d < 4$	4	4
$4 \leq d < 4.5$	3	7
$4.5 \leq d < 5$	8	15
$5 \leq d < 5.5$	7	22
$5.5 \leq d < 6$	9	31
$6 \leq d < 6.5$	8	39
$6.5 \leq d < 7$	0	39
$7 \leq d < 7.5$	1	40



c **i** Median ≈ 5.4 **ii** $Q_1 \approx 4.7$ **iii** $Q_3 \approx 5.9$



e All values of the five-number summary (min, Q_1 , median, Q_3 , and max) for the red car are higher than those for the blue car. This evidence is strongly against the view that the cars were made by the same machine.

MIXED QUESTIONS SET 6

1 $f(x) = 3 - 4^{-x}$

a $f(2) = 3 - 4^{-2} = 3 - \frac{1}{16}$
 $= 2\frac{15}{16}$
 $\therefore p = 2\frac{15}{16}$

$f(-2) = 3 - 4^2 = -13$
 $\therefore q = -13$

b i $f(0) = 3 - 4^0 = 2 \therefore$ the y -intercept is 2.

ii As $x \rightarrow \infty$, $4^{-x} \rightarrow 0$ and so $y \rightarrow 3$
 $\therefore y = 3$ is the horizontal asymptote.

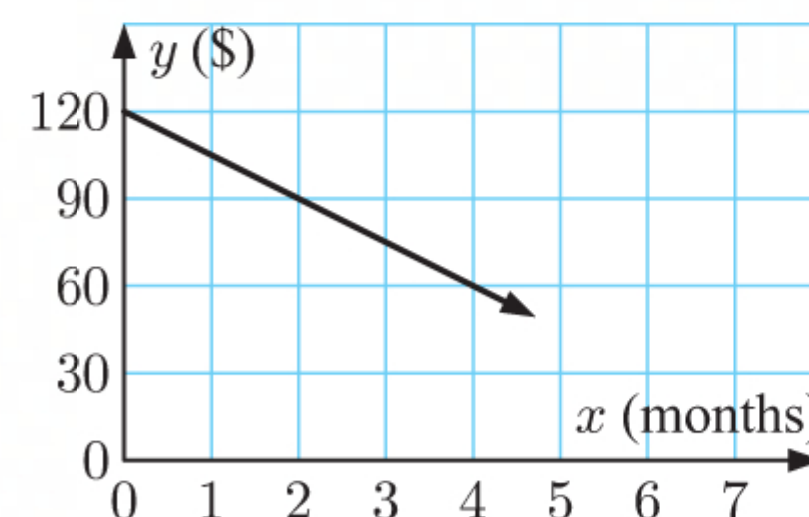
d The range is $\{y \mid y < 3\}$.

2 a The line passes through $(0, 120)$ and $(2, 90)$, so the gradient is

$$\frac{90 - 120}{2 - 0} = \frac{-30}{2} = -15.$$

This means that the amount of money left in the subscription account decreases by \$15 each month.

The y -intercept is 120. This means that the initial balance was \$120.



b The gradient is -15 and the y -intercept is 120, so the equation of the line is $y = -15x + 120$.

c The account runs out of money when $y = 0$

$$\therefore -15x + 120 = 0$$

$$\therefore 15x = 120$$

$$\therefore x = 8$$

The account will run out of money after 8 months.

3 a $v(t) = e^{2t} - 3e^t \text{ m s}^{-1}$

$$\therefore v(0) = e^0 - 3e^0 = 1 - 3 = -2 \text{ m s}^{-1}$$

$$\therefore \text{the initial velocity is } -2 \text{ m s}^{-1}.$$

b Now $v(t) = e^t(e^t - 3)$

Since $e^t > 0$ for all t , $v(t) = 0$ when $e^t = 3$
 which is when $t = \ln 3$

$$\therefore \text{the particle is stationary at } t = \ln 3 \text{ seconds.}$$

c $s(t) = \int v(t) dt$

$$= \int (e^{2t} - 3e^t) dt$$

$$= \frac{1}{2}e^{2t} - 3e^t + c \text{ metres}$$

But $s(0) = 1$, so $\frac{1}{2}e^0 - 3e^0 + c = 1$

$$\therefore \frac{1}{2} - 3 + c = 1$$

$$\therefore c = 3\frac{1}{2}$$

Thus $s(t) = \frac{1}{2}e^{2t} - 3e^t + \frac{7}{2} \text{ metres}$

$$\therefore s(\ln 5) = \frac{1}{2}e^{2\ln 5} - 3e^{\ln 5} + \frac{7}{2}$$

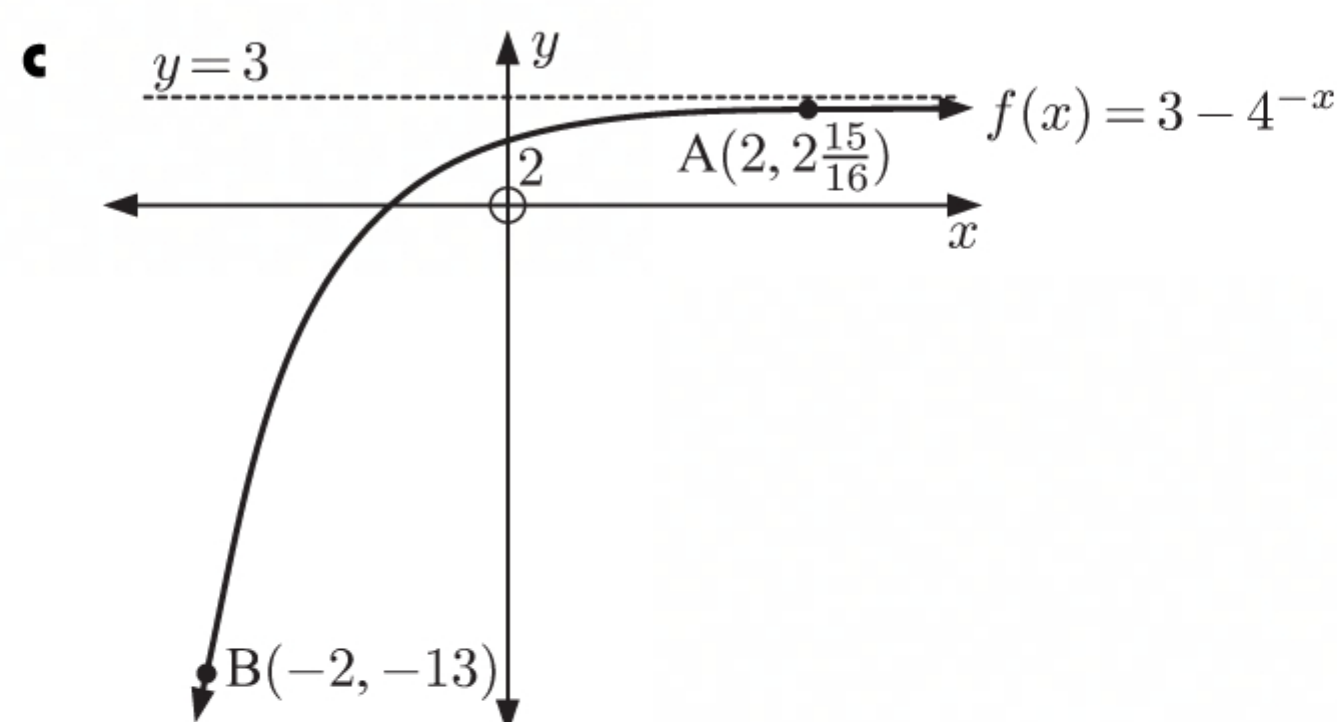
$$= \frac{1}{2}e^{\ln(5^2)} - 3(5) + \frac{7}{2}$$

$$= \frac{25}{2} - 15 + \frac{7}{2}$$

$$= 16 - 15$$

$$= 1 \text{ m}$$

So, the particle is 1 m to the right of O after $\ln 5$ seconds.

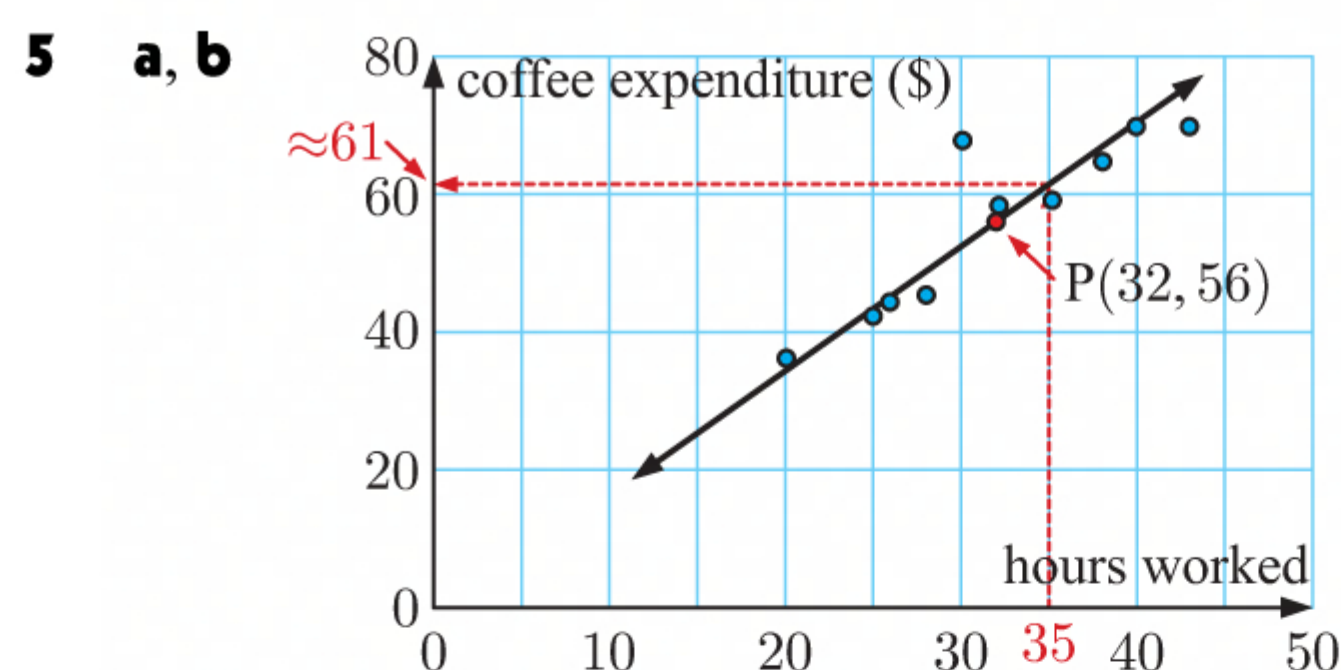
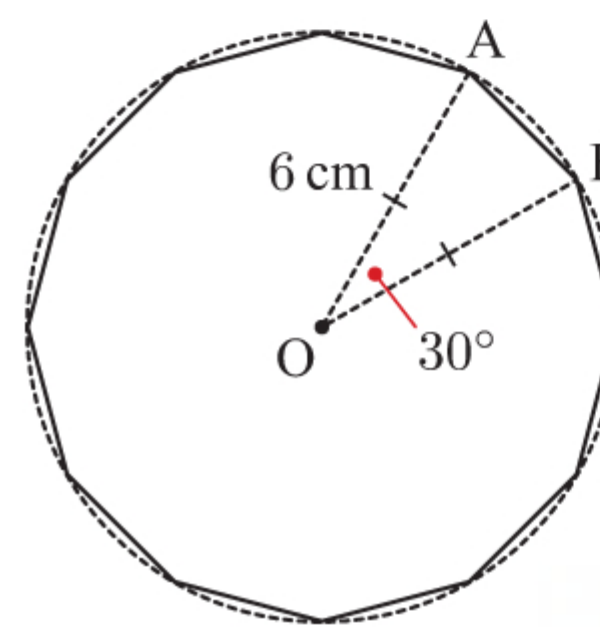


- 4 a There are twelve equal angles at the centre of the dodecagon.

$$\therefore \widehat{AOB} = \frac{360^\circ}{12} = 30^\circ$$

b Area of $\triangle AOB = \frac{1}{2} \times 6 \times 6 \times \sin 30^\circ$
 $= 9 \text{ cm}^2$

c Area of dodecagon $= 12 \times 9$
 $= 108 \text{ cm}^2$



- c From the graph, if James works a 35 hour week, he spends about \$61.

- d There is a strong positive linear relationship between the length of time James works and the amount he spends on coffee. Since the prediction in c was an interpolation on strongly correlated data, it is a reliable estimate.

- 6 a $\left(kx + \frac{1}{\sqrt{x}}\right)^9$ has general term

$$\begin{aligned} T_{r+1} &= \binom{9}{r} (kx)^{9-r} \left(\frac{1}{\sqrt{x}}\right)^r \\ &= \binom{9}{r} k^{9-r} x^{9-r} \frac{1}{x^{\frac{r}{2}}} \\ &= \binom{9}{r} k^{9-r} x^{9-\frac{3r}{2}} \end{aligned}$$

- b For the constant term, $9 - \frac{3r}{2} = 0$

$$\therefore \frac{3r}{2} = 9$$

$$\therefore r = 6$$

$$T_7 = \binom{9}{6} k^3 x^0$$

$$\therefore 84k^3 = -10\frac{1}{2} \quad \{\text{constant term} = -10\frac{1}{2}\}$$

$$\therefore k^3 = -\frac{1}{8}$$

$$\therefore k = -\frac{1}{2}$$

7 $\sin^2 t + \cos^2 t = 1$

$$\therefore \sin^2 t = 1 - \cos^2 t$$

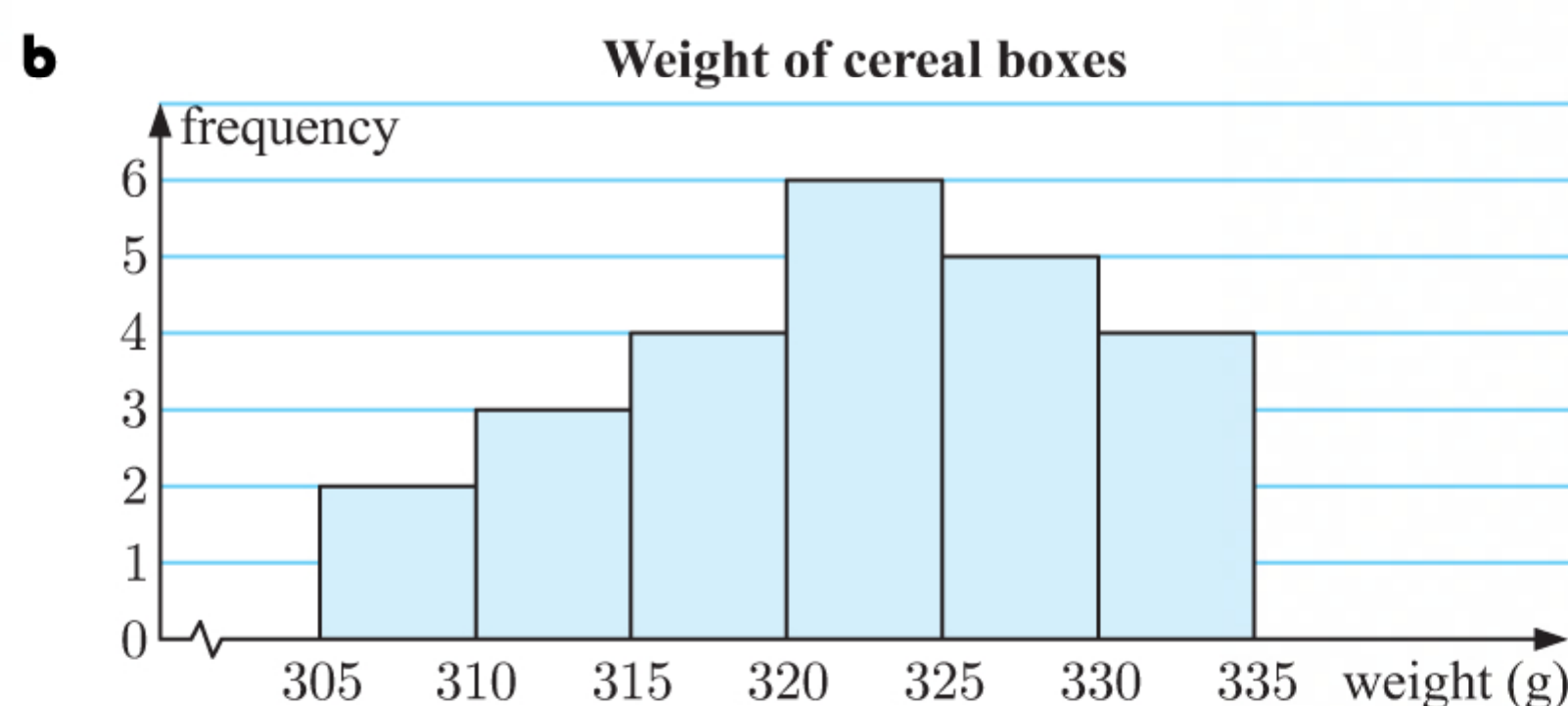
For $0 < t < \pi$, $\sin t > 0$

$$\therefore \sin t = \sqrt{1 - \cos^2 t}$$

$$\therefore \tan t = \frac{\sin t}{\cos t} = \frac{\sqrt{1 - \cos^2 t}}{\cos t} \quad \text{for } 0 < t < \pi$$

8 a

Weight (w g)	Frequency
$305 \leq w < 310$	2
$310 \leq w < 315$	3
$315 \leq w < 320$	4
$320 \leq w < 325$	6
$325 \leq w < 330$	5
$330 \leq w < 335$	4



- c The data is slightly negatively skewed.

- d The modal class is the interval $320 \leq w < 325$ because it has the highest frequency.

e Mean $= \frac{312 + 320 + \dots + 324}{24}$
 $= \frac{7696}{24}$
 ≈ 320.67

The mean of the sample is reasonably close to the average weight that the manufacturer claims.

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad y &= x(x^2 - 12x + 45) \\
 &= x^3 - 12x^2 + 45x \\
 \therefore \frac{dy}{dx} &= 3x^2 - 24x + 45
 \end{aligned}$$

$$\text{and } \frac{d^2y}{dx^2} = 6x - 24$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= 3x^2 - 24x + 45 \\
 &= 3(x^2 - 8x + 15) \\
 &= 3(x - 3)(x - 5)
 \end{aligned}$$

Sign diagram for $\frac{dy}{dx}$:

$$\begin{aligned}
 \text{When } x = 3, \quad y &= (3)^3 - 12(3)^2 + 45(3) \\
 &= 27 - 108 + 135 \\
 &= 54
 \end{aligned}$$

$$\begin{aligned}
 \text{and when } x = 5, \quad y &= (5)^3 - 12(5)^2 + 45(5) \\
 &= 125 - 300 + 225 \\
 &= 50
 \end{aligned}$$

\therefore there is a local maximum at $(3, 54)$ and a local minimum at $(5, 50)$.

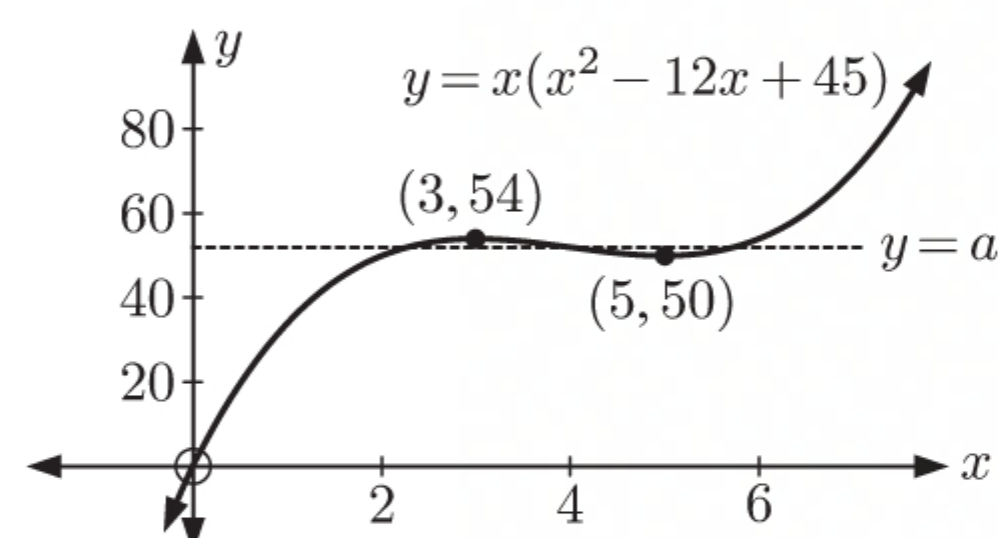
$$\begin{aligned}
 \mathbf{c} \quad \frac{d^2y}{dx^2} &= 6x - 24 \\
 \therefore \frac{d^2y}{dx^2} &= 0 \text{ when } x = 4
 \end{aligned}$$

Sign diagram for $\frac{d^2y}{dx^2}$ is:

$$\begin{aligned}
 \text{When } x = 4, \quad y &= 4^3 - 12(4)^2 + 45(4) \\
 &= 64 - 192 + 180 \\
 &= 52
 \end{aligned}$$

So, there is a non-stationary inflection point at $(4, 52)$.

\mathbf{d} The graph cuts the x and y -axes at 0.



\mathbf{e} The equation $x^3 - 12x^2 + 45x - a = 0$ has 3 real roots if $x(x^2 - 12x + 45) = a$ has 3 real roots. This occurs provided $y = x(x^2 - 12x + 45)$ meets $y = a$ in 3 places.
 $\therefore 50 < a < 54$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(25 - x^2) \\
 &= \frac{2}{\sqrt{25 - x^2}}
 \end{aligned}$$

The domain is $\{x \mid -5 < x < 5\}$.

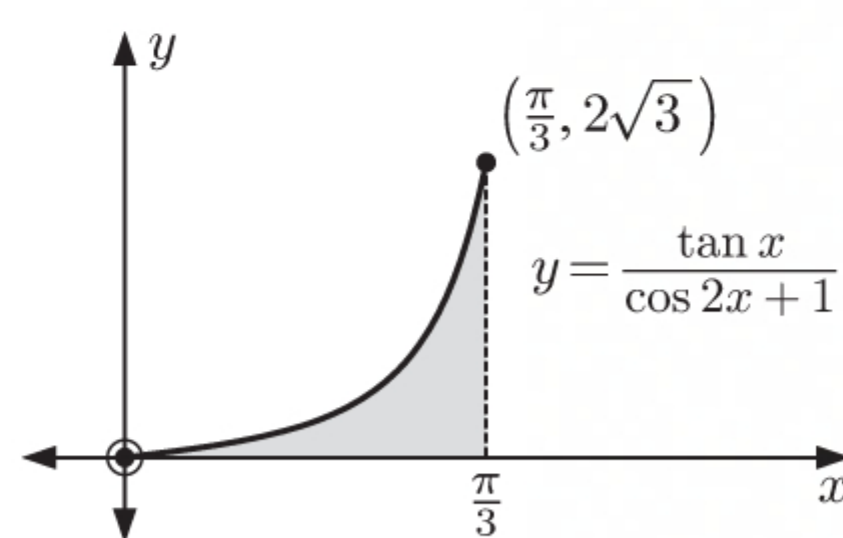
$$\begin{aligned}
 \mathbf{b} \quad (g \circ f)(x) &= 1 \\
 \therefore \frac{2}{\sqrt{25 - x^2}} &= 1 \quad \{\text{using } \mathbf{a}\} \\
 \therefore 2 &= \sqrt{25 - x^2} \\
 \therefore 4 &= 25 - x^2 \\
 \therefore x^2 &= 21 \\
 \therefore x &= \pm\sqrt{21}
 \end{aligned}$$

\mathbf{c} $\sqrt{25 - x^2} = 0$ when $x = \pm 5$, so $x = -5$ and $x = 5$ are the vertical asymptotes.

$$\begin{aligned}
 \text{11 a } \frac{\tan x}{\cos 2x + 1} &= \frac{\frac{\sin x}{\cos x}}{2 \cos^2 x - 1 + 1} \\
 &= \frac{\frac{\sin x}{\cos x}}{2 \cos^2 x} \\
 &= \frac{\sin x}{2 \cos^3 x}
 \end{aligned}$$

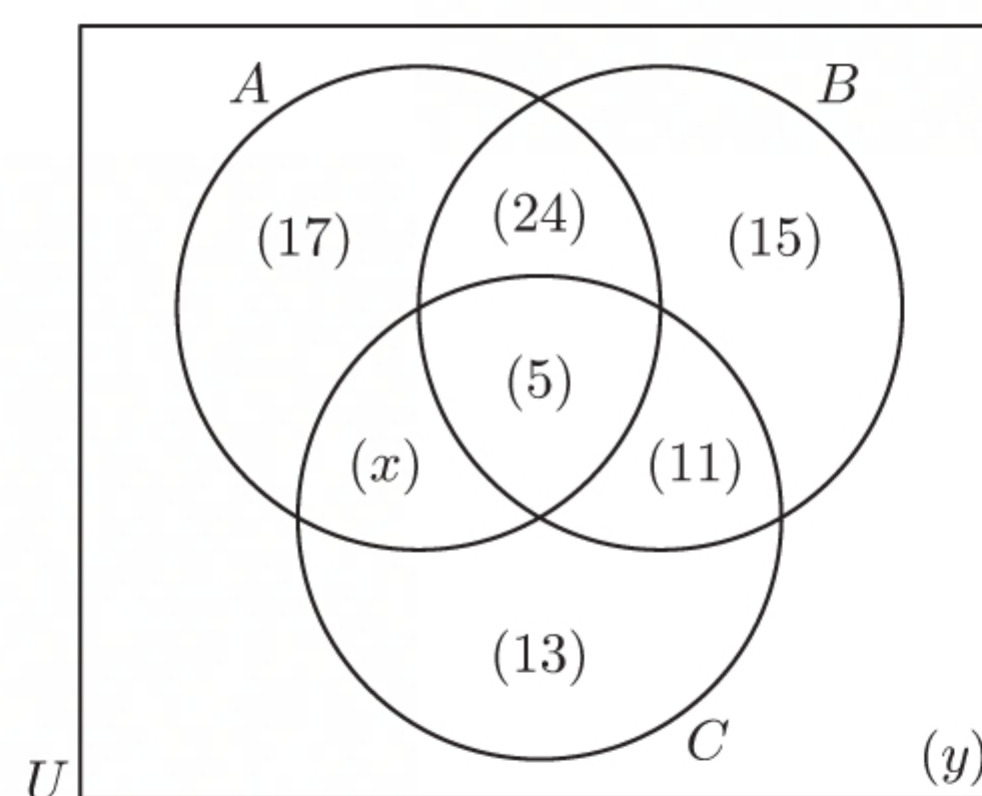
$$\begin{aligned}
 \text{b } \int \frac{\tan x}{\cos 2x + 1} dx &= \int \frac{\sin x}{2 \cos^3 x} dx \quad \{\text{using a}\} \\
 &= \int \frac{1}{2u^3} \left(-\frac{du}{dx}\right) dx \quad \left\{u = \cos x \quad \therefore \frac{du}{dx} = -\sin x\right\} \\
 &= -\frac{1}{2} \int u^{-3} du \\
 &= -\frac{1}{2} \left(-\frac{1}{2} u^{-2} + c\right) \\
 &= \frac{1}{4} u^{-2} + c \\
 &= \frac{1}{4 \cos^2 x} + c \quad \dots (*)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now shaded area} &= \int_0^{\frac{\pi}{3}} \frac{\tan x}{\cos 2x + 1} dx \\
 &= \left[\frac{1}{4 \cos^2 x} \right]_0^{\frac{\pi}{3}} \quad \{\text{using } (*)\} \\
 &= \frac{1}{4 \cos^2(\frac{\pi}{3})} - \frac{1}{4 \cos^2 0} \\
 &= \frac{1}{4(\frac{1}{2})^2} - \frac{1}{4(1)^2} \\
 &= 1 - \frac{1}{4} \\
 &= \frac{3}{4} \text{ units}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{12 a } n(A) &= 48 \\
 \therefore 17 + 24 + 5 + x &= 48 \\
 \therefore 46 + x &= 48 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } n(U) &= 100 \\
 \therefore 17 + 24 + 5 + 2 + 15 + 11 + 13 + y &= 100 \\
 \therefore 87 + y &= 100 \\
 \therefore y &= 13
 \end{aligned}$$



$$\text{b } n(A) = 48, \quad n(B) = 24 + 5 + 15 + 11 = 55, \quad n(C) = 2 + 5 + 11 + 13 = 31$$

\therefore course B was the most popular.

$$\begin{aligned}
 \text{c i } P(\text{liked all the courses}) &= \frac{5}{100} \\
 &= \frac{1}{20}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } P(B \cap C') &= \frac{24 + 15}{100} \\
 &= \frac{39}{100}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } P(\text{liked exactly 2 courses} \mid C) &= \frac{n(\text{liked } C \text{ and exactly one other course})}{n(C)} \\
 &= \frac{2 + 11}{2 + 5 + 11 + 13} \\
 &= \frac{13}{31}
 \end{aligned}$$

$$\begin{aligned}
 \text{iv } P(\text{liked none} \mid B') &= \frac{n(\text{liked none})}{n(B')} \\
 &= \frac{13}{17 + 2 + 13 + 13} \\
 &= \frac{13}{45}
 \end{aligned}$$

MIXED QUESTIONS SET 7

1 Let $k = 2n + 1$ where $n \in \mathbb{Z}$.

$$\begin{aligned}\therefore k^3 - k &= (2n + 1)^3 - (2n + 1) \\ &= (2n)^3 + 3(2n)^2(1) + 3(2n)(1)^2 + (1)^3 - 2n - 1 \quad \{\text{binomial theorem}\} \\ &= 8n^3 + 12n^2 + 6n + 1 - 2n - 1 \\ &= 8n^3 + 12n^2 + 4n \\ &= 4(2n^3 + 3n^2 + n) \quad \text{which is divisible by 4}\end{aligned}$$

2 a There are $n = 5 \times 4 = 20$ time periods.

Each time period the investment increases by $i = \frac{4.4\%}{4} = 1.1\%$.

$$\begin{aligned}\therefore \text{the amount after 5 years is } u_{20} &= u_0 \times (1 + i)^{20} \\ &= 2000 \times (1.011)^{20} \quad \{1.1\% = 0.011\} \\ &\approx 2489.16\end{aligned}$$

The final value of the investment is \$2489.16.

b Interest = \$2489.16 - \$2000
= \$489.16

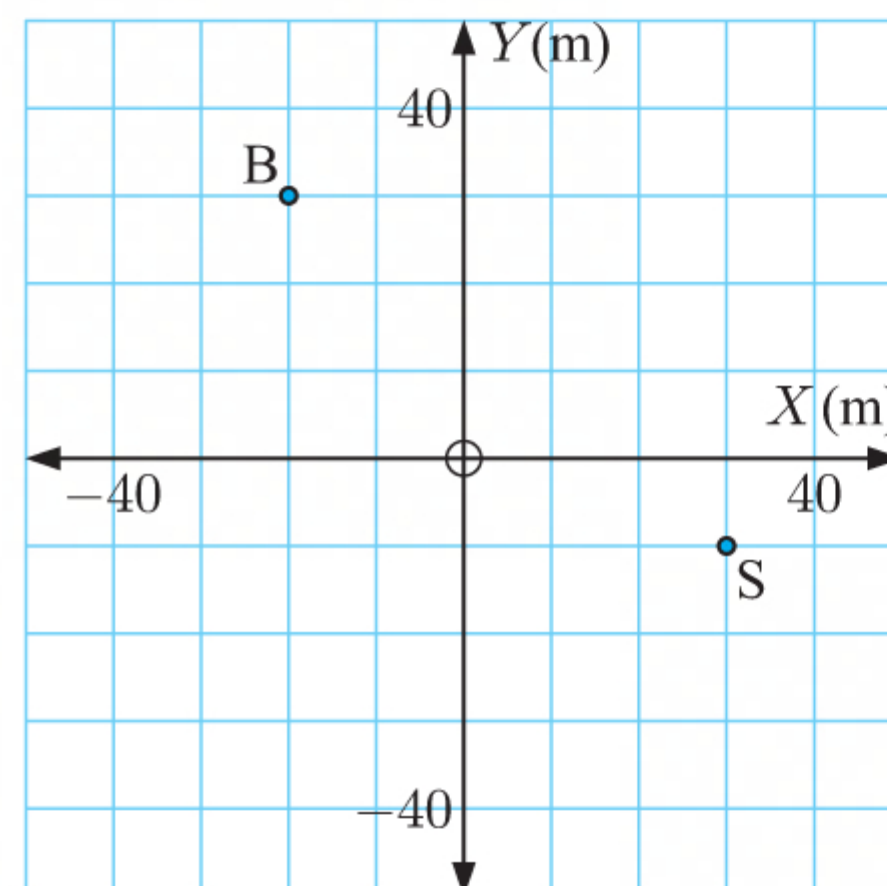
c real value $\times (1.025)^5 = \$2489.16$
 $\therefore \text{real value} = \frac{\$2489.16}{(1.025)^5}$
= \$2200.05

3 a i The anchor has coordinates $A(-20, 30, -50)$.

ii The shipwreck has coordinates $S(30, -10, -40)$.

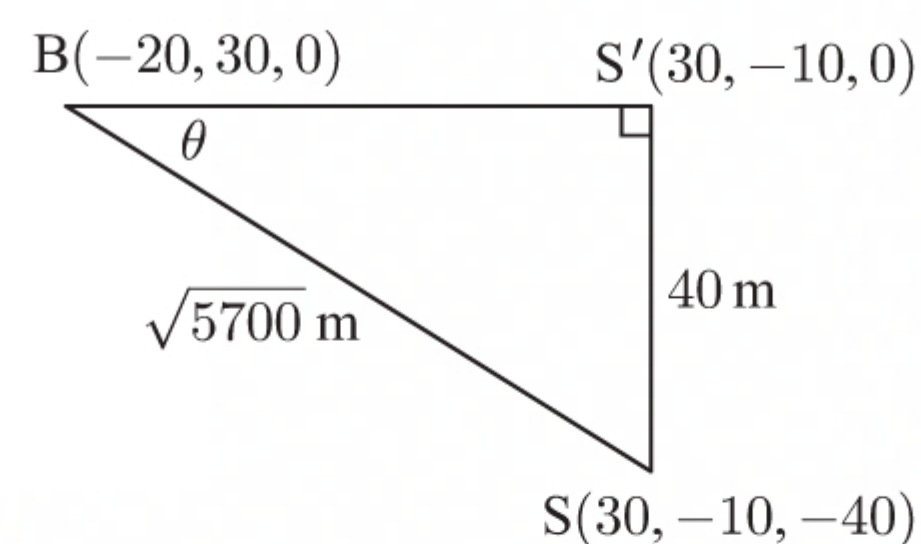
b $BS = \sqrt{(30 - (-20))^2 + (-10 - 30)^2 + (-40 - 0)^2}$
 $= \sqrt{50^2 + (-40)^2 + (-40)^2}$
 $= \sqrt{5700}$
 $\approx 75.5 \text{ m}$

\therefore the diver has to swim about 75.5 m.



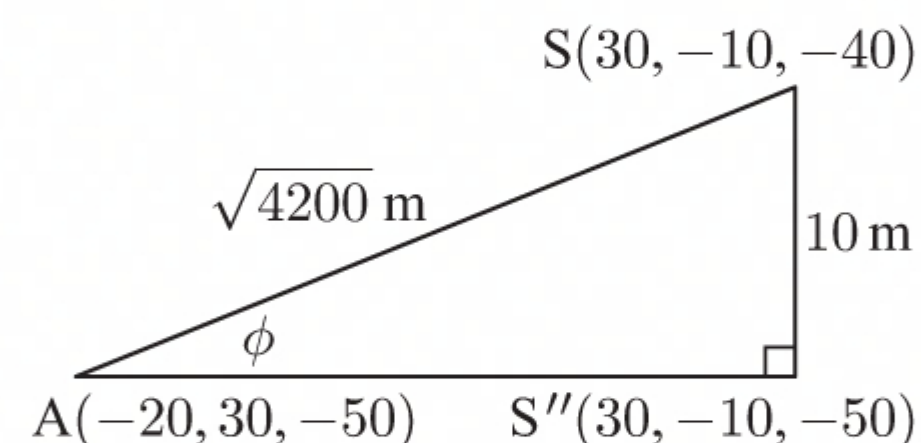
c i $\sin \theta = \frac{40}{\sqrt{5700}}$
 $\therefore \theta = \sin^{-1}\left(\frac{40}{\sqrt{5700}}\right)$
 $\therefore \theta \approx 32.0^\circ$

The angle of depression from the boat to the shipwreck is about 32.0° .



ii $AS = \sqrt{(-20 - 30)^2 + (30 - (-10))^2 + (-50 - (-40))^2}$
 $= \sqrt{(-50)^2 + 40^2 + (-10)^2}$
 $= \sqrt{4200} \text{ m}$
 $\therefore \sin \phi = \frac{10}{\sqrt{4200}}$
 $\therefore \phi = \sin^{-1}\left(\frac{10}{\sqrt{4200}}\right)$
 $\therefore \phi \approx 8.88^\circ$

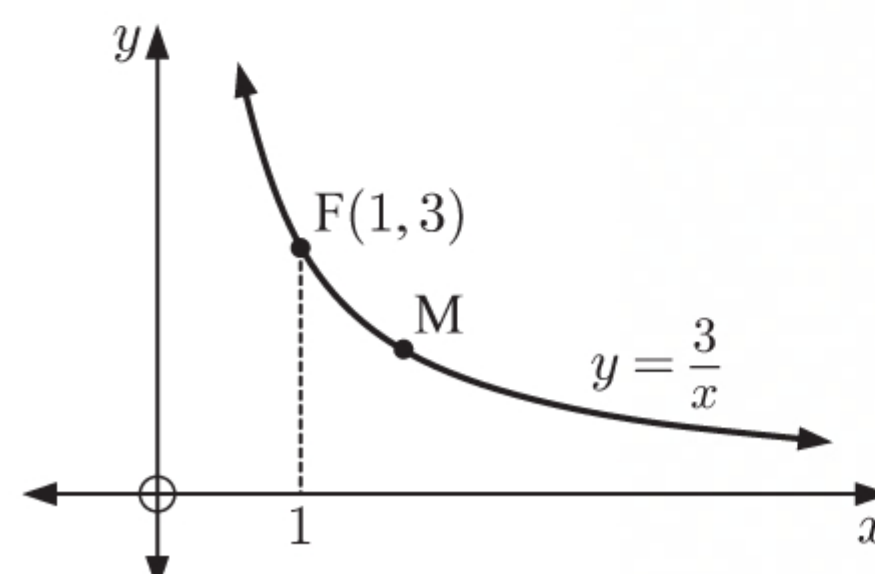
The angle of elevation from the anchor to the shipwreck is about 8.88° .



4 a M has x -coordinate $1 + h$.

\therefore M has y -coordinate $\frac{3}{1+h}$.

b $3 - \frac{3}{h+1} = 3\left(\frac{h+1}{h+1}\right) - \frac{3}{h+1}$
 $= \frac{3h+3-3}{h+1}$
 $= \frac{3h}{h+1}$



$$\begin{aligned}
 \text{c gradient of [FM]} &= \frac{y\text{-step}}{x\text{-step}} \\
 &= \frac{3 - \frac{3}{1+h}}{1 - (1+h)} \\
 &= \frac{3h}{h+1} \times \frac{1}{-h} \quad \{\text{using b}\} \\
 &= \frac{-3}{h+1} \quad \{h \neq 0\}
 \end{aligned}$$

d As $h \rightarrow 0$, gradient of [FM] \rightarrow gradient of tangent at F
 \therefore gradient of tangent at F $= \frac{-3}{1} = -3$.

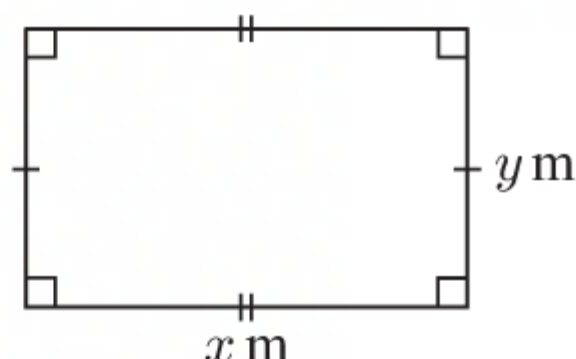
5 a Let the other side of the field be y m.

$$2x + 2y = 160$$

$$\therefore 2y = 160 - 2x$$

$$\therefore y = 80 - x$$

$$\therefore \text{area } A = x(80 - x) \text{ m}^2$$



b $A = 80x - x^2$

$$\therefore \frac{dA}{dx} = 80 - 2x$$

Now $\frac{dA}{dx} = 0$ when $80 - 2x = 0$

$$\therefore 2x = 80$$

$$\therefore x = 40$$

The sign diagram of $\frac{dA}{dx}$ is:

$\therefore A$ is maximised when $x = 40$, and $y = 80 - 40 = 40$.

\therefore the maximum area occurs when the field is a $40 \text{ m} \times 40 \text{ m}$ square.

c i $A = 1200$

$$\therefore x(80 - x) = 1200$$

$$\therefore x^2 - 80x + 1200 = 0$$

$$\therefore (x - 60)(x - 20) = 0$$

$$\therefore x = 60 \text{ or } 20$$

When $x = 60$, $y = 80 - 60 = 20$.

When $x = 20$, $y = 80 - 20 = 60$.

\therefore the field is $60 \text{ m} \times 20 \text{ m}$.

ii The maximum area is 1600 m^2 .

We lose $1600 - 1200 = 400 \text{ m}^2$ of productive land

$$\therefore \text{the lost production} = 400 \times 6.5 = 2600 \text{ kg}$$

6	Height (h cm)	Mid-interval value (x)	Frequency (f)	xf
	$80 \leq h < 90$	85	8	680
	$90 \leq h < 100$	95	12	1140
	$100 \leq h < 110$	105	17	1785
	$110 \leq h < 120$	115	30	3450
	$120 \leq h < 130$	125	13	1625
	Total		$\sum f = 80$	$\sum xf = 8680$

a $\bar{x} = \frac{\sum xf}{\sum f}$

$$= \frac{8680}{80}$$

$$= 108.5$$

b Using technology, $s \approx 12.1 \text{ cm}$.

We estimate the mean height to be about 108.5 cm .

7 When $x = 0$, $y = 0$

$$\therefore 0 = ke^0 + b$$

$$\therefore k = -b \quad \dots (*)$$

$$\text{Now shaded area} = \frac{3}{e^6} \text{ units}^2$$

$$\therefore - \int_{-3}^0 (ke^{2x} + b) dx = \frac{3}{e^6}$$

$$\therefore \int_{-3}^0 (ke^{2x} + b) dx = -3e^{-6}$$

$$\therefore \left[\frac{k}{2} e^{2x} + bx \right]_{-3}^0 = -3e^{-6}$$

$$\therefore \left(\frac{k}{2} e^0 + 0 \right) - \left(\frac{k}{2} e^{-6} - 3b \right) = -3e^{-6}$$

$$\therefore \frac{k}{2} - \frac{k}{2} e^{-6} + 3b = -3e^{-6}$$

$$\therefore -\frac{k}{2} e^{-6} + \left(\frac{k}{2} + 3b \right) = -3e^{-6}$$

$$\therefore \frac{b}{2} e^{-6} + \left(-\frac{b}{2} + 3b \right) = -3e^{-6} \quad \{\text{using } (*)\}$$

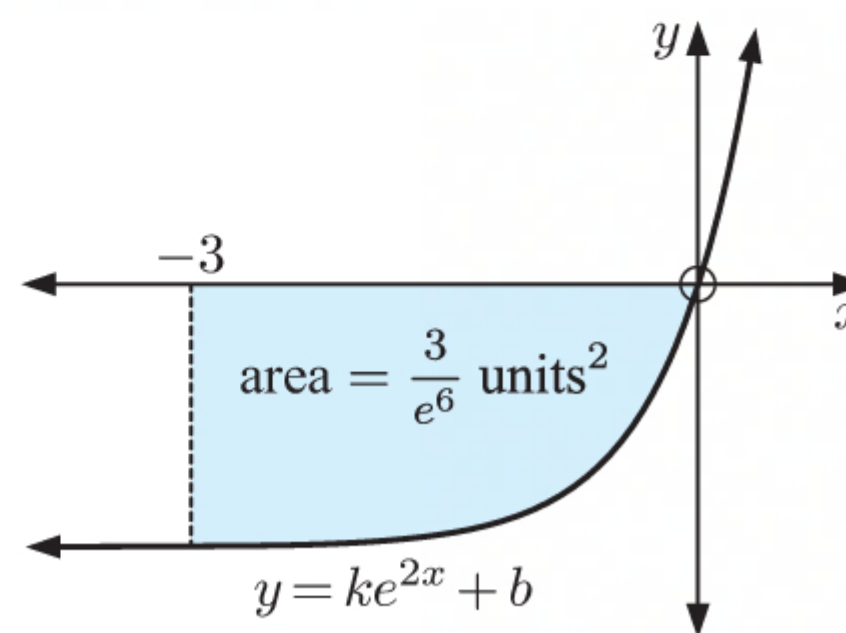
$$\therefore \frac{b}{2} e^{-6} + \frac{5}{2} b = -3e^{-6}$$

$$\therefore b + 5be^6 = -6$$

$$\therefore b(1 + 5e^6) = -6$$

$$\therefore b = \frac{-6}{1 + 5e^6}$$

$$\therefore k = \frac{6}{1 + 5e^6} \quad \{\text{using } (*)\}$$



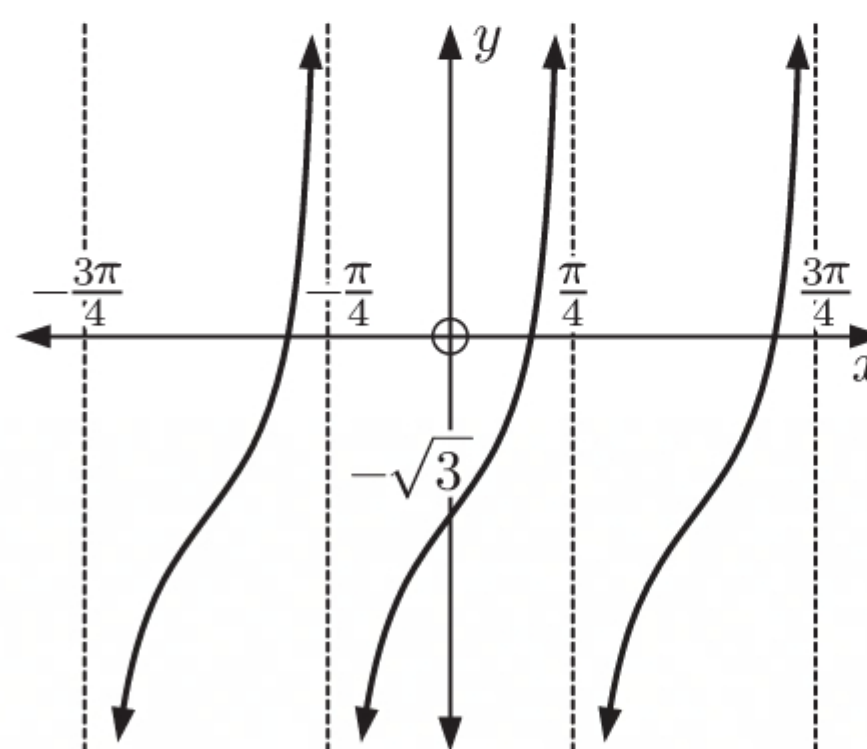
8 a The period is $\frac{\pi}{a} = \frac{\pi}{2}$

$$\therefore a = 2$$

The y -intercept is $-\sqrt{3}$.

$$\therefore b = -\sqrt{3}$$

$$\therefore a = 2, \quad b = -\sqrt{3}$$



b x -intercepts occur when $\tan 2x - \sqrt{3} = 0$

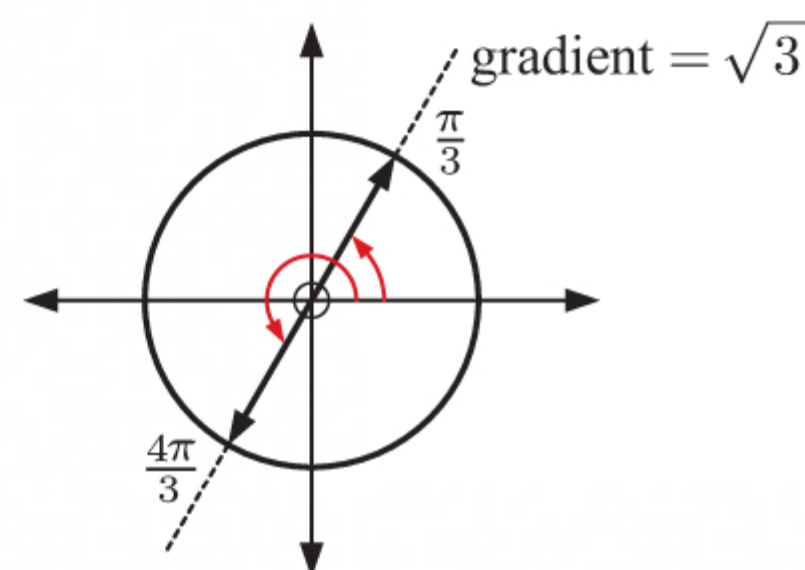
$$\therefore \tan 2x = \sqrt{3}$$

$$\text{Now, if } -\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4},$$

$$\text{then } -\frac{3\pi}{2} \leq 2x \leq \frac{3\pi}{2}$$

$$\therefore 2x = -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}$$

$$\therefore x = -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$$



9 $P(B) = 0.3$ and $P(A \cup B) = 0.55$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ {addition law of probability}

$$\therefore P(A \cup B) = P(A) + P(B) \quad \{A \text{ and } B \text{ are mutually exclusive}\}$$

$$\therefore 0.55 = P(A) + 0.3$$

$$\therefore P(A) = 0.25$$

$$10 \quad 3x^2 + 2x \xrightarrow{\text{vertical stretch scale factor 2}} 2(3x^2 + 2x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ -1 \end{pmatrix}} 2(3(x-3)^2 + 2(x-3)) - 1$$

The image has equation

$$\begin{aligned} y &= 2(3(x-3)^2 + 2(x-3)) - 1 \\ \therefore y &= 2(3(x^2 - 6x + 9) + 2x - 6) - 1 \\ \therefore y &= 2(3x^2 - 18x + 27 + 2x - 6) - 1 \\ \therefore y &= 2(3x^2 - 16x + 21) - 1 \\ \therefore y &= 6x^2 - 32x + 42 - 1 \\ \therefore y &= 6x^2 - 32x + 41 \end{aligned}$$

$$\begin{aligned} 11 \quad a \quad f(\theta) &= \frac{2 - \cos \theta}{\sin \theta}, \quad 0 < \theta \leq \frac{\pi}{2} \\ \therefore f'(\theta) &= \frac{\sin \theta \times \sin \theta - (2 - \cos \theta) \cos \theta}{\sin^2 \theta} \quad \{\text{quotient rule}\} \\ &= \frac{\sin^2 \theta - 2 \cos \theta + \cos^2 \theta}{\sin^2 \theta} \\ &= \frac{1 - 2 \cos \theta}{\sin^2 \theta} \end{aligned}$$

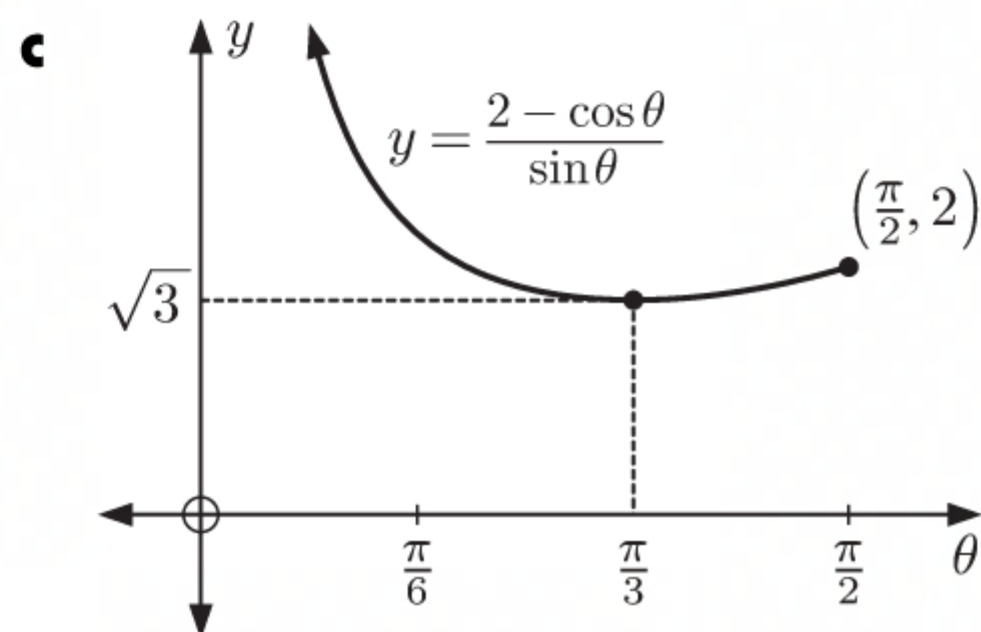
$$b \quad f'(\theta) = 0 \text{ when } \cos \theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3} \quad \{0 < \theta \leq \frac{\pi}{2}\}$$

The sign diagram for $f'(\theta)$ is:

$\therefore f(\theta)$ is a minimum when $\theta = \frac{\pi}{3}$.

$$\text{Now } f\left(\frac{\pi}{3}\right) = \frac{2 - \cos \frac{\pi}{3}}{\sin \frac{\pi}{3}} = \frac{2 - \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$$

\therefore the minimum value is $\sqrt{3}$ when $\theta = \frac{\pi}{3}$.



$$\begin{aligned} 12 \quad a \quad P(\text{win}) &= P(\text{1st ball red} \cap \text{2nd ball red} \cap \text{3rd ball red}) \\ &= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \\ &= \frac{1}{6} \end{aligned}$$

b If X is the number of wins when the game is played 60 times, then $X \sim B(60, \frac{1}{6})$.

$$\begin{aligned} i \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\ &= 60\left(\frac{1}{6}\right) & &= \sqrt{60\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} \\ &= 10 & &= \sqrt{\frac{25}{3}} \\ & & &\approx 2.89 \end{aligned}$$

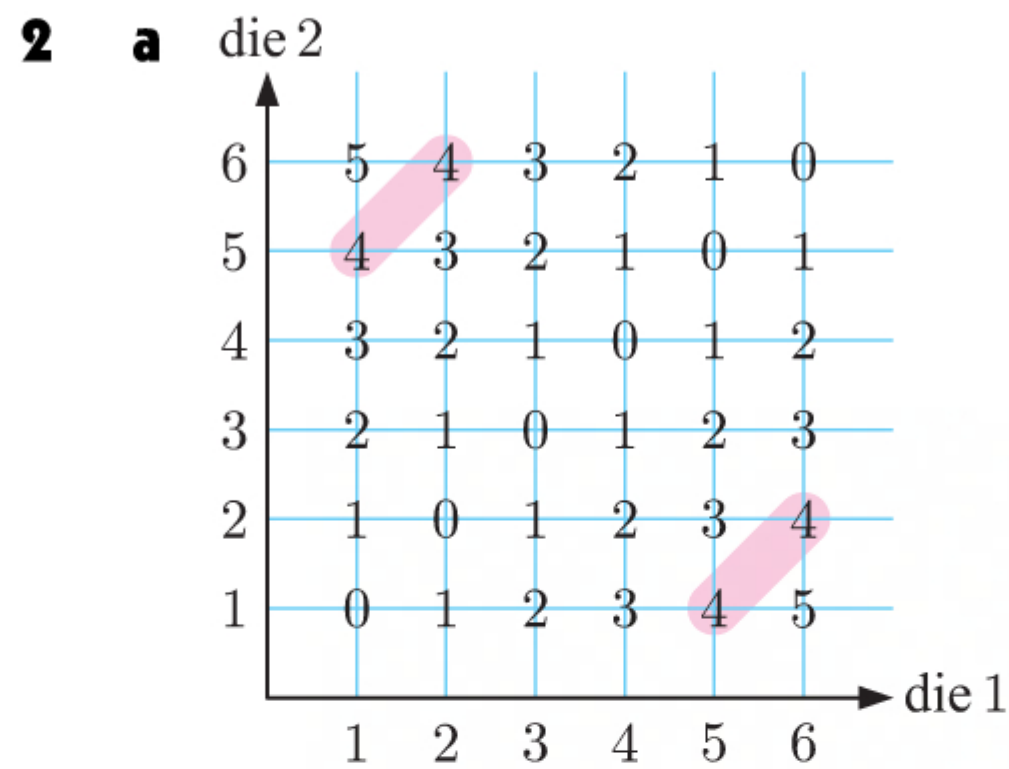
$$\begin{aligned} ii \quad P(X = \mu) &= P(X = 10) \\ &= \binom{60}{10} \left(\frac{1}{6}\right)^{10} \left(\frac{5}{6}\right)^{50} \\ &\approx 0.137 \end{aligned}$$

$$\begin{aligned} iii \quad P(\mu - \sigma \leq X \leq \mu + \sigma) &= P\left(10 - \sqrt{\frac{25}{3}} \leq X \leq 10 + \sqrt{\frac{25}{3}}\right) \\ &= P(7.11 \leq X \leq 12.9) \\ &= P(8 \leq X \leq 12) \\ &\approx 0.614 \quad \{\text{using technology}\} \end{aligned}$$

MIXED QUESTIONS SET 8

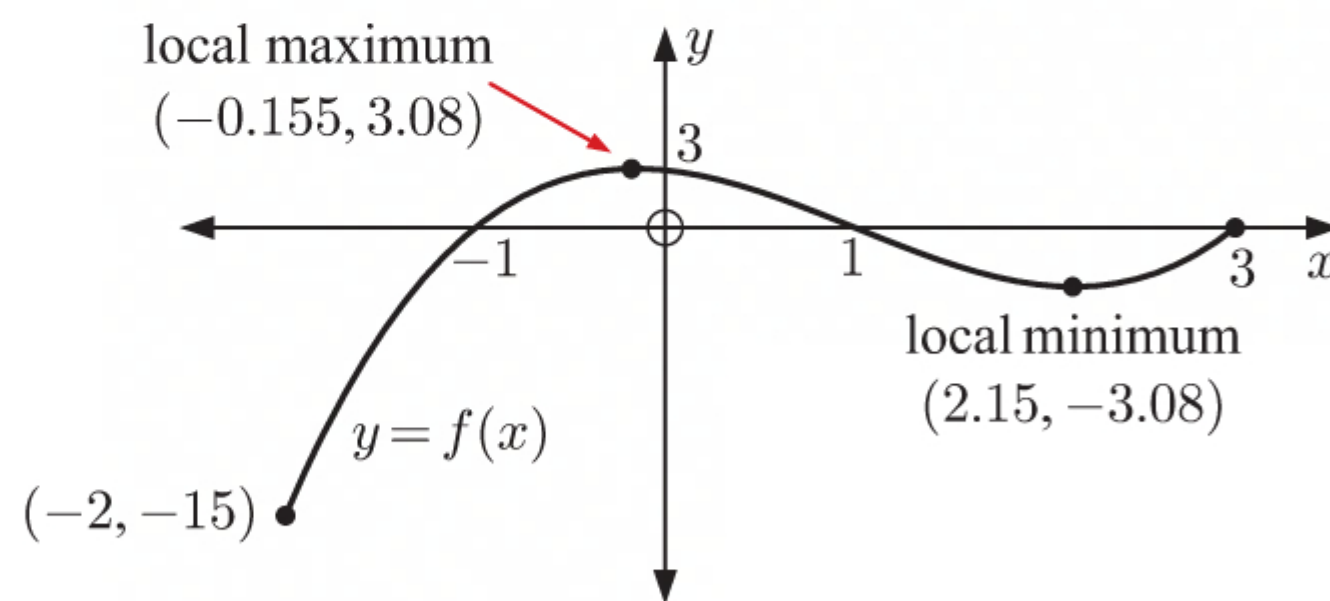
$$\begin{aligned}
 1 \quad & 2^{x-1} = 3^{2-x} \\
 & \therefore \log(2^{x-1}) = \log(3^{2-x}) \\
 & \therefore (x-1)\log 2 = (2-x)\log 3 \\
 & \therefore x\log 2 - \log 2 = 2\log 3 - x\log 3 \\
 & \therefore x\log 2 + x\log 3 = 2\log 3 + \log 2 \\
 & \therefore x(\log 2 + \log 3) = \log(3^2) + \log 2 \\
 & \therefore x\log 6 = \log 18 \\
 & \therefore x = \frac{\log 18}{\log 6} \\
 & \therefore x = \log_6 18 \quad \{\text{change of base rule}\}
 \end{aligned}$$

So, $a = 6$ and $b = 18$.



- b There are 4 outcomes where the difference is 4.
- As all outcomes are equally possible, the probability of the difference being 4 is $\frac{4}{36} = \frac{1}{9}$.

3 a $f(x) = x^3 - 3x^2 - x + 3, \quad -2 \leq x \leq 3$



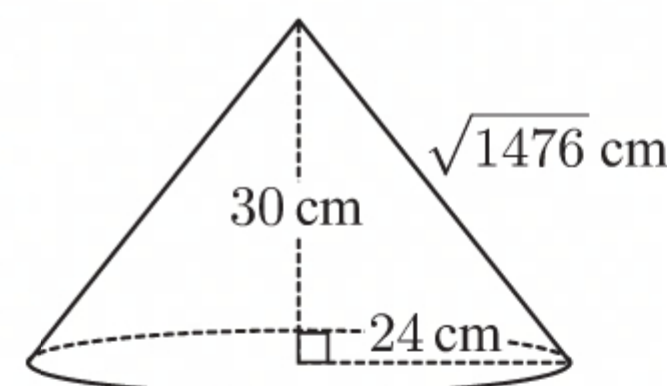
- b The range of f is $\{y \mid -15 \leq y \leq 3.08\}$.

4 a

$$\begin{aligned}
 s^2 &= 24^2 + 30^2 \quad \{\text{Pythagoras}\} \\
 \therefore s^2 &= 1476 \\
 \therefore s &= \sqrt{1476} \quad \{s > 0\} \\
 \therefore s &\approx 38.4 \text{ cm}
 \end{aligned}$$

b Surface area $= \pi r s + \pi r^2$

$$\begin{aligned}
 &= \pi(24)\sqrt{1476} + \pi(24)^2 \\
 &\approx 4710 \text{ cm}^2 \\
 &\approx 4.71 \times 10^3 \text{ cm}^2
 \end{aligned}$$



5 a

Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	p
20 - 24	5
25 - 29	2
Total	50

Total number of sample spots = 50

$$\begin{aligned}
 \therefore 9 + 15 + 10 + p + 5 + 2 &= 50 \\
 \therefore p + 41 &= 50 \\
 \therefore p &= 9
 \end{aligned}$$

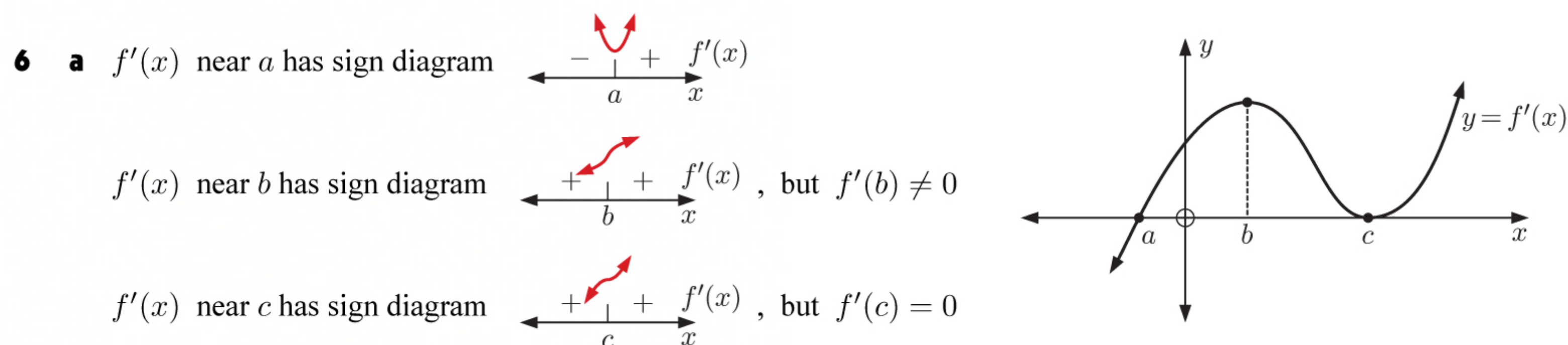
b

Number of weeds	Midpoint (x)	Frequency (f)	xf
0 - 4	2	9	18
5 - 9	7	15	105
10 - 14	12	10	120
15 - 19	17	9	153
20 - 24	22	5	110
25 - 29	27	2	54
Total		$\sum f = 50$	$\sum xf = 560$

$$\begin{aligned}\bar{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{560}{50} \\ &= 11.2\end{aligned}$$

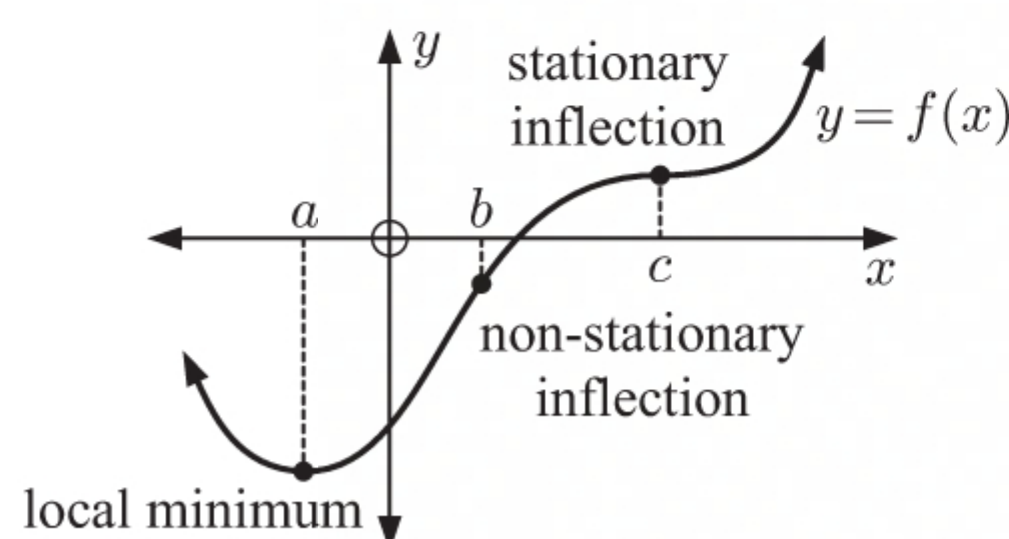
We estimate the mean number of weeds per spot to be 11.2.

c Percentage fewer than 10 weeds $= \frac{9+15}{50} \times 100\%$
 $= \frac{24}{50} \times 100\%$
 $= 48\%$



\therefore there is a local minimum at $x = a$, a non-stationary inflection at $x = b$, and a stationary inflection at $x = c$.

So a possible curve for $y = f(x)$ is:



b $f(x) = f_1(x) + k$ where k is a constant.

7 $W(t) = 5 \times (0.965)^t$ grams, $t \geq 0$

a The weight of the radioactive substance at the *end* of each year forms a geometric sequence with common ratio $r = 0.965$.

So, percentage decrease $= (1 - r) \times 100\%$
 $= 0.035 \times 100\%$
 $= 3.5\%$

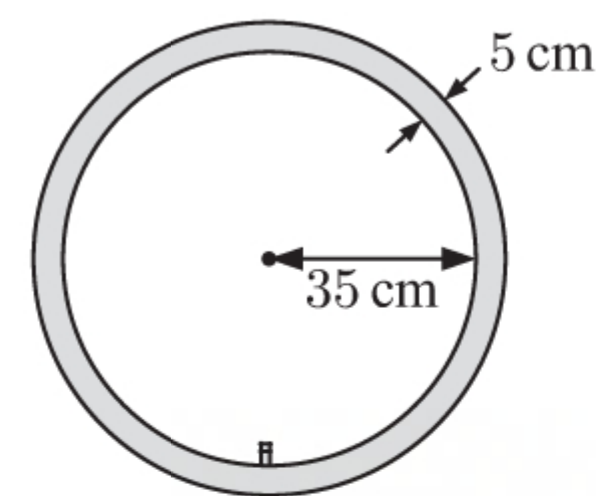
b $W(300) = 5 \times (0.965)^{300}$
 ≈ 0.000114
 $\approx 1.14 \times 10^{-4}$

The weight of the substance after 300 years is about 1.14×10^{-4} grams.

c We need to solve $W(t) = 1$
 $\therefore 5 \times (0.965)^t = 1$
 $\therefore (0.965)^t = 0.2$
 $\therefore t \log(0.965) = \log(0.2)$
 $\therefore t = \frac{\log(0.2)}{\log(0.965)}$
 ≈ 45.2 years

\therefore it will take about 45.2 years for the weight of the substance to fall below 1 g.

- 8 a i** At 0 seconds, the valve is at its lowest position, closest to the road, so the height of the valve above the road is 5 cm.



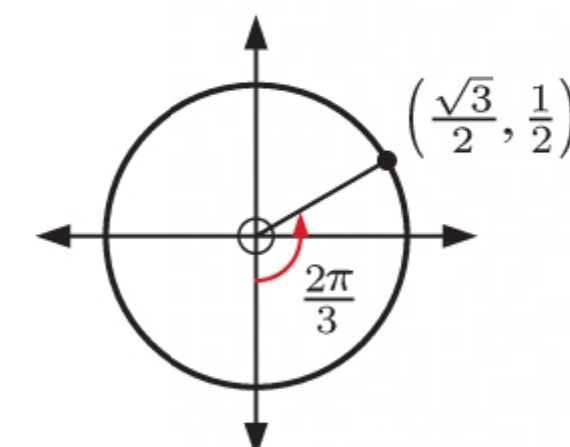
- ii** The wheel rotates at a constant speed of 4 revolutions per second.

\therefore after $\frac{1}{12}$ second, the wheel has rotated $\frac{1}{12} \times 4 = \frac{1}{3}$ revolution.

\therefore the valve will have moved through an angle of $\frac{2\pi}{3}$.

So, the valve will be at a height of $1\frac{1}{2}$ times the radius of the wheel, plus 5 cm.

\therefore the height of the valve above the road $= 1.5 \times 35 + 5$
 $= 57.5$ cm



b $H(t) = a \sin(b(t - c)) + d$ cm

- i** Amplitude = radius of the wheel
 $= 35$ cm

$\therefore a = 35$

- iii** There are 4 revolutions per second, so the period is $\frac{1}{4}$ second.

The period $= \frac{2\pi}{b}$

$\therefore \frac{1}{4} = \frac{2\pi}{b}$

$\therefore b = 8\pi$

- ii** The centre of the wheel is 40 cm above the ground, so the principal axis is at $H = 40$ cm.

$\therefore d = 40$

iv $H(t) = 35 \sin(8\pi(t - c)) + 40$

Now from **a i**, $H(0) = 5$

$\therefore 5 = 35 \sin(8\pi(-c)) + 40$

$\therefore -1 = \sin(8\pi(-c))$

$\therefore -\frac{\pi}{2} = 8\pi(-c)$

$\therefore c = \frac{1}{16}$

c From **b**, $H(t) = 35 \sin(8\pi(t - \frac{1}{16})) + 40$

Now $H(t) = 60$ when $35 \sin(8\pi(t - \frac{1}{16})) + 40 = 60$

$\therefore \sin(8\pi(t - \frac{1}{16})) = \frac{20}{35}$

$\therefore t \approx 0.0867$ {using technology}

It takes approximately 0.0867 seconds for the valve to rise to 60 cm above the road.

- 9** The bin has capacity 500 litres = 500 000 mL

$\therefore \pi r^2 h = 500\,000$

$\therefore h = \frac{500\,000}{\pi r^2}$

Surface area $A = 2\pi r h + \pi r^2$

$= 2\pi r \left(\frac{500\,000}{\pi r^2} \right) + \pi r^2$

$= 1\,000\,000 r^{-1} + \pi r^2$

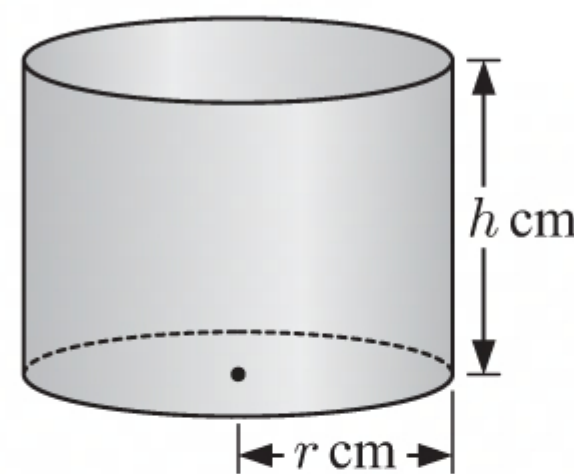
$\therefore \frac{dA}{dr} = -\frac{1\,000\,000}{r^2} + 2\pi r$

Now $\frac{dA}{dr} = 0$ when $2\pi r = \frac{1\,000\,000}{r^2}$

$\therefore 2\pi r^3 = 1\,000\,000$

$\therefore r = \sqrt[3]{\frac{1\,000\,000}{2\pi}} \approx 54.2$

and $h \approx \frac{500\,000}{\pi(54.2)^2} \approx 54.2$

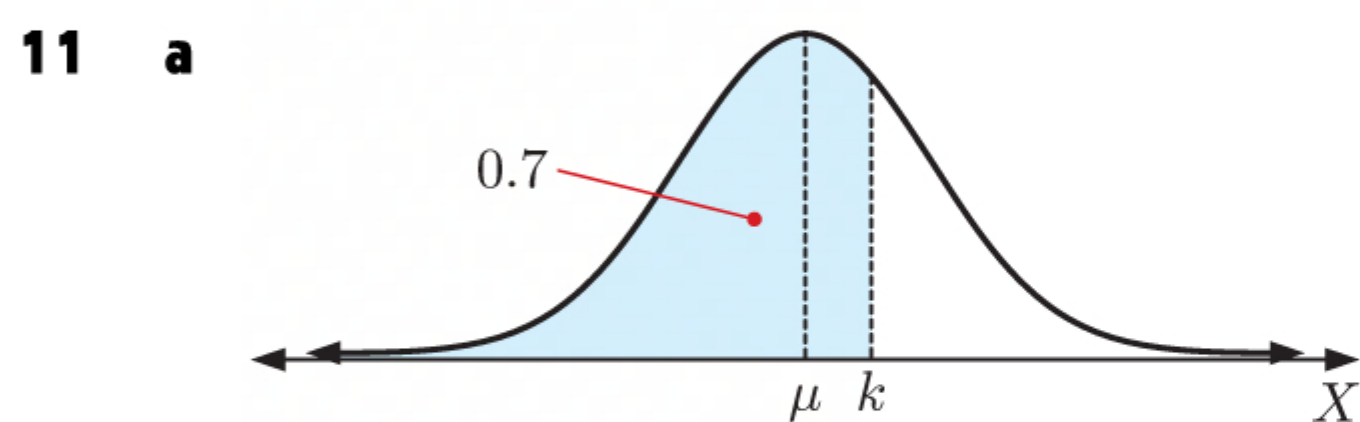
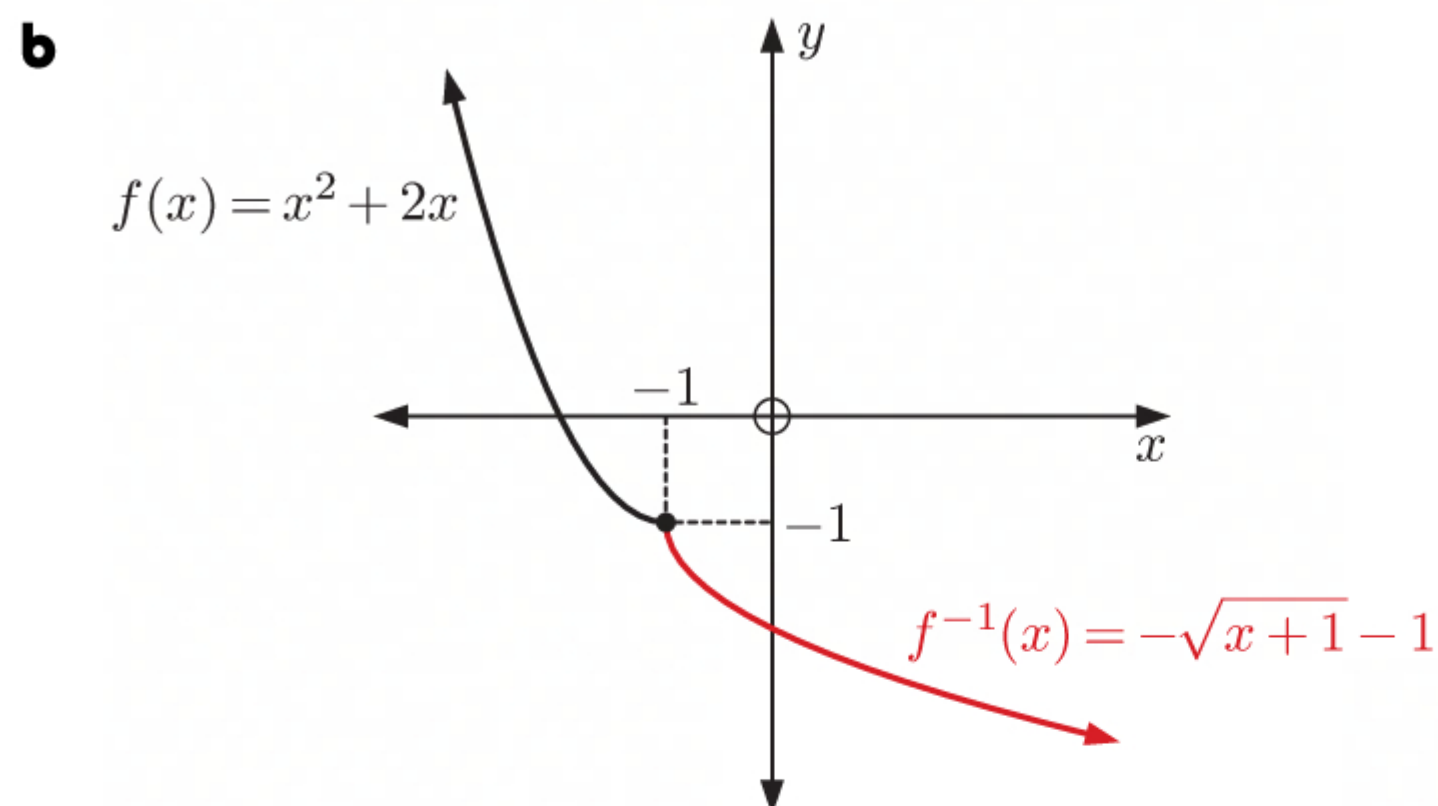


So, the surface area of the bin is minimised when the bin has a base radius and height of about 54.2 cm.

10 a f is $y = x^2 + 2x$, $x \leq -1$
 $\therefore f^{-1}$ is $x = y^2 + 2y$, $y \leq -1$
 $\therefore x + 1 = y^2 + 2y + 1$
 $\therefore x + 1 = (y + 1)^2$
 $\therefore \pm\sqrt{x+1} = y + 1$
 $\therefore y = -\sqrt{x+1} - 1$ {as $y \leq -1$ }
so $f^{-1}(x) = -\sqrt{x+1} - 1$

Now f has domain $\{x \mid x \leq -1\}$ and range $\{y \mid y \geq -1\}$

$\therefore f^{-1}$ has domain $\{x \mid x \geq -1\}$ and range $\{y \mid y \leq -1\}$



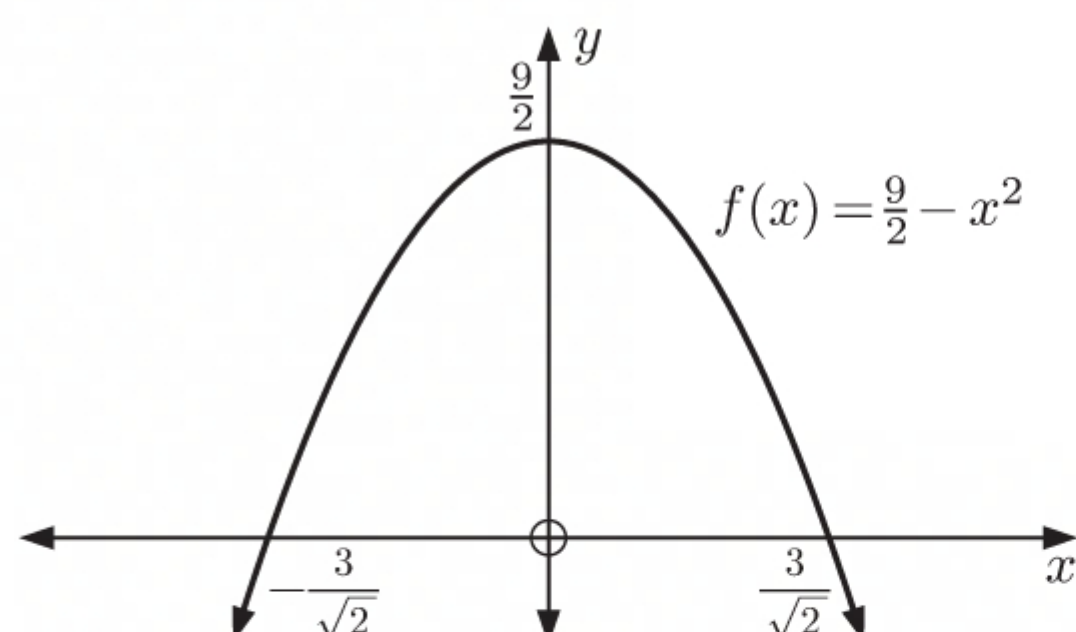
b i $P(X > k) = 1 - P(X < k)$
 $= 1 - 0.7$
 $= 0.3$

ii $P(\mu < X < k) = P(X < k) - P(X \leq \mu)$
 $= 0.7 - 0.5$
 $= 0.2$

iii $P(\mu - \sigma < X < k) = P(\mu - \sigma < X < \mu) + P(\mu < X < k)$
 $\approx 0.341 + 0.2$
 ≈ 0.541

c $P(k \leq X \leq t) = P(X \leq t) - P(X < k)$
 $= 1 - P(X \geq t) - P(X < k)$
 $= 1 - 0.2 - 0.7$
 $= 0.1$

12 a $f(x) = \frac{9}{2} - x^2$
The vertex is $(0, \frac{9}{2})$, and the y -intercept is $\frac{9}{2}$.
When $y = 0$, $\frac{9}{2} - x^2 = 0$
 $\therefore x^2 = \frac{9}{2}$
 $\therefore x = \pm\frac{3}{\sqrt{2}}$



b $f(x) = \frac{9}{2} - x^2$
 $\therefore f'(x) = -2x$
 \therefore the tangent at $P(a, f(a))$ has gradient $-2a$.
 \therefore the normal at $P(a, f(a))$ has gradient $\frac{1}{2a}$.

The equation of normal L is $y = \frac{1}{2a}(x - a) + f(a)$

$$\therefore y = \frac{1}{2a}x - \frac{1}{2} + \frac{9}{2} - a^2$$

$$\therefore y = \frac{1}{2a}x + 4 - a^2$$

c L passes through the origin.

$$\therefore \text{ when } x = 0, \quad y = 0$$

$$\therefore \frac{1}{2a}(0) + 4 - a^2 = 0$$

$$\therefore 4 - a^2 = 0$$

$$\therefore a^2 = 4$$

$$\therefore a = \pm 2$$

When $a = 2$, L has equation $y = \frac{1}{4}x$.

Now L intersects the graph of $y = f(x)$ where $\frac{1}{4}x = \frac{9}{2} - x^2$

$$\therefore x^2 + \frac{1}{4}x - \frac{9}{2} = 0$$

$$\therefore 4x^2 + x - 18 = 0$$

$$\therefore 4x^2 - 8x + 9x - 18 = 0$$

$$\therefore 4x(x - 2) + 9(x - 2) = 0$$

$$\therefore (x - 2)(4x + 9) = 0$$

$$\therefore x = 2 \text{ or } -\frac{9}{4}$$

$$\text{Since } f(x) \geq \frac{1}{4}x \text{ for } -\frac{9}{4} \leq x \leq 2, \quad \text{area} = \int_{-\frac{9}{4}}^2 \left[\left(\frac{9}{2} - x^2 \right) - \frac{1}{4}x \right] dx$$

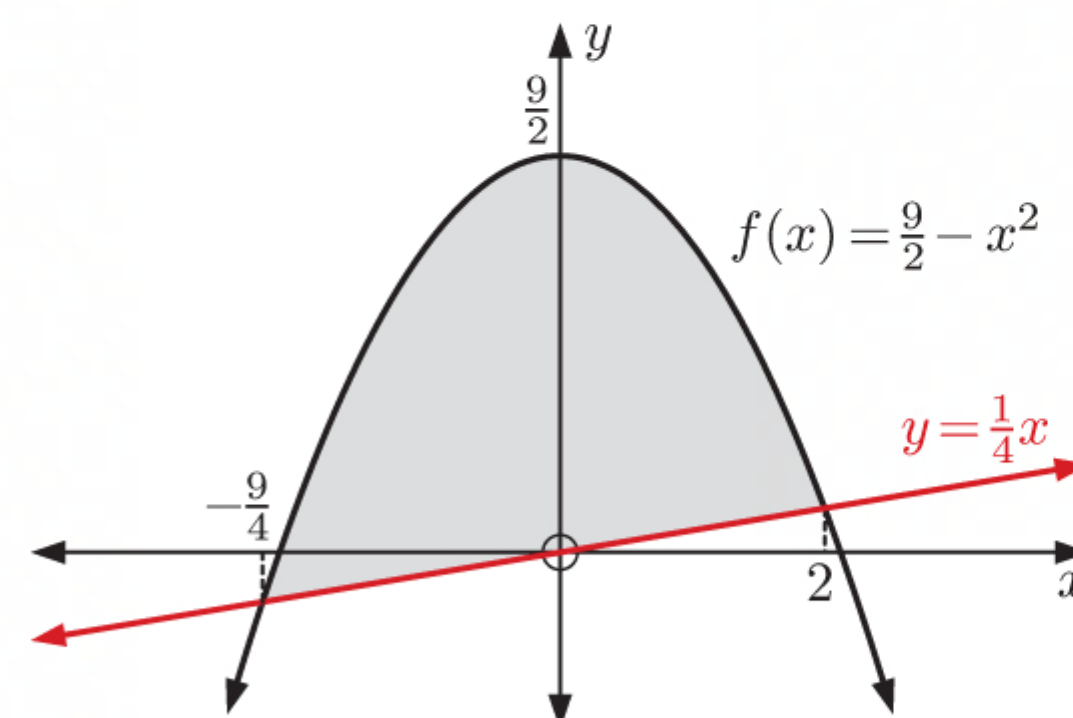
$$= \int_{-\frac{9}{4}}^2 \left(-x^2 - \frac{1}{4}x + \frac{9}{2} \right) dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{8} + \frac{9}{2}x \right]_{-\frac{9}{4}}^2$$

$$= \left(-\frac{8}{3} - \frac{4}{8} + 9 \right) - \left(\frac{243}{64} - \frac{81}{128} - \frac{81}{8} \right)$$

$$= \frac{35}{6} + \frac{891}{128}$$

$$\approx 12.8 \text{ units}^2$$



Now since $y = f(x)$ is symmetric about its axis of symmetry $x = 0$, and L passes through the origin, when $a = -2$, the area enclosed by L and $y = f(x)$ is also $\approx 12.8 \text{ units}^2$.

MIXED QUESTIONS SET 9

1 $f(x) = ax^2 + bx + 7$

a We have $f(2) = 7$ and $f(4) = 23$

$$\therefore 4a + 2b + 7 = 7 \quad 16a + 4b + 7 = 23$$

$$\therefore 4a + 2b = 0 \quad 16a + 4b = 16$$

$$\therefore 2a + b = 0 \quad \dots (1) \quad \therefore 4a + b = 4 \quad \dots (2)$$

b Subtracting (1) from (2), we get $2a = 4$

$$\therefore a = 2$$

Substituting $a = 2$ into (1) gives $2(2) + b = 0$

$$\therefore b = -4$$

c Using **b**, $f(x) = 2x^2 - 4x + 7$

$$\therefore f(-1) = 2(-1)^2 - 4(-1) + 7$$

$$= 2 + 4 + 7$$

$$= 13$$

2 $f(x) = (x^2 + 1)e^{-x}$

$$\therefore f(1) = (1^2 + 1)e^{-1}$$

$$= \frac{2}{e}$$

$$\therefore \text{ the point of contact is } \left(1, \frac{2}{e} \right).$$

Now $f(x) = (x^2 + 1)e^{-x}$ has derivative $f'(x) = 2xe^{-x} - (x^2 + 1)e^{-x}$ {product rule}

$$= e^{-x}(2x - (x^2 + 1))$$

$$\therefore \text{ the tangent at } \left(1, \frac{2}{e} \right) \text{ has gradient } e^{-1}(2 - 2) = 0.$$

$$\therefore \text{ the tangent is horizontal and has equation } y = \frac{2}{e}.$$

- 3 a** Let u_n km be the distance Hayley cycled in the n th week, and v_n km be the distance Patrick cycled in the n th week.

Hayley cycled an additional 20 km each week.

$$\therefore u_n = 60 + 20(n - 1)$$

$$\therefore u_5 = 60 + 20 \times 4 = 140$$

So, Hayley cycled 140 km in the 5th week of training.

Patrick increased his distance by 20% each week.

$$\begin{aligned}\therefore v_n &= 60(1 + 0.2)^{n-1} \quad \{20\% = 0.2\} \\ &= 60(1.2)^{n-1}\end{aligned}$$

$$\therefore v_5 = 60(1.2)^4 \approx 124$$

So, Patrick cycled about 124 km in the 5th week of training.

$$\begin{aligned}\mathbf{b} \quad u_n &= 210 \quad \text{where} \quad 60 + 20(n - 1) = 210 \quad \text{and} \quad v_n = 210 \quad \text{where} \quad 60(1.2)^{n-1} = 210 \\ &\therefore 20(n - 1) = 150 & \therefore (1.2)^{n-1} = \frac{7}{2} \\ &\therefore n - 1 = 7.5 & \therefore \log(1.2)^{n-1} = \log\left(\frac{7}{2}\right) \\ &\therefore n = 8.5 & \therefore (n - 1) \log(1.2) = \log\left(\frac{7}{2}\right) \\ & & \therefore n - 1 = \frac{\log(\frac{7}{2})}{\log(1.2)} \\ & & \therefore n = 1 + \frac{\log(\frac{7}{2})}{\log(1.2)} \approx 7.87\end{aligned}$$

So, Hayley first cycled 210 km in the 9th week, and Patrick first cycled 210 km in the 8th week.

\therefore Patrick was the first to cycle 210 km in one week.

- c** u_n is an arithmetic sequence with $u_1 = 60$ and $d = 20$.

$$\begin{aligned}\therefore \text{the total distance Hayley cycled in the first 12 weeks} &= \frac{n}{2}(2u_1 + (n - 1)d) \\ &= \frac{12}{2}(2 \times 60 + 11 \times 20) \\ &= 2040 \text{ km}\end{aligned}$$

v_n is a geometric sequence with $v_1 = 60$ and $r = 1.2$.

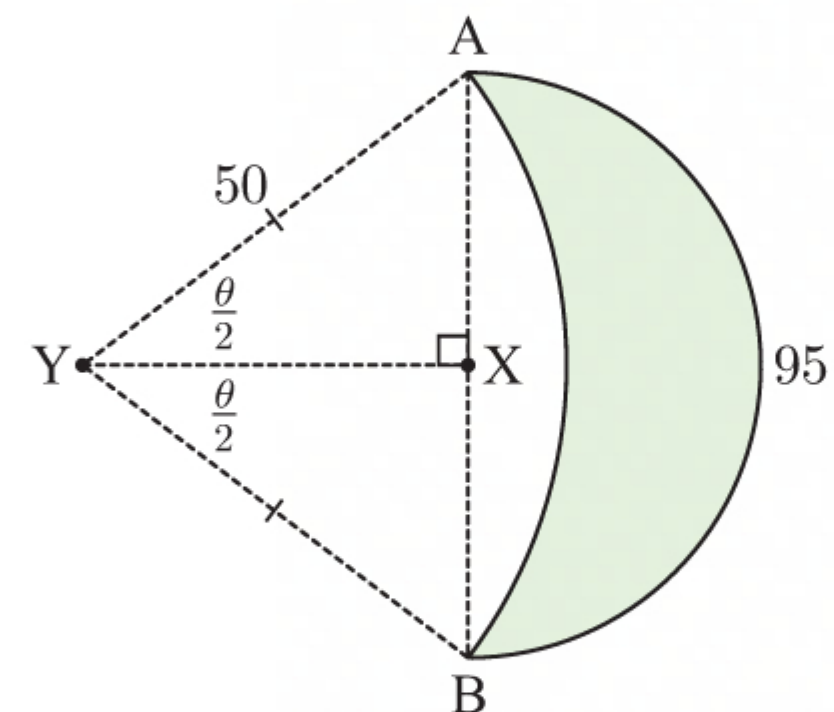
$$\begin{aligned}\therefore \text{the total distance Patrick cycled in the first 12 weeks} &= \frac{v_1(1 - r^n)}{1 - r} \\ &= \frac{60(1 - (1.2)^{12})}{1 - 1.2} \\ &\approx 2375 \text{ km}\end{aligned}$$

So, Patrick cycled a greater total distance in the first 12 weeks.

- 4 a** The circle centred at X has radius [AX].

The length of the large arc [AB] is half of the circumference.

$$\begin{aligned}\therefore \frac{1}{2}(2 \times \pi \times AX) &= 95 \\ \therefore \pi \times AX &= 95 \\ \therefore AX &= \frac{95}{\pi} \text{ units} \\ &\approx 30.2 \text{ units}\end{aligned}$$



$$\begin{aligned}\mathbf{b} \quad \text{In } \triangle AXY, \quad \sin \frac{\theta}{2} &= \frac{AX}{AY} \\ &= \frac{\frac{95}{\pi}}{50} \\ &= \frac{19}{10\pi} \\ \therefore \frac{\theta}{2} &= \sin^{-1}\left(\frac{19}{10\pi}\right) \\ \therefore \theta &= 2 \sin^{-1}\left(\frac{19}{10\pi}\right) \\ &\approx 1.30\end{aligned}$$

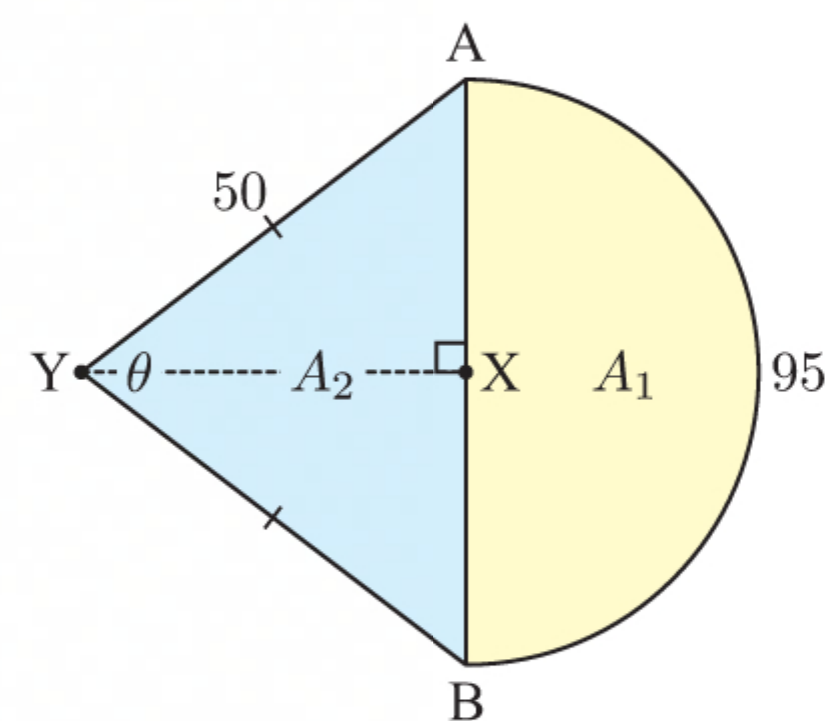
- c** We first divide the figure into areas A_1 and A_2 .

Now A_1 = area of semi-circle centred at X

$$\begin{aligned} &= \frac{1}{2} \times \pi \times \left(\frac{95}{\pi}\right)^2 \quad \{\text{from a}\} \\ &= \frac{9025}{2\pi} \text{ units}^2 \end{aligned}$$

and A_2 = area of $\triangle ABY$

$$\begin{aligned} &= \frac{1}{2} \times 50 \times 50 \times \sin \theta \\ &= 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) \quad \{\text{from b}\} \\ &= 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) \text{ units}^2 \end{aligned}$$



So, total area of figure = $A_1 + A_2$

$$= \frac{9025}{2\pi} + 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) \text{ units}^2$$

Now, shaded area = total area of figure – sector ABY

$$\begin{aligned} &= \frac{9025}{2\pi} + 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) - \frac{1}{2} \times \theta \times 50^2 \\ &= \frac{9025}{2\pi} + 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) - 1250 \times 2 \sin^{-1} \left(\frac{19}{10\pi} \right) \\ &= \frac{9025}{2\pi} + 1250 \times \sin \left(2 \sin^{-1} \left(\frac{19}{10\pi} \right) \right) - 2500 \sin^{-1} \left(\frac{19}{10\pi} \right) \\ &\approx 1020 \text{ units}^2 \end{aligned}$$

5 $f(x) = \log_3(x+1) + 2$

- a** A translation through the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ maps $y = \log_3 x$ to $y = f(x)$.

- b** $\log_3(x+1)$ is defined when $x+1 > 0$, that is, when $x > -1$.

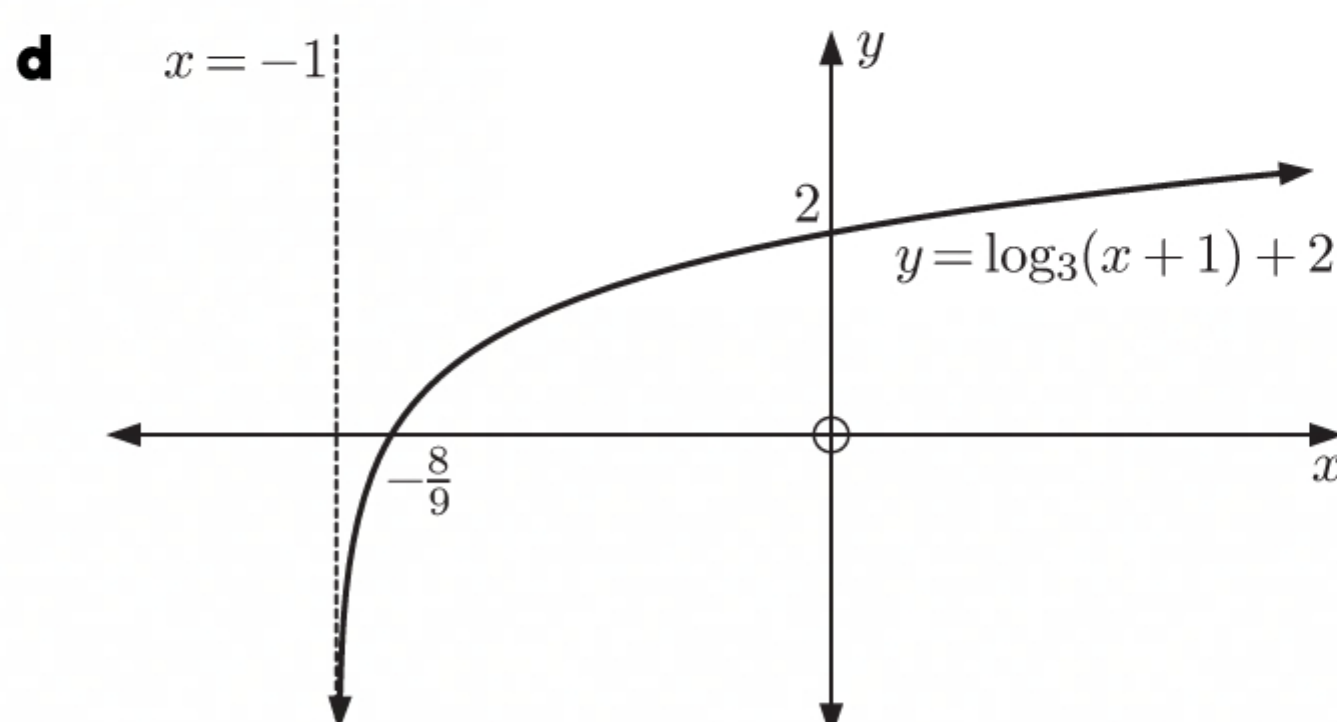
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

- c** When $x = 0$, $\log_3 1 + 2 = 2$, so the y -intercept is 2.

When $y = 0$, $\log_3(x+1) + 2 = 0$

$$\begin{aligned} \therefore \log_3(x+1) &= -2 \\ \therefore x+1 &= 3^{-2} \\ \therefore x+1 &= \frac{1}{9} \\ \therefore x &= -\frac{8}{9} \end{aligned}$$

So, the x -intercept is $-\frac{8}{9}$.



e

$$\begin{aligned} f &\text{ is } y = \log_3(x+1) + 2 \\ \therefore f^{-1} &\text{ is } x = \log_3(y+1) + 2 \\ \therefore x-2 &= \log_3(y+1) \\ \therefore 3^{x-2} &= 3^{\log_3(y+1)} \\ \therefore 3^{x-2} &= y+1 \\ \therefore y &= 3^{x-2} - 1 \\ \therefore f^{-1}(x) &= 3^{x-2} - 1 \end{aligned}$$

6 $(x-1)(2-x)^9 = -(x-1)(x-2)^9$

$$\begin{aligned} &= -(x-1) \left[x^9 + \binom{9}{1}x^8(-2) + \binom{9}{2}x^7(-2)^2 + \binom{9}{3}x^6(-2)^3 + \dots \right] \\ &= -(x-1) \left[x^9 + \binom{9}{1}(-2)x^8 + \binom{9}{2}(-2)^2x^7 + \binom{9}{3}(-2)^3x^6 + \dots \right] \end{aligned}$$

$\xrightarrow{\quad (2) \quad} \xrightarrow{\quad (1) \quad}$

So, the terms containing x^7 are $-\binom{9}{3}(-2)^3x^7$ from (1)

and $\binom{9}{2}(-2)^2x^7$ from (2)

\therefore the coefficient of x^7 is $-\binom{9}{3}(-2)^3 + \binom{9}{2}(-2)^2 = 816$.

7 The ordered data set is:

132 140 149 155 159 160 161 163 164 165 (20 data values)
169 171 173 181 185 191 200 207 212 303

a Since $n = 20$, $\frac{n+1}{2} = 10.5$ \therefore the median is the average of the 10th and 11th value.

~~132 140 149 155 159 160 161 163 164 165~~
~~169 171 173 181 185 191 200 207 212 303~~

$$\begin{aligned}\therefore \text{median} &= \frac{10\text{th value} + 11\text{th value}}{2} \\ &= \frac{\$165 + \$169}{2} \\ &= \$167\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set in two.

lower half
132 140 149 155 159 160 161 163 164 165
169 171 173 181 185 191 200 207 212 303
upper half

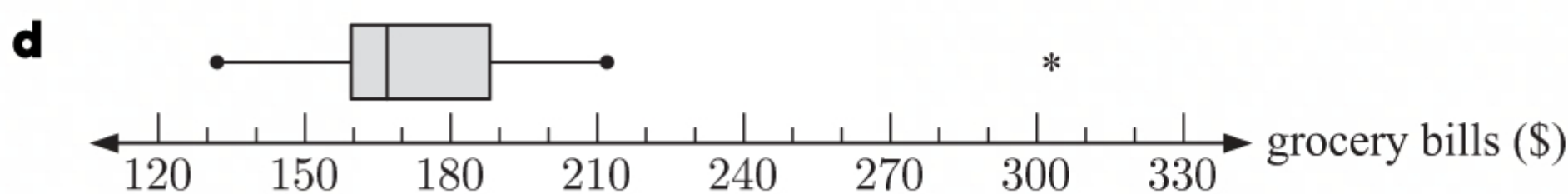
$$Q_1 = \text{median of lower half} = \frac{\$159 + \$160}{2} = \$159.50$$

$$Q_3 = \text{median of upper half} = \frac{\$185 + \$191}{2} = \$188$$

b $\text{IQR} = Q_3 - Q_1$
 $= \$188 - \159.50
 $= \$28.50$

c *Test for outliers:* upper boundary and lower boundary
 $= \text{upper quartile} + 1.5 \times \text{IQR}$ $= \text{lower quartile} - 1.5 \times \text{IQR}$
 $= \$188 + 1.5 \times 28.50$ $= \$159.50 - 1.5 \times 28.50$
 $= \$230.75$ $= \$116.75$

\$303 is above the upper boundary, so it is an outlier.

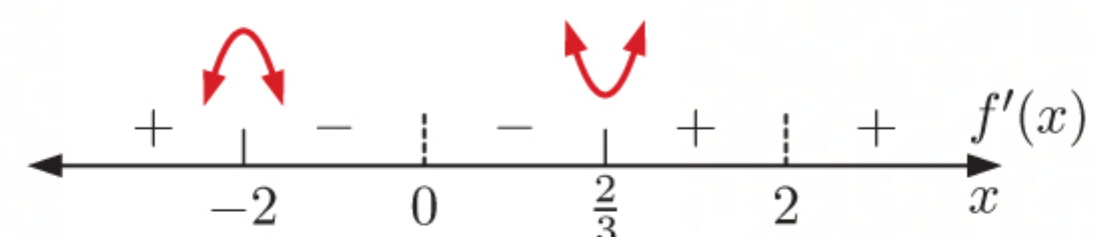


8 a $f(x) = \frac{1}{x} - \frac{4}{x-2}$
 $= \frac{x-2-4x}{x(x-2)}$
 $= \frac{-3x-2}{x(x-2)}$

So, $f(x) = 0$ when $-3x - 2 = 0$
 $\therefore x = -\frac{2}{3}$

b $f(x) = x^{-1} - 4(x-2)^{-1}$
 $\therefore f'(x) = -x^{-2} + 4(x-2)^{-2}$
 $= -\frac{1}{x^2} + \frac{4}{(x-2)^2}$
 $= \frac{4x^2 - (x-2)^2}{x^2(x-2)^2}$
 $= \frac{(2x + (x-2))(2x - (x-2))}{x^2(x-2)^2}$
 $= \frac{(3x-2)(x+2)}{x^2(x-2)^2}$

$f'(x)$ has sign diagram:



$$f(-2) = \frac{1}{2} \quad \text{and} \quad f\left(\frac{2}{3}\right) = 4\frac{1}{2}$$

\therefore there is a local maximum at $(-2, \frac{1}{2})$ and a local minimum at $(\frac{2}{3}, 4\frac{1}{2})$.

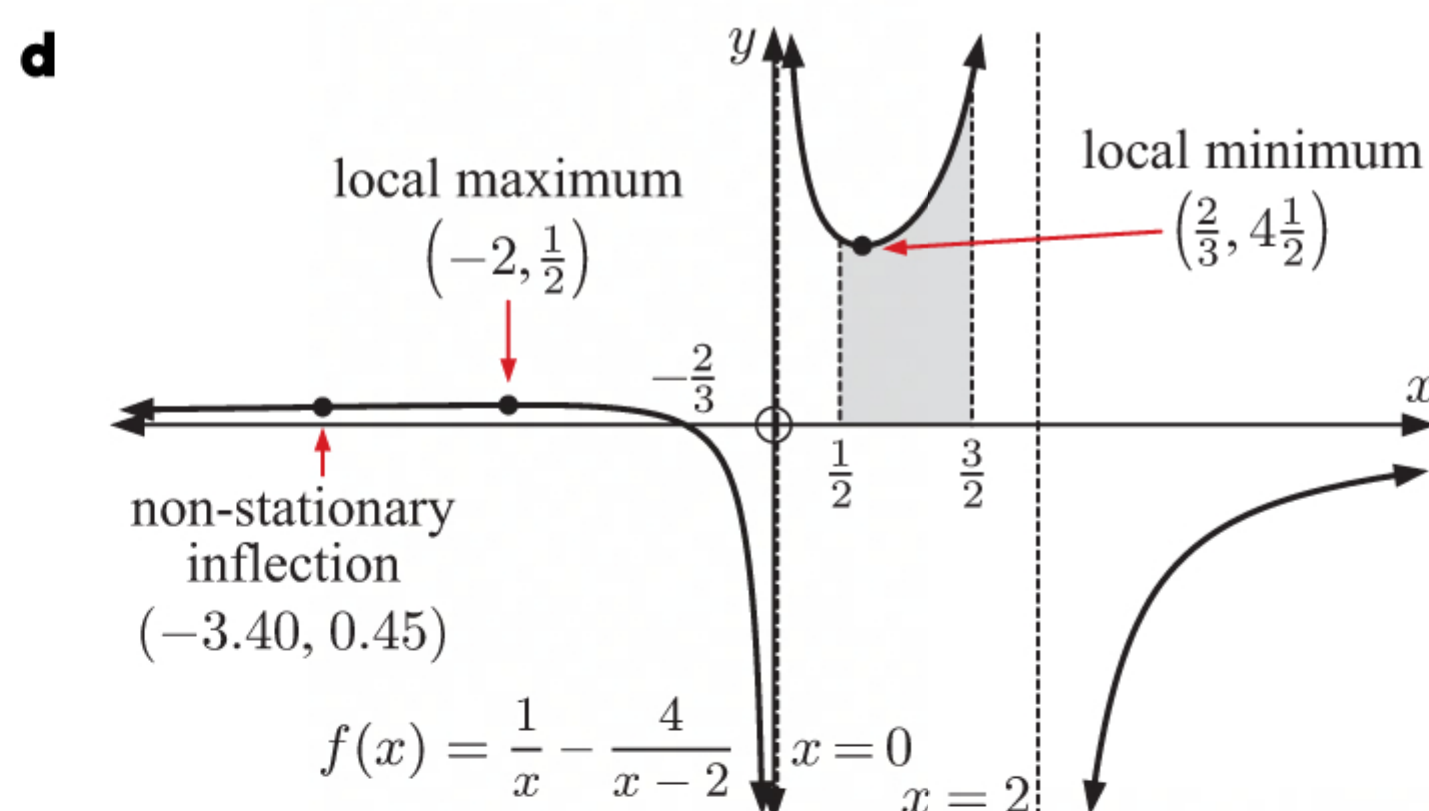
$$\begin{aligned} \text{c } f''(x) &= 2x^{-3} - 8(x-2)^{-3} \\ &= \frac{2}{x^3} - \frac{8}{(x-2)^3} \end{aligned}$$

Points of inflection occur where $f''(x) = 0$

$$\begin{aligned} \therefore \frac{2}{x^3} &= \frac{8}{(x-2)^3} \\ \therefore (x-2)^3 &= 4x^3 \\ \therefore x &\approx -3.4048 \quad \{\text{using technology}\} \end{aligned}$$

$$f(-3.4048) \approx 0.4464$$

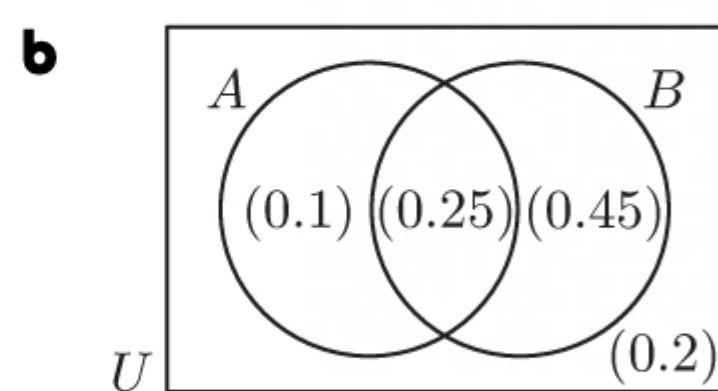
So, $(-3.40, 0.45)$ is a non-stationary point of inflection.



$$\begin{aligned} \text{e Area} &= \int_{\frac{1}{2}}^{\frac{3}{2}} \left(\frac{1}{x} - \frac{4}{x-2} \right) dx \\ &= \left[\ln|x| - 4 \ln|x-2| \right]_{\frac{1}{2}}^{\frac{3}{2}} \\ &= (\ln \frac{3}{2} - 4 \ln \frac{1}{2}) - (\ln \frac{1}{2} - 4 \ln \frac{3}{2}) \\ &= \ln \frac{3}{2} - 4 \ln \frac{1}{2} - \ln \frac{1}{2} + 4 \ln \frac{3}{2} \\ &= 5 \ln \frac{3}{2} - 5 \ln \frac{1}{2} \\ &= 5(\ln \frac{3}{2} - \ln \frac{1}{2}) \\ &= 5 \ln 3 \text{ units}^2 \end{aligned}$$

9 $P(A) = 0.35$, $P(B) = 0.7$, $P(A \cup B) = 0.8$

a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $\therefore 0.8 = 0.35 + 0.7 - P(A \cap B)$
 $\therefore P(A \cap B) = 1.05 - 0.8 = 0.25$



c i $P(A' \cap B') = 0.2$

ii $P(A | B) = \frac{P(A \cap B)}{P(B)}$
 $= \frac{0.25}{0.7}$
 $= \frac{5}{14} \approx 0.357$

d From **c ii**, $P(A | B) \neq P(A) = 0.35$.

So, events A and B are not independent.

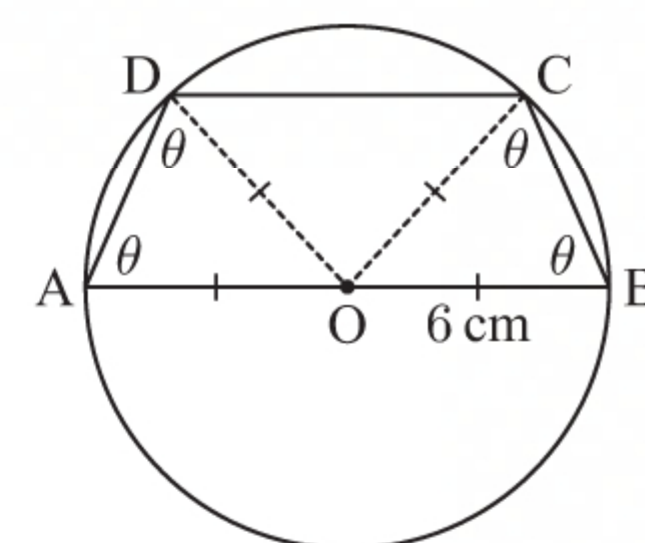
10 a $\widehat{ADO} = \theta$ and $\widehat{OCB} = \theta$ $\{\triangle ADO \text{ and } \triangle OCB \text{ are isosceles triangles}\}$

Now $\widehat{AOD} = \pi - 2\theta$ $\{\text{angles in } \triangle ADO\}$

$\widehat{COB} = \pi - 2\theta$ $\{\text{angles in } \triangle OCB\}$

$$\begin{aligned} \therefore \widehat{DOC} &= \pi - 2(\pi - 2\theta) \quad \{\text{angles in a line}\} \\ &= \pi - 2\pi + 4\theta \\ &= 4\theta - \pi \end{aligned}$$

$$\begin{aligned} \therefore \text{area } A &= 2 \times \frac{1}{2}(6)^2 \sin(\pi - 2\theta) + \frac{1}{2}(6)^2 \sin(4\theta - \pi) \\ &= 36 \sin(\pi - 2\theta) + 18 \sin(4\theta - \pi) \\ &= 36 \sin 2\theta - 18 \sin 4\theta \quad \{\sin(\pi - x) = \sin x \text{ and } \sin(x - \pi) = -\sin x\} \\ &= 18(2 \sin 2\theta - \sin 4\theta) \end{aligned}$$



$$\mathbf{b} \quad \frac{dA}{d\theta} = 18(4 \cos 2\theta - 4 \cos 4\theta)$$

$$\text{Now } \frac{dA}{d\theta} = 0 \quad \text{where} \quad 18(4 \cos 2\theta - 4 \cos 4\theta) = 0$$

$$\therefore 4 \cos 2\theta - 4 \cos 4\theta = 0$$

$$\therefore \cos 2\theta - \cos 4\theta = 0$$

$$\therefore \cos 2\theta - (2 \cos^2 2\theta - 1) = 0$$

$$\therefore 2 \cos^2 2\theta - \cos 2\theta - 1 = 0$$

$$\therefore 2 \cos^2 2\theta - 2 \cos 2\theta + \cos 2\theta - 1 = 0$$

$$\therefore 2 \cos 2\theta (\cos 2\theta - 1) + (\cos 2\theta - 1) = 0$$

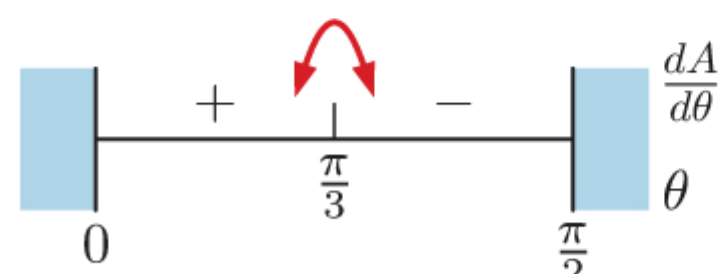
$$\therefore (\cos 2\theta - 1)(2 \cos 2\theta + 1) = 0$$

$$\therefore \cos 2\theta = 1 \quad \text{or} \quad \cos 2\theta = -\frac{1}{2}$$

$$\therefore 2\theta = 0 \quad \text{or} \quad 2\theta = \frac{2\pi}{3}$$

$$\therefore \theta = 0 \quad \text{or} \quad \theta = \frac{\pi}{3}$$

The sign diagram of $\frac{dA}{d\theta}$ is:



$\therefore A$ is maximised when $\theta = \frac{\pi}{3}$.

$\therefore \theta = \frac{\pi}{3}$ maximises the area of ABCD.

$$\mathbf{11} \quad f(x) = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}, \quad 0 \leq x \leq \frac{\pi}{2}$$

$$\begin{aligned} \mathbf{a} \quad f(x) &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\ &= \frac{1}{\sin x \cos x} \times \frac{2}{2} \\ &= \frac{2}{\sin 2x} \end{aligned}$$

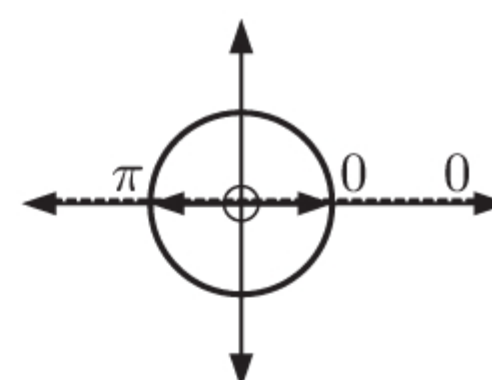
$$\mathbf{b} \quad \sin 2x = 0$$

$$\text{Now } 0 \leq x \leq \frac{\pi}{2}$$

$$\therefore 0 \leq 2x \leq \pi$$

$$\text{So, } 2x = 0 \quad \text{or} \quad \pi$$

$$\therefore x = 0 \quad \text{or} \quad \frac{\pi}{2}$$



\therefore the vertical asymptotes of $f(x)$ are $x = 0$ and $x = \frac{\pi}{2}$.

\mathbf{c} The greatest value of $\sin 2x$ on $0 \leq x \leq \frac{\pi}{2}$ is 1.

\therefore the least value of $f(x)$ is $\frac{2}{1} = 2$.

This occurs when $\sin 2x = 1$

$$\therefore 2x = \frac{\pi}{2} \quad \{0 \leq 2x \leq \pi\}$$

$$\therefore x = \frac{\pi}{4}$$

$$\mathbf{d} \quad \sin a = \frac{1}{3}$$

$$\therefore a = \sin^{-1}\left(\frac{1}{3}\right)$$

$$\begin{aligned} \therefore f(2a) &= \frac{2}{\sin(4a)} \\ &= \frac{2}{\sin\left(4 \sin^{-1}\left(\frac{1}{3}\right)\right)} \\ &\approx 2.046 \end{aligned}$$

12	Time spent training (t hours)	2.5	1	2.5	3.5	4	2.5	2	3	3	2	1.5
	Points scored (y)	2	0	5	16	9	8	2	6	10	0	2

\mathbf{a} The number of points scored can be counted exactly, whereas the time spent training is usually estimated or cannot be measured exactly.

Since the response variable y is more precisely measured than the explanatory variable t , it would be appropriate to use the regression line of t against y in this case.

$$\mathbf{b} \quad \begin{array}{l} \text{LinearReg}(ax+b) \\ a = 0.1398305 \\ b = 1.73728813 \\ r = 0.80201018 \\ r^2 = 0.64322033 \\ \text{MSe} = 0.29731638 \\ y = ax + b \end{array}$$

The regression line of t against y is $t \approx 0.140y + 1.74$ hours.

$\mathbf{c} \quad \mathbf{i}$ When $y = 7$, $t \approx 0.140(7) + 1.74$
 ≈ 2.72

We expect a player who scored 7 points to have spent about 2.72 hours training.

$$\begin{aligned}\text{ii When } t = 5, \quad 5 &\approx 0.140y + 1.74 \\ \therefore 3.26 &\approx 0.140y \\ \therefore y &\approx 23.3\end{aligned}$$

We expect a player who spent 5 hours training to score about 23 points.

d We expect that the estimate in **c i** to be reliable because it is an interpolation.

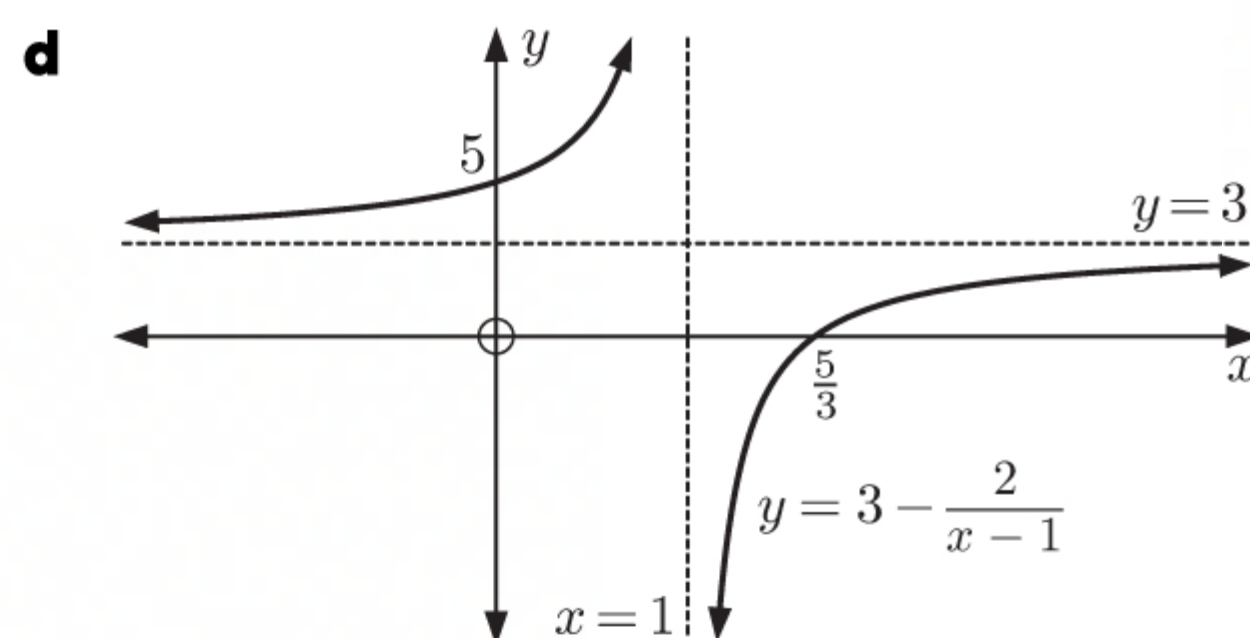
The estimate in **c ii** is not likely to be reliable because it is an extrapolation and the linear model may not extend beyond the poles.

MIXED QUESTIONS SET 10

$$\begin{aligned}\text{1 a } y = 3 - \frac{k}{x-1} \text{ has } x\text{-intercept } \frac{5}{3} \\ \therefore 3 - \frac{k}{\frac{5}{3}-1} &= 0 \\ \therefore 3 - \frac{k}{\frac{2}{3}} &= 0 \\ \therefore \frac{3k}{2} &= 3 \\ \therefore k &= 2\end{aligned}$$

c The vertical asymptote is $x = 1$.
The horizontal asymptote is $y = 3$.

$$\begin{aligned}\text{b The } y\text{-intercept occurs where } x &= 0 \\ \therefore y &= 3 - \frac{2}{0-1} \\ &= 3 + 2 \\ &= 5 \\ \therefore \text{the } y\text{-intercept is } 5.\end{aligned}$$

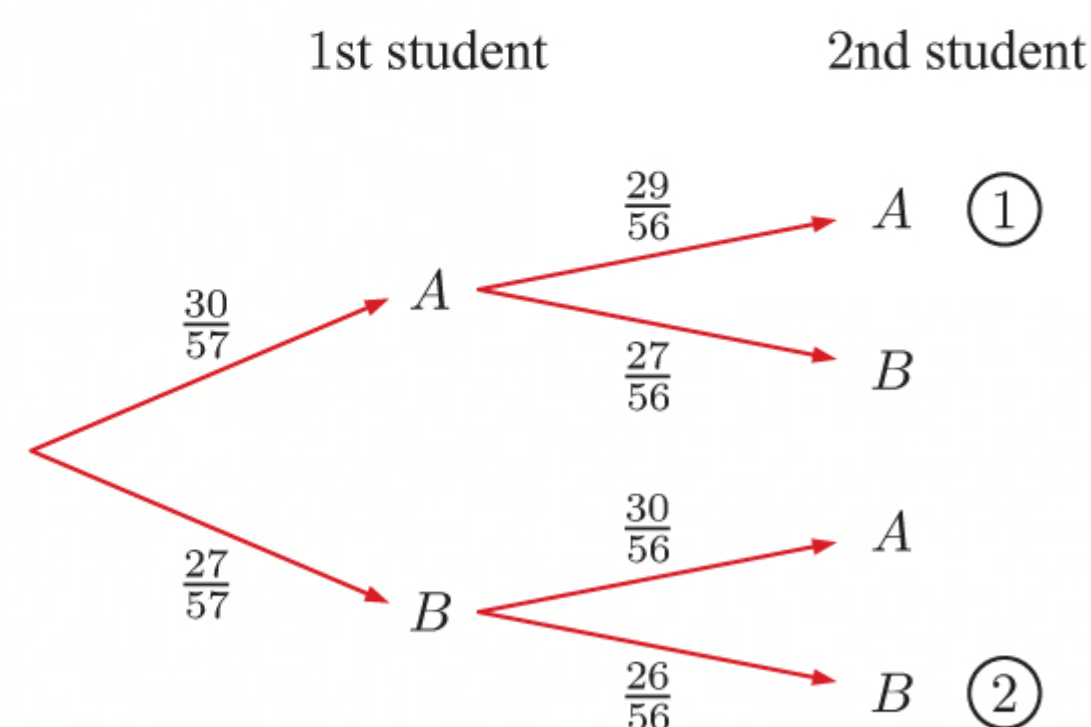


$$\begin{aligned}\text{2 a } 0.\overline{34} &= 0.343\,434\, \dots \\ &= \frac{34}{10^2} + \frac{34}{10^4} + \frac{34}{10^6} + \dots \\ \text{is an infinite geometric series with } u_1 &= \frac{34}{100} \text{ and } r = \frac{1}{100}.\end{aligned}$$

$$\begin{aligned}\text{b } S &= \frac{u_1}{1-r} \\ &= \frac{\frac{34}{100}}{1 - \frac{1}{100}} \\ &= \frac{34}{99} \\ \therefore 0.\overline{34} &= \frac{34}{99}\end{aligned}$$

3 a Total number of Year 7 students = $30 + 27 = 57$
Let A represent a student selected from class A and
 B represent a student selected from class B.

$$\begin{aligned}P(\text{same class}) &= P(AA \text{ or } BB) \\ &= \underbrace{\frac{30}{57} \times \frac{29}{56}}_{\textcircled{1}} + \underbrace{\frac{27}{57} \times \frac{26}{56}}_{\textcircled{2}} \\ &= \frac{131}{266} \\ &\approx 0.492\end{aligned}$$



\therefore the probability that in any given week the two selected students are in the same class is $\frac{131}{266} \approx 0.492$.

b Let X be the number of weeks out of 20 that the two selected students are in the same class.

$$\therefore X \sim B\left(20, \frac{131}{266}\right) \quad \{\text{using a}\}$$

$$\begin{aligned}E(X) &= np \\ &= 20 \times \frac{131}{266} \\ &\approx 9.85\end{aligned}$$

\therefore we expect that the two selected students are in the same class about 9.85 times out of 20.

$$\begin{aligned}
 4 \quad 2^a 8^b &= \frac{1}{2} & \text{and} & \quad \frac{3^{-a}}{3^{b+1}} = 9 \\
 \therefore 2^a (2^3)^b &= 2^{-1} & \therefore 3^{-a} 3^{-(b+1)} &= 3^2 \\
 \therefore 2^{a+3b} &= 2^{-1} & \therefore 3^{-a-b-1} &= 3^2 \\
 \therefore a+3b &= -1 \quad \dots (1) & \therefore -a-b-1 &= 2 \\
 & & \therefore a &= -3-b \quad \dots (2)
 \end{aligned}$$

Substituting (2) into (1), $-3-b+3b = -1$

$$\begin{aligned}
 \therefore 2b &= 2 \\
 \therefore b &= 1 \\
 \therefore a &= -4 \quad \{\text{using (2)}\}
 \end{aligned}$$

- 5 a Let θ be the angle that L_1 makes with the positive x -axis.

$$\begin{aligned}
 \therefore \tan \theta &= \frac{3}{4} \\
 \therefore \theta &= \tan^{-1}\left(\frac{3}{4}\right) \\
 &\approx 36.9^\circ
 \end{aligned}$$

Let ϕ be the angle that L_2 makes with the positive x -axis.

$$\begin{aligned}
 \therefore \tan \phi &= -1 \\
 \therefore \phi &= 135^\circ
 \end{aligned}$$

- b Let α be the angle between L_1 and L_2 shown.

Using vertically opposite angles and the exterior angle of a triangle theorem,

$$\begin{aligned}
 \alpha + \theta &= \phi \\
 \therefore \alpha &\approx 135^\circ - 36.9^\circ \\
 &\approx 98.1^\circ
 \end{aligned}$$

\therefore the acute angle between L_1 and $L_2 \approx 180^\circ - 98.1^\circ \approx 81.9^\circ$.

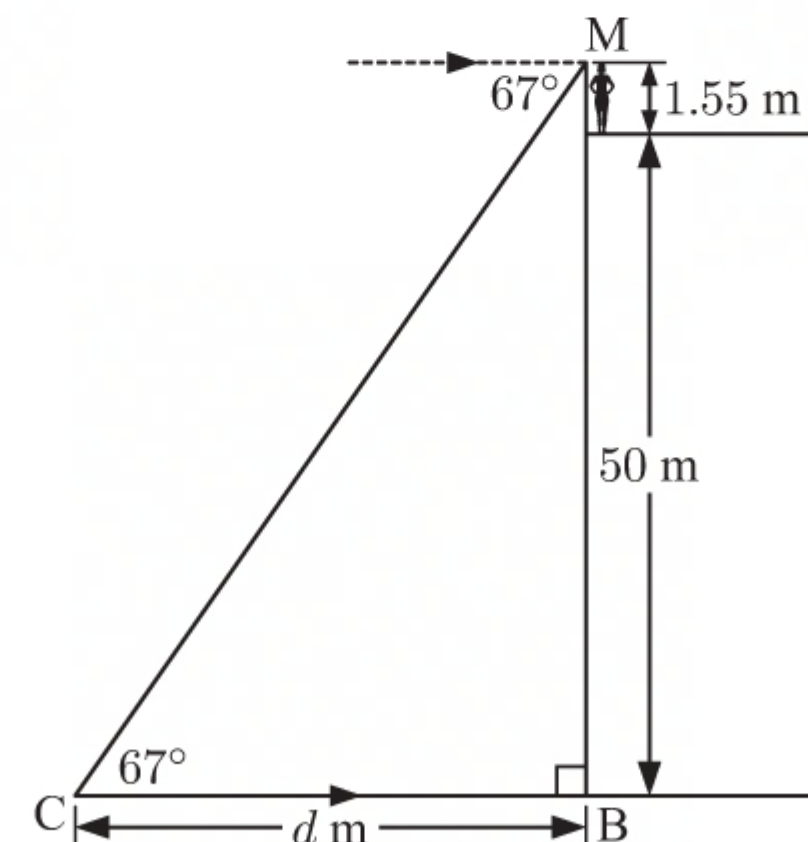
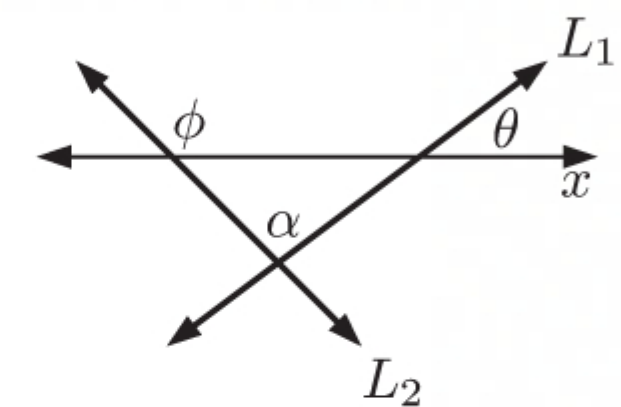
- 6 a Let Maggie's eye level be at M, the car be at C, and the base of the building be at B.

$$\begin{aligned}
 MB &= \text{Maggie's height} + \text{building height} \\
 &= 51.55 \text{ m}
 \end{aligned}$$

Now $\widehat{MCB} = 67^\circ$ {alternate angles}

$$\begin{aligned}
 \therefore \tan 67^\circ &= \frac{51.55}{d} \\
 \therefore d &= \frac{51.55}{\tan 67^\circ} \approx 21.9
 \end{aligned}$$

So the car is about 21.9 m away from the base of the building.



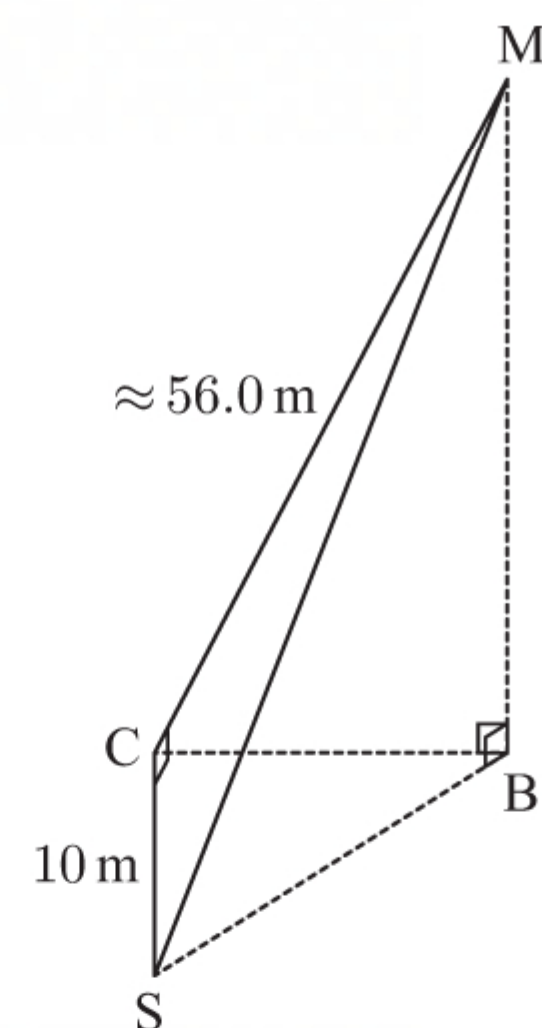
- b Let S be Sven's location.

$$\begin{aligned}
 \text{i In } \triangle MBC, \quad \sin 67^\circ &= \frac{51.55}{MC} \\
 \therefore MC &= \frac{51.55}{\sin 67^\circ} \\
 &\approx 56.0 \text{ m}
 \end{aligned}$$

Since the car is directly opposite to Maggie, $\triangle MCS$ is right angled at C.

$$\begin{aligned}
 \therefore MS^2 &\approx 10^2 + 56.0^2 \quad \{\text{Pythagoras}\} \\
 \therefore MS &\approx \sqrt{3236} \approx 56.9 \text{ m}
 \end{aligned}$$

The distance between Maggie and Sven is about 56.9 m.



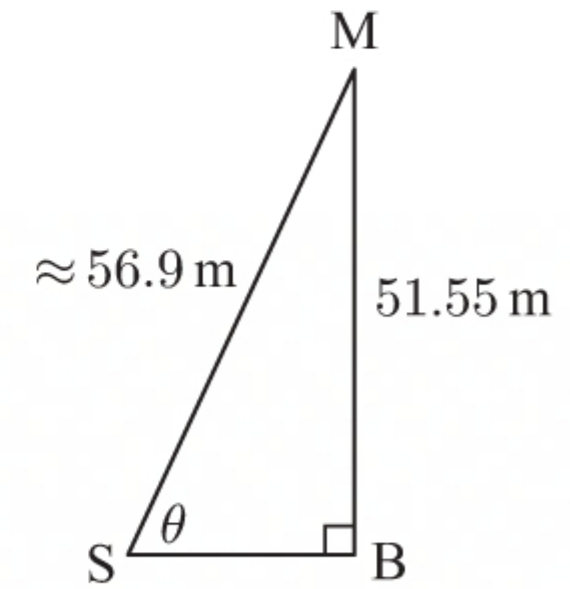
- ii Let θ be the angle of elevation from Sven to Maggie.

Now $\triangle MBS$ is right angled at B.

$$\therefore \sin \theta \approx \frac{51.55}{56.9}$$

$$\therefore \theta \approx \sin^{-1}\left(\frac{51.55}{56.9}\right) \approx 65.0^\circ$$

Sven needs to look up at an angle of about 65.0° to see Maggie.



7 a f is $y = x - 2$

$$\therefore f^{-1} \text{ is } x = y - 2$$

$$\therefore x + 2 = y$$

$$\therefore f^{-1}(x) = x + 2$$

b $(g \circ f)(x) = g(f(x))$

$$= g(x - 2)$$

$$= 3 - (x - 2) - 2(x - 2)^2$$

$$= 3 - x + 2 - 2(x^2 - 4x + 4)$$

$$= -2x^2 + 7x - 3$$

c Using **b**, $(g \circ f)(-1) = -2(-1)^2 + 7(-1) - 3$
 $= -12$

8 $v = \frac{20}{\sqrt{2t+1}} \text{ m s}^{-1}, \quad 0 \leq t \leq 10$

a The brakes are applied when $t = 0$

$$\therefore v = \frac{20}{\sqrt{0+1}}$$

$$= 20 \text{ m s}^{-1}$$

\therefore the speed of the truck when the brakes are applied is 20 m s^{-1} .

b $v = 20(2t+1)^{-\frac{1}{2}} \text{ m s}^{-1}$

Now $a = \frac{dv}{dt}$

$$= 20 \times \left(-\frac{1}{2}\right)(2t+1)^{-\frac{3}{2}}(2) \quad \{\text{chain rule}\}$$

$$= \frac{-20}{(2t+1)^{\frac{3}{2}}} \text{ m s}^{-2}$$

c The truck has acceleration $a = -2.5 \text{ m s}^{-1}$ when

$$\frac{-20}{(2t+1)^{\frac{3}{2}}} = -2.5$$

$$\therefore (2t+1)^{\frac{3}{2}} = \frac{20}{2.5} = 8$$

$$\therefore 2t+1 = 8^{\frac{2}{3}} = 4$$

$$\therefore 2t = 3$$

$$\therefore t = \frac{3}{2} \text{ seconds}$$

d Distance travelled $= \int_0^{10} |v| dt$

$$= \int_0^{10} \frac{20}{\sqrt{2t+1}} dt$$

$$= \int_0^{10} 20(2t+1)^{-\frac{1}{2}} dt$$

$$= \left[40(2t+1)^{\frac{1}{2}}\right]_0^{10}$$

$$= 40\sqrt{20+1} - 40\sqrt{1}$$

$$= 40\sqrt{21} - 40$$

$$= 40(\sqrt{21} - 1)$$

$$\approx 143 \text{ metres}$$

- 9 a** If X kg is the mass of a sea lion, then $X \sim N(\mu, \sigma^2)$.

We start by finding z_1 and z_2 which correspond to $x_1 = 500$ and $x_2 = 900$.

$$P(X < 500) = 0.15$$

$$\therefore P\left(\frac{X-\mu}{\sigma} < \frac{500-\mu}{\sigma}\right) = 0.15$$

$$\therefore P\left(Z < \frac{500-\mu}{\sigma}\right) = 0.15$$

$$\therefore z_1 = \frac{500-\mu}{\sigma} \approx -1.0364$$

$$\therefore 500 - \mu \approx -1.0364\sigma \quad \dots (1)$$

$$\text{Also } P(X > 900) = 0.1$$

$$\therefore P(X \leq 900) = 0.9$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \leq \frac{900-\mu}{\sigma}\right) = 0.9$$

$$\therefore P\left(Z \leq \frac{900-\mu}{\sigma}\right) = 0.9$$

$$\therefore z_2 = \frac{900-\mu}{\sigma} \approx 1.2816$$

$$\therefore 900 - \mu \approx 1.2816\sigma \quad \dots (2)$$

Solving (1) and (2) simultaneously, we obtain $\mu \approx 679 \text{ kg}$ and $\sigma \approx 173 \text{ kg}$.

$$\begin{aligned}
 \text{b } P(X < 850 \mid X > 800) &= \frac{P((X < 850) \cap (X > 800))}{P(X > 800)} \\
 &= \frac{P(800 < X < 850)}{P(X > 800)} \\
 &\approx \frac{0.0807}{0.242} \\
 &\approx 0.333
 \end{aligned}$$

So, the probability that a randomly selected sea lion weighing more than 800 kg weighs less than 850 kg is approximately 0.333.

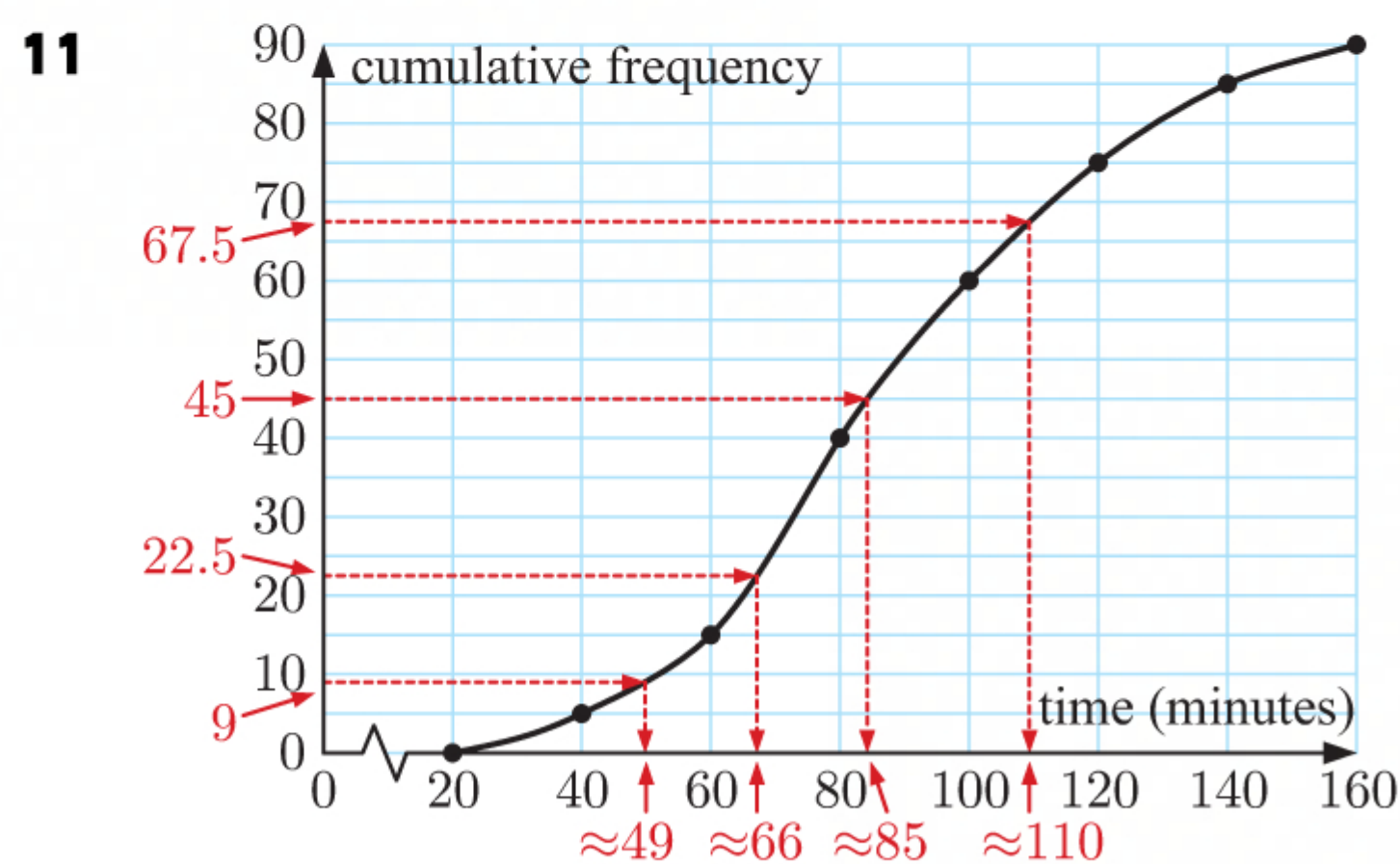
$$\begin{aligned}
 \text{10 a } f(x) &= \sin^2 x - \cos^2 x \\
 \therefore f'(x) &= 2 \sin x \cos x - 2 \cos x(-\sin x) \quad \{\text{chain rule}\} \\
 &= 2 \sin x \cos x + 2 \sin x \cos x \\
 &= \sin 2x + \sin 2x \quad \{\text{double angle formula}\} \\
 &= 2 \sin 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f''(x) &= 4 \cos 2x \\
 \therefore 4[f'(x)]^2 + [f''(x)]^2 &= 4(2 \sin 2x)^2 + (4 \cos 2x)^2 \\
 &= 4(4 \sin^2 2x) + 16 \cos^2 2x \\
 &= 16 \sin^2 2x + 16 \cos^2 2x \\
 &= 16(\sin^2 2x + \cos^2 2x) \\
 &= 16
 \end{aligned}$$

$$\begin{aligned}
 \text{c } f(\theta) &= \sin^2 \theta - \cos^2 \theta = -\cos 2\theta \quad \{\text{double angle formula}\} \\
 f'(\theta) &= 2 \sin 2\theta \\
 f''(\theta) &= 4 \cos 2\theta
 \end{aligned}$$

Now $f(\theta)$, $f'(\theta)$, and $f''(\theta)$ are consecutive terms in an arithmetic sequence.

$$\begin{aligned}
 \therefore f'(\theta) - f(\theta) &= f''(\theta) - f'(\theta) \quad \{\text{equating common differences}\} \\
 \therefore 2 \sin 2\theta - (-\cos 2\theta) &= 4 \cos 2\theta - 2 \sin 2\theta \\
 \therefore 4 \sin 2\theta &= 3 \cos 2\theta \\
 \therefore \frac{\sin 2\theta}{\cos 2\theta} &= \frac{3}{4} \\
 \therefore \tan 2\theta &= \frac{3}{4}
 \end{aligned}$$



a From the graph, 90 games were played.

b The median corresponds to cumulative frequency = 45.

Hence, the median game length is about 85 minutes.

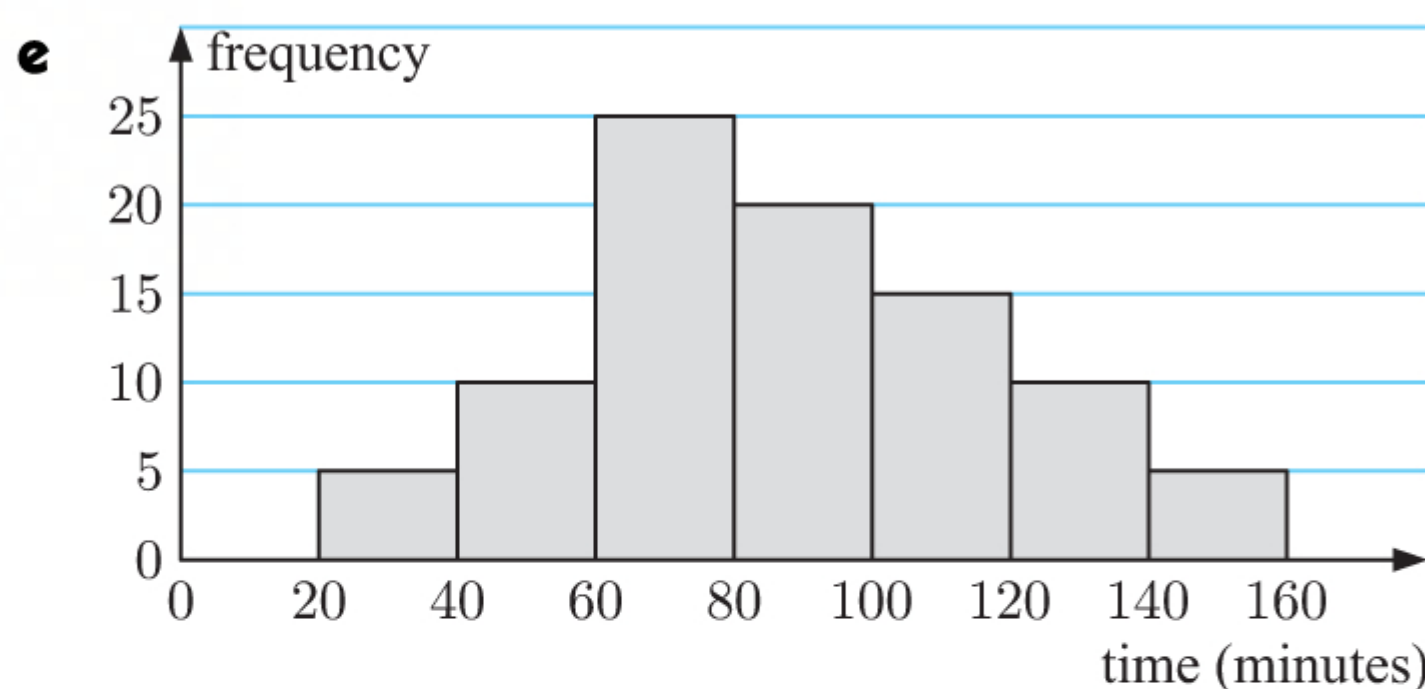
c $IQR = Q_3 - Q_1$

Q_3 corresponds to cumulative frequency = 67.5 which is ≈ 110 minutes.

Q_1 corresponds to cumulative frequency = 22.5 which is ≈ 66 minutes.

$\therefore IQR \approx 110 - 66 = 44$ minutes.

d The 10th percentile corresponds to cumulative frequency = 9 which is ≈ 49 minutes.



- 12 a** Suppose the line and the arc meet at point P where $x = k$.

\therefore P has coordinates $(k, \sqrt{3}k)$.

$$\tan \theta = \frac{\sqrt{3}k}{k} = \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3} \quad \text{and} \quad \alpha = \frac{\pi}{2} - \theta = \frac{\pi}{6}$$

b Area $A = \frac{1}{2}\alpha r^2 = \frac{1}{2}\left(\frac{\pi}{6}\right)(4)^2 = \frac{4\pi}{3} \text{ units}^2$

c Using Pythagoras, $k^2 + (\sqrt{3}k)^2 = 4^2$

$$\therefore k^2(1+3) = 16$$

$$\therefore k^2 = 4$$

$$\therefore k = 2 \quad \{k > 0\}$$

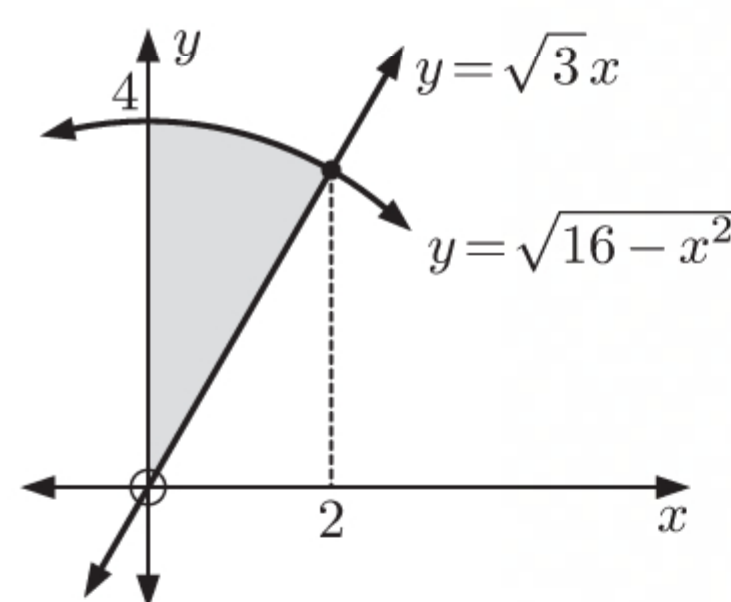
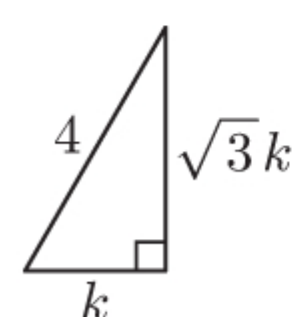
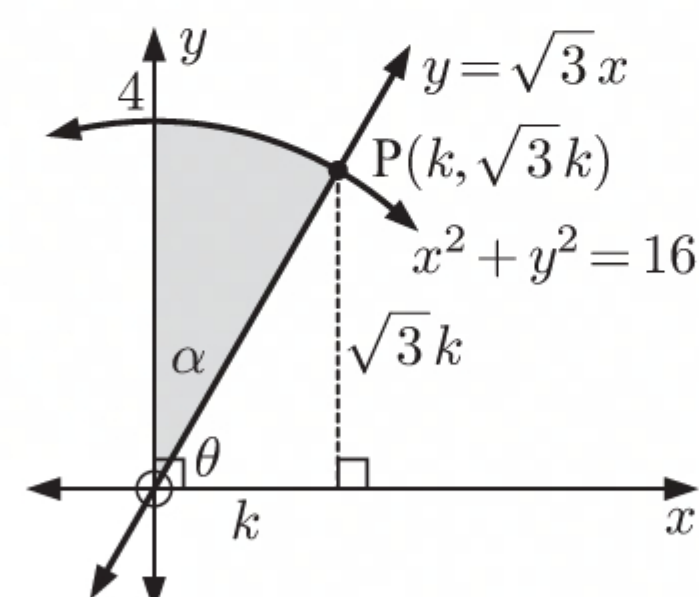
The equation of the arc is $y^2 = 16 - x^2$

$$\therefore y = \sqrt{16 - x^2} \quad \{y > 0\}$$

$$\begin{aligned} \text{Now } A &= \int_0^2 (\sqrt{16 - x^2} - \sqrt{3}x) dx \\ &= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \int_0^2 x dx \\ &= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \left[\frac{1}{2}x^2 \right]_0^2 \\ &= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \times \frac{1}{2}(2)^2 \\ &= \int_0^2 \sqrt{16 - x^2} dx - 2\sqrt{3} \quad \text{as required} \end{aligned}$$

d Using **b** and **c**, $A = \frac{4\pi}{3} = \int_0^2 \sqrt{16 - x^2} dx - 2\sqrt{3}$

$$\therefore \int_0^2 \sqrt{16 - x^2} dx = \frac{4\pi}{3} + 2\sqrt{3}$$



TRIAL EXAMINATION 1

PAPER 1

Section A

1 a There are $n + 1$ terms in this expansion. A1

b Using the binomial theorem, we have

$$(1 - 2x)^n = (1)^n + \binom{n}{1}(1)^{n-1}(-2x)^1 + \binom{n}{2}(1)^{n-2}(-2x)^2 + \dots$$
M1

So, the coefficient of x^2 is $\binom{n}{2}(1)^{n-2}(-2)^2 = 144$ M1A1

$$\frac{n!}{2!(n-2)!} \times 4 = 144$$

$$\therefore n(n-1) = 72$$
M1

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \quad \{n > 0\}$$
A1

Total [6 marks]

2 a i The inverse function is $x = e^{2y-4}$ M1

$$\therefore \ln x = 2y - 4$$

$$\therefore 4 + \ln x = 2y$$

$$\therefore y = 2 + \frac{1}{2} \ln x$$

$$\therefore y = 2 + \ln \sqrt{x}$$
A1

Thus, $f^{-1}(x) = 2 + \ln \sqrt{x}$. AG

ii The domain of $f^{-1}(x)$ is $x > 0$. A1

b $2 + \ln \sqrt{x} = \frac{1}{2} \ln 2$

$$\therefore \ln \sqrt{x} = \frac{1}{2} \ln 2 - 2$$

$$\therefore \ln \sqrt{x} = \ln \sqrt{2} - \ln(e^2)$$
M1

$$\therefore \ln \sqrt{x} = \ln \left(\frac{\sqrt{2}}{e^2} \right)$$

$$\therefore \sqrt{x} = \frac{\sqrt{2}}{e^2}$$
M1

$$\therefore x = \frac{2}{e^4}$$
A1

Total [6 marks]

3 a Since $\cos \widehat{ABC} = \cos \widehat{ABD} = \frac{12}{13} = \frac{AB}{BD}$, triangle ABD must be a right-angled triangle with the right angle at A. M1

$$\text{Thus, } AD = \sqrt{13^2 - 12^2}$$

$$= \sqrt{25}$$

$$= 5 \text{ cm}$$
A1

b Since triangle ABD is a right-angled triangle,

$$\sin \widehat{ADC} = \frac{12}{13}$$
A1

c Area = $\frac{1}{2} \times 5 \times 5 \times \sin \widehat{ADC}$ M1

$$= \frac{25}{2} \times \frac{12}{13}$$

$$= \frac{150}{13} \text{ cm}^2$$
A1

Total [5 marks]

- 4 a** $l = r\theta = 7 \times \frac{\pi}{3} = \frac{7\pi}{3}$ M1A1
- b** Area of the larger sector $= \frac{1}{2} \times 7^2 \times \frac{\pi}{3} = \frac{49\pi}{6}$ A1
 Area of the smaller sector $= \frac{1}{2} \times 4^2 \times \frac{\pi}{3} = \frac{16\pi}{6}$ A1
 \therefore area of the shaded region $= \frac{49\pi}{6} - \frac{16\pi}{6}$ M1
 $= \frac{11\pi}{2}$ A1
- c** Using **a**, the length of the longer arc is $\frac{7\pi}{3}$.
 The length of the shorter arc is
- $$4 \times \frac{\pi}{3} = \frac{4\pi}{3}$$
- \therefore the perimeter of the shaded region $= 3 + 3 + \frac{7\pi}{3} + \frac{4\pi}{3}$ M1
 $= \frac{11\pi}{3} + 6$ AG
- Total [8 marks]**

- 5 a** The first three terms of the geometric sequence are
- $$u_1 = \ln x^{27} = 27 \ln x, \quad u_2 = \ln x^9 = 9 \ln x, \quad u_3 = \ln x^3 = 3 \ln x$$
- A1
- Thus, the common ratio of consecutive terms is $r = \frac{3 \ln x}{9 \ln x} = \frac{9 \ln x}{27 \ln x} = \frac{1}{3}$ M1A1
- b** $\sum_{k=1}^{\infty} 3^{4-k} \ln x = 3^3 \ln x + 3^2 \ln x + 3^1 \ln x + \dots = 27 \ln x + 9 \ln x + 3 \ln x + \dots$ M1
- Using **a**, this is an infinite geometric series with $r = \frac{1}{3}$. M1
- $$\therefore \sum_{k=1}^{\infty} 3^{4-k} \ln x = \frac{27 \ln x}{1 - \frac{1}{3}}$$
- $$= \frac{81}{2} \ln x$$
- A1
- Therefore, $\frac{81}{2} \ln x = 324$ M1
 $\therefore \ln x = 8$
 $\therefore x = e^8$ A1
- Total [8 marks]**

- 6** $\sin \theta = \cos 2\theta$
- $$\therefore \sin \theta = 1 - 2 \sin^2 \theta$$
- A1
- $$\therefore 2 \sin^2 \theta + \sin \theta - 1 = 0$$
- $$\therefore (\sin \theta + 1)(2 \sin \theta - 1) = 0$$
- M1
- $$\therefore \sin \theta = -1 \text{ or } \frac{1}{2}$$
- A1A1
- Given $0 < \theta < 2\pi$, $\sin \theta = -1$ when $\theta = \frac{3\pi}{2}$ A1
 and $\sin \theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ A1A1
- Total [7 marks]**

Section B

- 7 a i** $p = 1 - \frac{1}{6} = \frac{5}{6}$ A1
- ii** $q = 1 - \frac{2}{3} = \frac{1}{3}$ A1
- b** $P(C \cap T) = \frac{1}{2} \times \frac{1}{3}$ M1A1
 $= \frac{1}{6}$ A1
- c** $P(T') = P(P \cap T') + P(C \cap T')$ M1
 $= \frac{1}{2} \times \frac{5}{6} + \frac{1}{2} \times \frac{2}{3}$ (M1)A1
 $= \frac{5}{12} + \frac{1}{3}$
 $= \frac{3}{4}$ A1

$$\begin{aligned}
 \text{d } P(P | T') &= \frac{P(P \cap T')}{P(T')} && \text{M1} \\
 &= \frac{\frac{1}{2} \times \frac{5}{6}}{\frac{3}{4}} && \text{A1} \\
 &= \frac{\left(\frac{5}{12}\right)}{\left(\frac{9}{12}\right)} \\
 &= \frac{5}{9} && \text{A1}
 \end{aligned}$$

e Let X be the number of times Hassan is on time.

$$\therefore X \sim B\left(3, \frac{1}{4}\right) \quad \text{M1}$$

$$P(X \geq 1) = 1 - P(X = 0) \quad \text{M1}$$

$$= 1 - \binom{3}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^3 \quad \text{A1}$$

$$= 1 - \frac{27}{64}$$

$$= \frac{37}{64} \quad \text{A1}$$

Total [16 marks]

8 a Using the product rule,

M1

$$f'(x) = a(x+p) + a(x+q) = 2ax + ap + aq \quad \text{A1AG}$$

b Given the gradient of the normal, we find the gradient of the tangent at the two given points: M1

At $x = -2$, the gradient of the tangent is $-\frac{3}{2}$. A1

$$\therefore 2a(2) + ap + aq = \frac{5}{2}$$

$$\therefore 4a + ap + aq = \frac{5}{2} \quad \dots (1) \quad \text{A1}$$

At $x = 2$, the gradient of the tangent is $\frac{5}{2}$. A1

$$\therefore 2a(-2) + ap + aq = -\frac{3}{2}$$

$$\therefore -4a + ap + aq = -\frac{3}{2} \quad \dots (2) \quad \text{A1}$$

Subtracting (2) from (1) gives $8a = 4$

$$\therefore a = \frac{1}{2} \quad \text{A1}$$

c $f(x)$ has x -intercepts $-p$ and $-q$.

The axis of symmetry occurs at the midpoint of the x -intercepts.

M1

$$\therefore \frac{-p-q}{2} = -\frac{1}{2}$$

A1

$$\therefore -(p+q) = -1$$

$$\therefore p+q = 1 \quad \text{AG}$$

$$\text{d } f(0) = -6 \quad \text{M1}$$

$$\therefore \frac{1}{2}(0+p)(0+q) = -6$$

$$\therefore pq = -12$$

A1AG

e If $p+q = 1$ then $q = 1-p$.

Substituting $q = 1-p$ into $pq = -12$ gives

M1

$$p(1-p) = -12$$

$$\therefore p - p^2 = -12$$

$$\therefore p^2 - p - 12 = 0$$

$$\therefore (p-4)(p+3) = 0$$

A1

So, we have $p = 4$, $q = 1 - 4 = -3$

or $p = -3$, $q = 1 - (-3) = 4$

A1A1

f The curve $f(x) = \frac{1}{2}(x-3)(x+4)$ and the line $y = kx - 8$ intersect when

$$\frac{1}{2}(x-3)(x+4) = kx - 8 \quad \text{M1}$$

$$\therefore \frac{1}{2}x^2 + \frac{1}{2}x - 6 = kx - 8$$

$$\therefore \frac{1}{2}x^2 + \left(\frac{1}{2} - k\right)x + 2 = 0 \quad \text{M1}$$

For the line $y = kx - 8$ to be a tangent to the graph of $f(x)$, this quadratic must have discriminant 0. M1

$$\therefore \left(\frac{1}{2} - k\right)^2 - 4\left(\frac{1}{2}\right)(2) = 0 \quad \text{M1}$$

$$\therefore \frac{1}{4} - k + k^2 - 4 = 0$$

$$\therefore k^2 - k - \frac{15}{4} = 0 \quad \text{A1}$$

$$\therefore 4k^2 - 4k - 15 = 0$$

$$\therefore (2k+3)(2k-5) = 0 \quad \text{A1}$$

$$\therefore k = -\frac{3}{2} \text{ or } k = \frac{5}{2} \quad \text{A1A1}$$

Total [24 marks]

PAPER 2

Section A

1 a $0.10 + 0.13 + 0.14 + k + 0.16 + 0.12 + 0.11 + 0.08 = 1$ M1

$$\therefore k = 0.16 \quad \text{A1}$$

b $E(X) = 1 \times 0.10 + 2 \times 0.13 + 3 \times 0.14 + 4 \times 0.16 + 5 \times 0.16 + 6 \times 0.12 + 7 \times 0.11 + 8 \times 0.08$ M1

$$= 0.10 + 0.26 + 0.42 + 0.64 + 0.8 + 0.72 + 0.77 + 0.64$$

$$= 4.35 \quad \text{A1}$$

c $P(\text{rolling a 5 or 6}) = 0.16 + 0.12 = 0.28$ M1

$$\therefore \text{expected number of times} = 75 \times 0.28 = 21 \quad \text{A1}$$

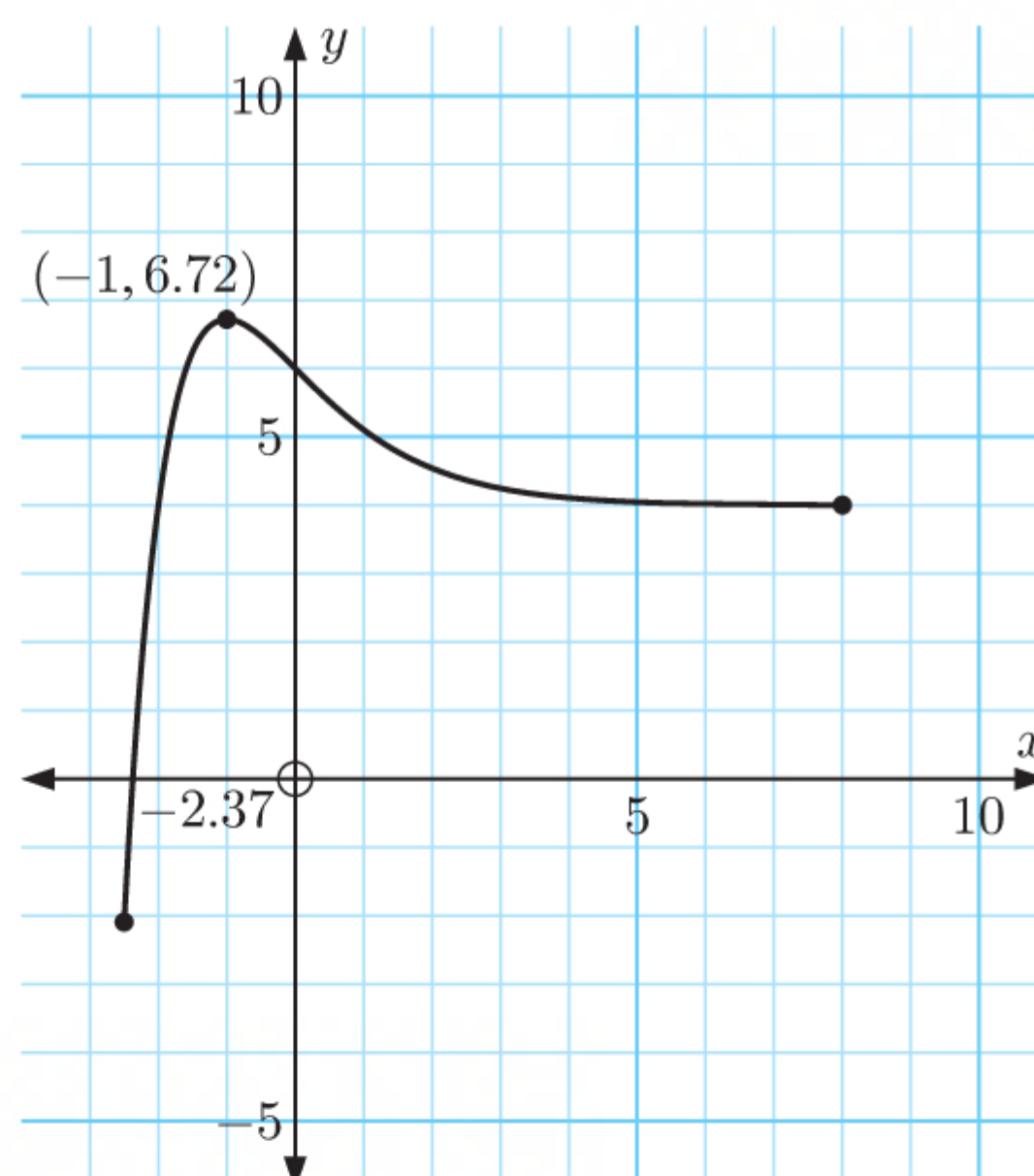
Total [6 marks]

2 a Using technology, we find that the coordinates of point A are $(-1, 6.72)$. A1A1

b At the x -intercept, $f(x) = 0$. M1

Using technology, the x -intercept is ≈ -2.37 . A1

c A1A1A1



Total [7 marks]

3 a $4 + 16 + p = 44$ A1

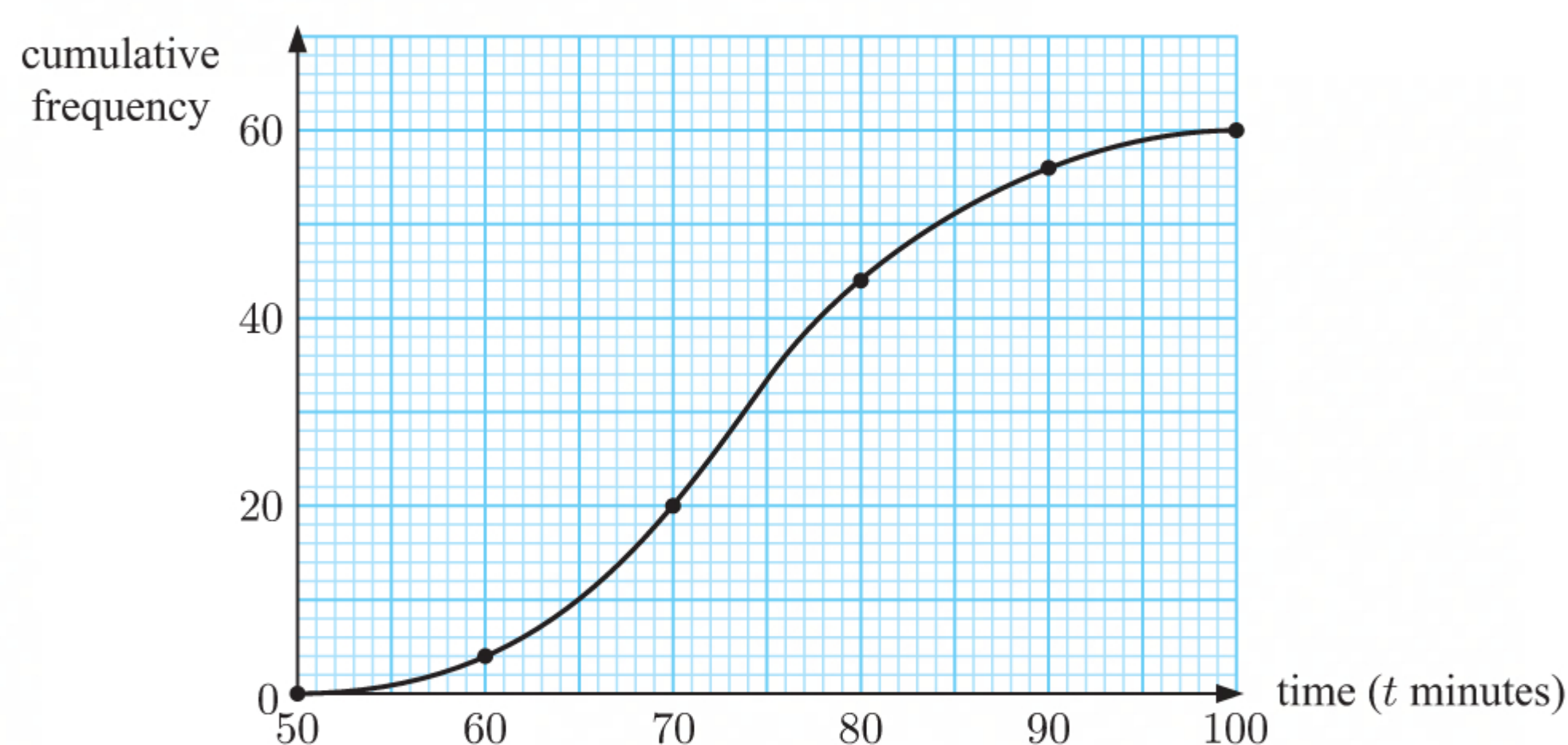
$$\therefore p = 24$$

b i $4 + 16 + p + q + 4 = 60$ M1

$$\therefore 4 + 16 + 24 + q + 4 = 60$$

$$\therefore q = 12 \quad \text{A1}$$

ii $r = 60 - 4 = 56$ M1A1

c**A1A1****Total [7 marks]**

- 4 a** Using technology, we find the equation of the linear regression line is $y \approx 0.801x + 21.1$

M1

$$\therefore a_1 \approx 0.801 \text{ and } b_1 \approx 21.1$$

A1A1

- b** Using technology, $r \approx 0.969$

A1

- c** When $x = 3.5$, $y_2 = (0.780\ 36)(3.5) + 20.470\ 71 = 23.201\ 97 \approx 23.2$

M1

So, the mean length of a three and a half month old female baby is approximately 23.2 cm.

A1**Total [6 marks]**

- 5 a** The height of the ball after its sixth bounce is 5×0.85^6
 ≈ 1.89 m

A1**A1**

- b** The height of the ball after it bounces k times is 5×0.85^k .

We must therefore solve $5 \times 0.85^k < 2$

M1

$$\therefore 0.85^k < 0.4$$

$$\therefore k \log 0.85 < \log 0.4$$

$$\therefore k > \frac{\log 0.4}{\log 0.85} \quad \{\log 0.85 < 0\}$$

$$\therefore k > 5.64$$

A1

$$\text{Thus, } k = 6$$

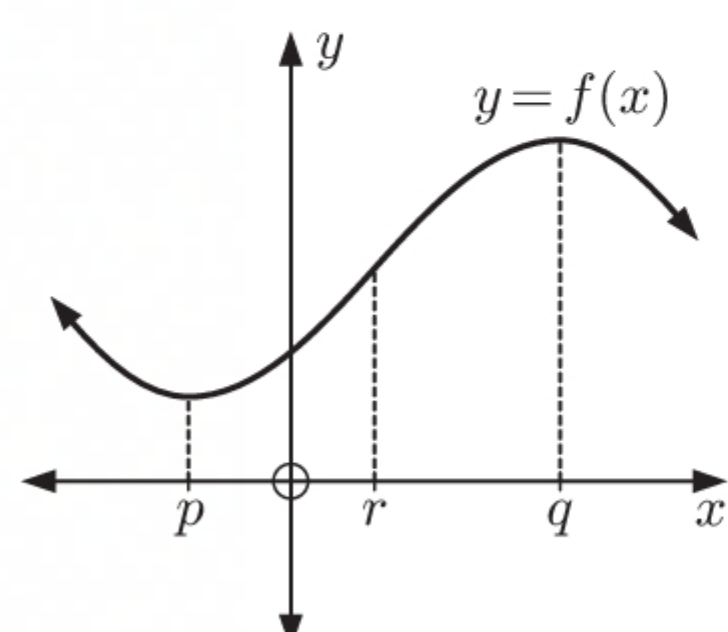
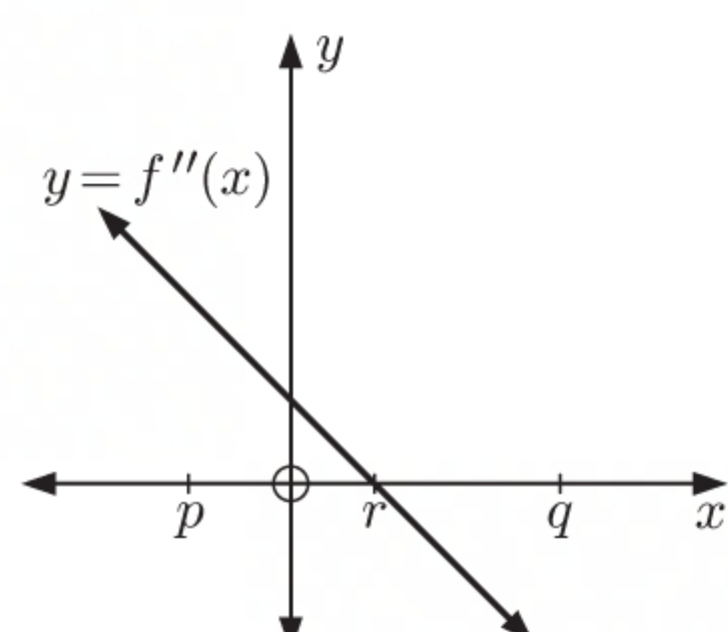
A1

- c** The distance the ball travels downwards forms an infinite geometric series with $u_1 = 5$ and $r = 0.85$.
 The distance the ball travels upwards forms an infinite geometric series with $u_1 = 5 \times 0.85 = 4.25$
 and $r = 0.85$.

M1

Thus, the total distance travelled by the ball is

$$\frac{5}{1 - 0.85} + \frac{4.25}{1 - 0.85} = \frac{185}{3} \text{ m or } 61.\bar{6} \text{ m}$$

A1A1**Total [8 marks]****6 a i****A1A1A1****ii****A1**

- b** The graph of $f(x)$ is concave down when $f''(x) \leq 0$
 $\therefore x \geq r$

M1

A1

Total [6 marks]

Section B

7 a $r = \frac{\left(\frac{\ln x}{\ln 16}\right)}{\left(\frac{\ln x}{\ln 4}\right)}$ M1

$$= \frac{\ln x}{\ln 16} \times \frac{\ln 4}{\ln x}$$

$$= \frac{\ln 4}{\ln 16}$$

$$= \log_{16} 4$$

$$= \frac{1}{2}$$
 A1

b Using the formula $u_n = u_1 \times r^{n-1}$, M1

$$u_8 = \frac{\ln x}{\ln 4} \times \left(\frac{1}{2}\right)^7$$

$$= \frac{\ln x}{2 \ln 2} \times \frac{1}{2^7}$$

$$= \frac{\ln x}{2^8 \ln 2}$$
 A1AG

c Using the formula $S_n = \frac{u_1(1-r^n)}{1-r}$, M1

$$S_8 = \frac{\frac{\ln x}{\ln 4} \left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$$

$$= \frac{\frac{\ln x}{2 \ln 2} \left(1 - \frac{1}{256}\right)}{\frac{1}{2}}$$

$$= \frac{255 \ln x}{256 \ln 2}$$
 A1

d Using the formula $S_\infty = \frac{u_1}{1-r}$, M1

$$S_\infty = \frac{\frac{\ln x}{\ln 4}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{\ln x}{2 \ln 2}}{\frac{1}{2}}$$

$$= \frac{\ln x}{\ln 2}$$
 A1

So, $a = 2$. A1

Total [10 marks]

8 a $h(x) = (f \circ g)(x)$

$$= f(2 \sin(2x))$$
 M1

$$= \frac{2 \sin(2x)}{4} + 2$$

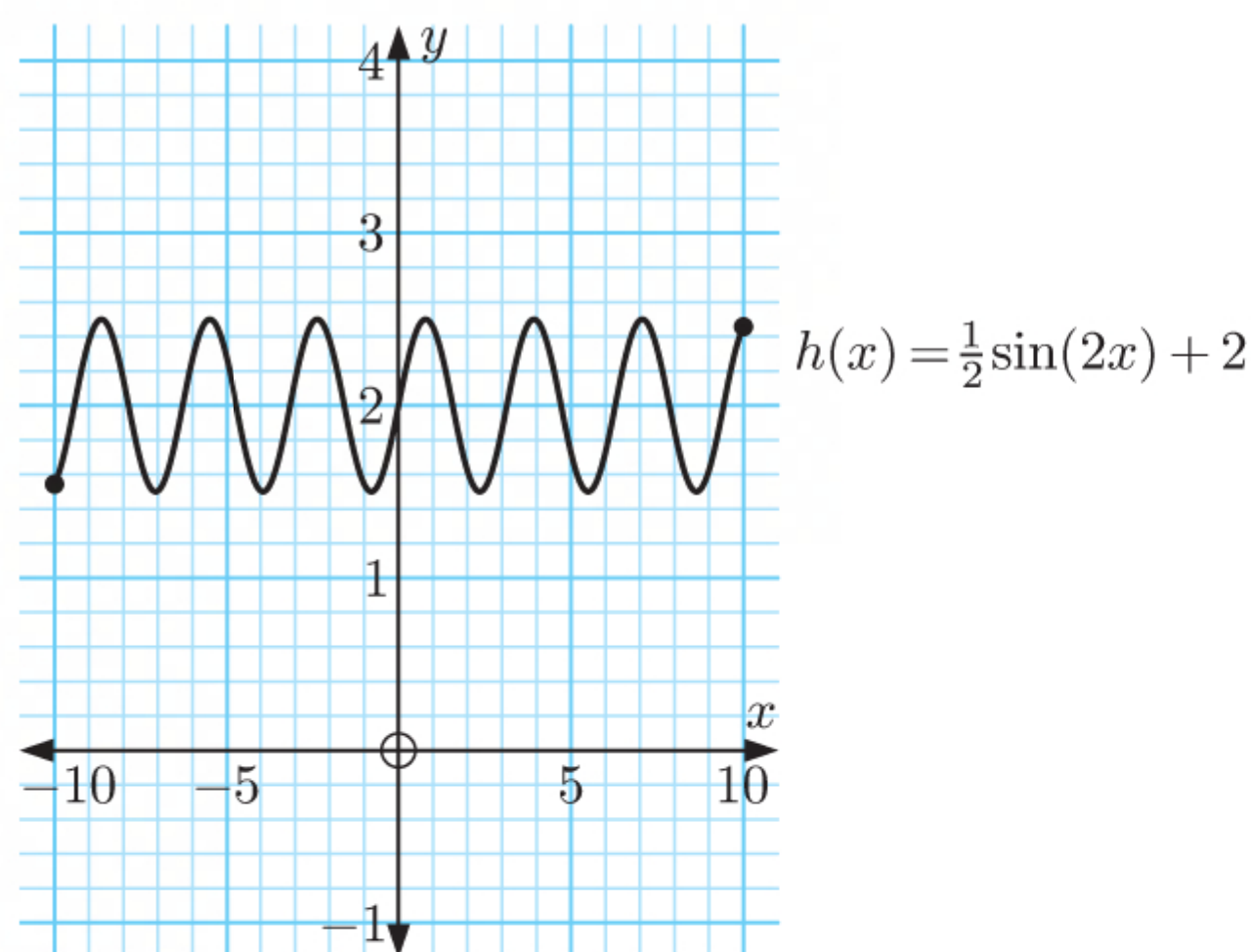
$$= \frac{1}{2} \sin(2x) + 2$$
 A1

b i The period of $h(x)$ is $\frac{2\pi}{2} = \pi$. A1

ii The amplitude of $h(x)$ is $\frac{1}{2}$. A1

c i The scale factor of the vertical stretch is $p = \frac{1}{4}$. A1

ii The vertical translation is $b = 2$. A1

d**A1A1****Total [8 marks]**

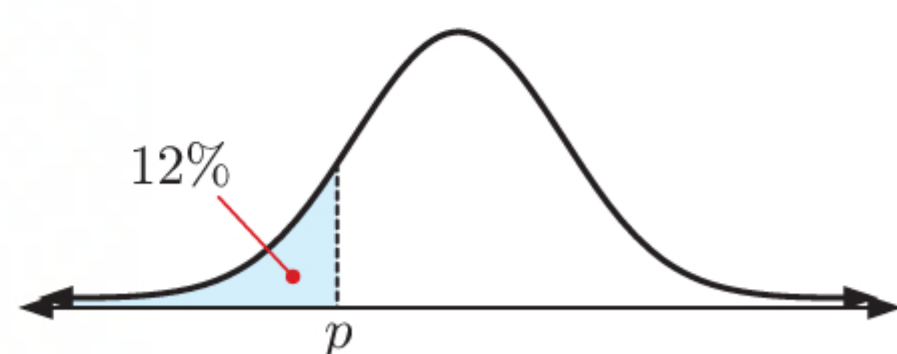
- 9 a i** If X represents the weight of the cookies in grams, then $X \sim N(15, 1.2^2)$

M1

$$\therefore P(X < 17) \approx 0.952$$

A1

- ii** $P(12 < X < 17) \approx 0.946$

A1**b i****A1**

- ii** Using the inverse normal distribution function on a calculator,

$$\text{If } P(X < p) = 0.12, \text{ then}$$

M1

$$p \approx 13.6$$

A1

- c i** Using technology, we find the z -score:

$$\text{If } P(Z < z) = 0.03, \text{ then}$$

M1

$$z = -1.880\,793\,6 \dots$$

A1

$$\therefore \frac{x - \mu}{\sigma} = -1.880\,793\,6 \dots$$

A1

$$\therefore \frac{9 - 13}{\sigma} = -1.880\,793\,6 \dots$$

$$\therefore \sigma \approx 2.13 \text{ grams}$$

A1

- ii** Let Y be the number of cookies weighing less than 9 grams.

$$\therefore Y \sim B(20, 0.03)$$

M1

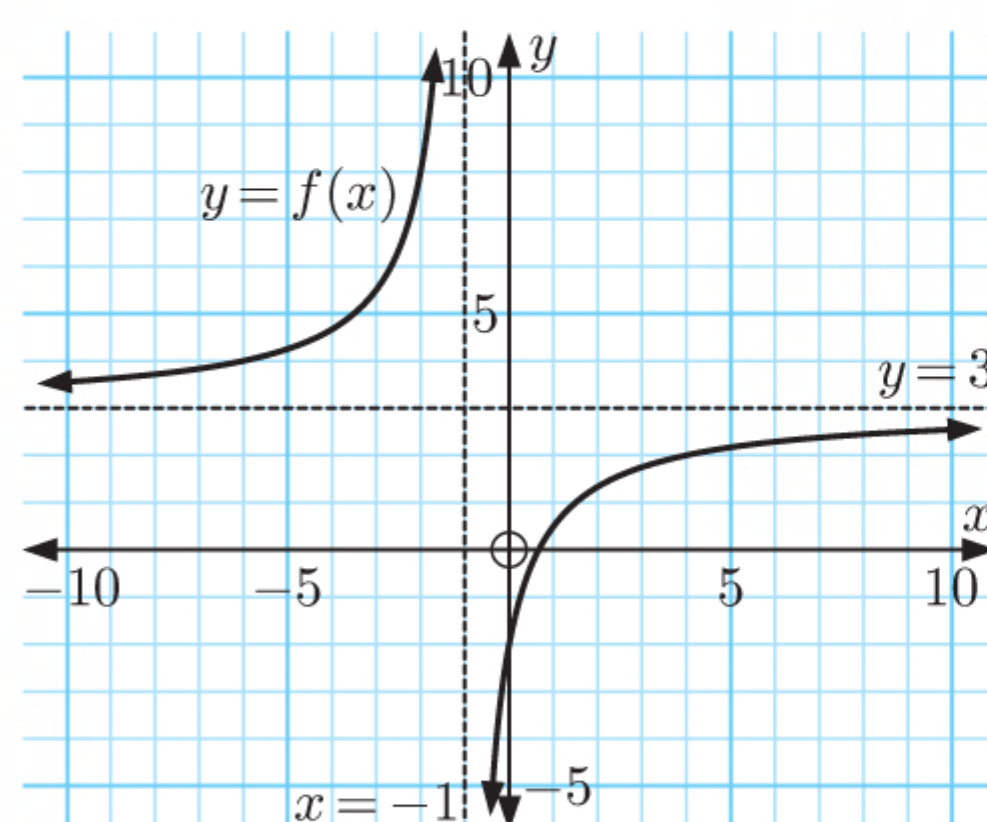
$$\text{Using technology, } P(Y = 2) \approx 0.0988$$

A1**Total [12 marks]**

- 10 a i** $y = 3$

A1

- ii** $x = -1$

A1**b****A1A1A1A1**

c The x -intercept occurs when $f(x) = 0$

M1

$$\therefore \frac{3x-2}{x+1} = 0$$

$$\therefore 3x - 2 = 0$$

$$\therefore x = \frac{2}{3}$$

A1

d Area $= -\int_0^{\frac{2}{3}} \frac{3x-2}{x+1} dx + \int_{\frac{2}{3}}^5 \frac{3x-2}{x+1} dx$

A1

$$= -(-0.5541 \dots) + (6.5953 \dots) \quad \{\text{using technology}\}$$

$$\approx 7.15 \text{ units}^2$$

A1

Total [10 marks]

TRIAL EXAMINATION 2

PAPER 1

Section A

1 40, 20, 10, 5,

a i $r = \frac{1}{2}$

A1

ii as $|r| < 1$, S_∞ exists

R1AG

b $S_\infty = \frac{u_1}{1-r}$
 $= \frac{40}{1-\frac{1}{2}}$
 $= 80$

M1

A1

Total [4 marks]

2 **a** For $n \in \mathbb{Z}^+$, $2n$ and $4n$ must be even.

$\therefore 2n-1$ and $4n+1$ must be odd.

R1

b $(2n-1)(4n+1)$
 $= 8n^2 + 2n - 4n - 1$
 $= 8n^2 - 2n - 1$

A1

c $(2n-1)(4n+1) = 8n^2 - 2n - 1$
 $= 2(4n^2 - n) - 1$
 $= 2N - 1$ where $N = 4n^2 - n \in \mathbb{Z}$

M1

A1

$\therefore (2n-1)(4n+1)$ is always odd.

R1

Total [5 marks]

3 $\bar{x} = 18$, $\sigma^2 = 9$

a $\sigma = \sqrt{\sigma^2}$
 $= \sqrt{9}$
 $= 3$

A1

b i $\bar{x}_{\text{new}} = \frac{1}{2}(18)$
 $= 9$

A1

ii $\sigma_{\text{new}} = \frac{1}{2}(3)$
 $= \frac{3}{2}$

M1A1

$\therefore \sigma_{\text{new}}^2 = \frac{9}{4}$

A1

Total [5 marks]

4 **a** $f(x) = xe^x$, $x \in \mathbb{R}$

$\therefore f'(x) = (1)e^x + xe^x$ {product rule}
 $= e^x + xe^x$

M1

A1A1

b From **a**, $\int (e^x + xe^x) dx = xe^x + c$

M1

$\therefore \int e^x dx + \int xe^x dx = xe^x + c$

M1

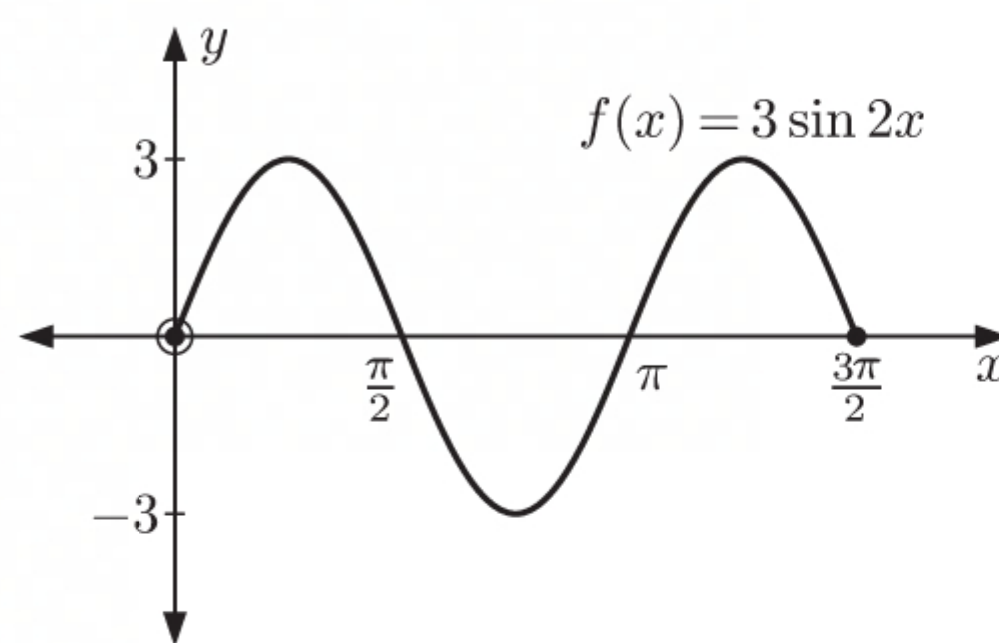
$\int xe^x dx = xe^x - e^x + c$

A2

Total [7 marks]

5 a Amplitude = 3

$$\text{period} = \frac{2\pi}{2} = \pi$$



A1 : end points
A1 : zeros
A1 : shape, min/max

b $f(x) + 2 = 0$

$$\therefore f(x) = -2$$

From the graph, there are 2 solutions on $0 \leq x \leq \frac{3\pi}{2}$.

A2

Total [5 marks]

6 $g'(x) = \frac{2}{3-x} dx$

a $g(x) = \int \frac{2}{3-x} dx$

$$= -2 \ln|3-x| + c$$

Now $g(2) = -1$

$$\therefore -2 \ln 1 + c = -1$$

$$\therefore c = -1$$

$$\therefore g(x) = -2 \ln|3-x| - 1$$

b Domain is $\{x \mid x \neq 3\}$, range is $\{y \mid y \in \mathbb{R}\}$

M1

A1

M1

A1

A1

A1A1

Total [7 marks]

7 $3 \log_2 x + 2 \log_4(2x+1) = 6 \log_8 x$

$$\therefore 3 \log_2 x + \frac{2 \log_2(2x+1)}{\log_2 4} = \frac{6 \log_2 x}{\log_2 8} \quad \{\text{change of base}\}$$

$$\therefore 3 \log_2 x + \frac{2 \log_2(2x+1)}{2} = \frac{6 \log_2 x}{3}$$

$$\therefore 3 \log_2 x + \log_2(2x+1) = 2 \log_2 x$$

$$\therefore \log_2 x + \log_2(2x+1) = 0$$

$$\therefore \log_2(2x^2 + x) = 0$$

$$\therefore 2x^2 + x = 1$$

$$\therefore 2x^2 + x - 1 = 0$$

$$\therefore (2x-1)(x+1) = 0$$

$$\text{But } x > 0, \text{ so } x = \frac{1}{2}.$$

M1A1

A1

M1

A1

M1

A1A1

Total [8 marks]

Section B

8 $f(x) = \frac{x+4}{2x-1}, x \neq \frac{1}{2}$

a The horizontal asymptote is $y = \frac{1}{2}$.

A1

The vertical asymptote is $x = \frac{1}{2}$.

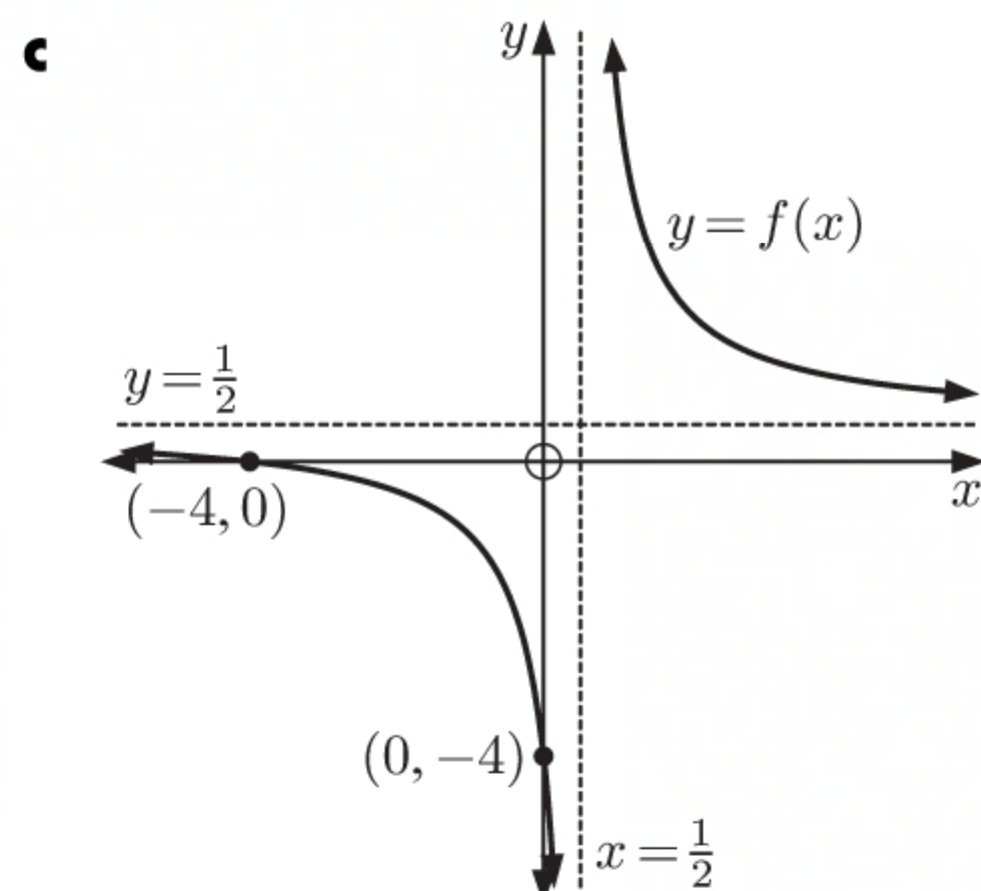
A1

b When $y = 0$, $x = -4$

When $x = 0$, $y = -4$

$\therefore f(x)$ cuts the x -axis at $(-4, 0)$ and the y -axis at $(0, -4)$.

A1A1



A1 : asymptotes

A1 : intercepts

A1 : two curves

d $f(x) < 0$ when $-4 < x < \frac{1}{2}$

A2

e $g(x) = 2f(x - 1) - 1$

g is obtained from f by a:

- horizontal translation of 1 unit right
- vertical stretch with scale factor 2
- vertical translation of 1 unit down.

M1

$A(2, 2) \rightarrow (3, 2) \rightarrow (3, 4) \rightarrow A'(3, 3)$

A1A1

Total [12 marks]

9 a $(x + 3)^2 = (2x)^2 + 5^2 - 2(2x)(5) \cos 60^\circ$
 $\therefore (x + 3)^2 = 4x^2 + 25 - 10x$
 $\therefore x^2 + 6x + 9 = 4x^2 + 25 - 10x$
 $\therefore 3x^2 - 16x + 16 = 0$
 $\therefore (3x - 4)(x - 4) = 0$
 $\therefore x = \frac{4}{3}$ or 4

M1A1

A1

b If $x = \frac{4}{3}$, $A = \frac{1}{2} \left(\frac{8}{3} \right) (5) \sin 60^\circ$
 $= \frac{4}{3} (5) \frac{\sqrt{3}}{2}$
 $= \frac{10\sqrt{3}}{3} \text{ cm}^2$

A1

If $x = 4$, $A = \frac{1}{2} (8) (5) \sin 60^\circ$
 $= 20 \frac{\sqrt{3}}{2}$
 $= 10\sqrt{3} \text{ cm}^2$

A1

c Triangle ABC has the larger area when $x = 4$.

$$\cos \theta = \frac{5^2 + 7^2 - 8^2}{2 \times 5 \times 7}$$

$$= \frac{10}{70} = \frac{1}{7}$$

A1

M1

A1

Now $\sin^2 \theta + \cos^2 \theta = 1$

$$\therefore \sin^2 \theta + \frac{1}{49} = 1$$

$$\therefore \sin^2 \theta = \frac{48}{49}$$

$$\therefore \sin \theta = \frac{4\sqrt{3}}{7} \quad \{\sin \theta > 0\}$$

A1

So, $\sin 2\theta = 2 \sin \theta \cos \theta$

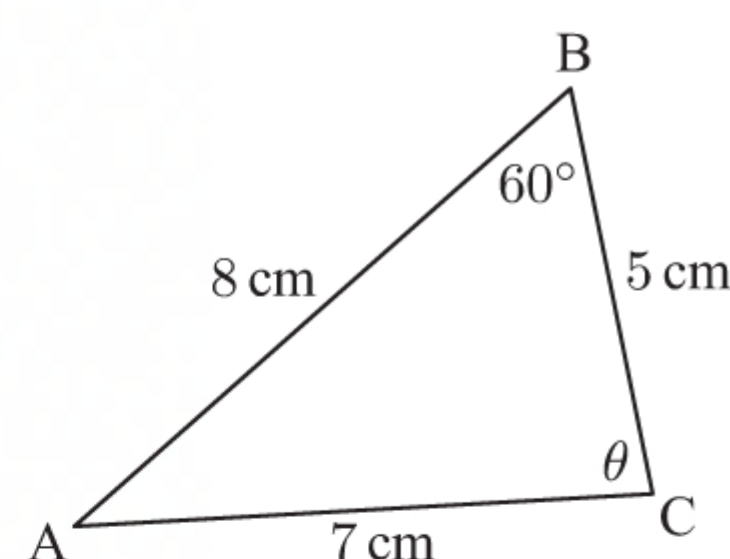
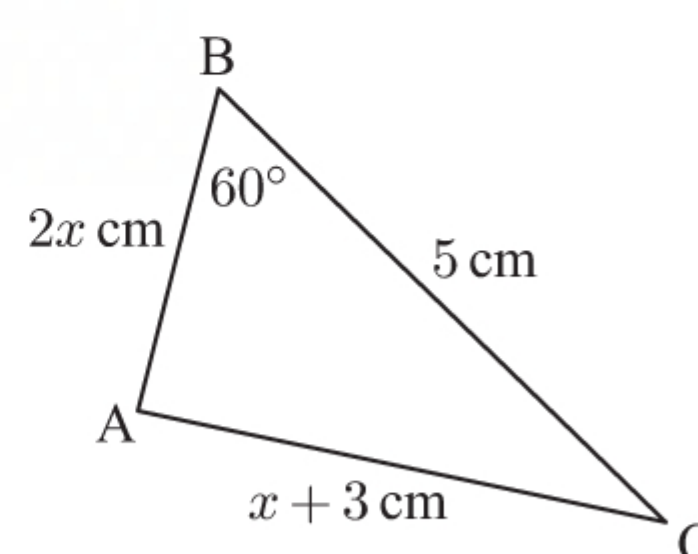
M1

$$= 2 \left(\frac{4\sqrt{3}}{7} \right) \left(\frac{1}{7} \right)$$

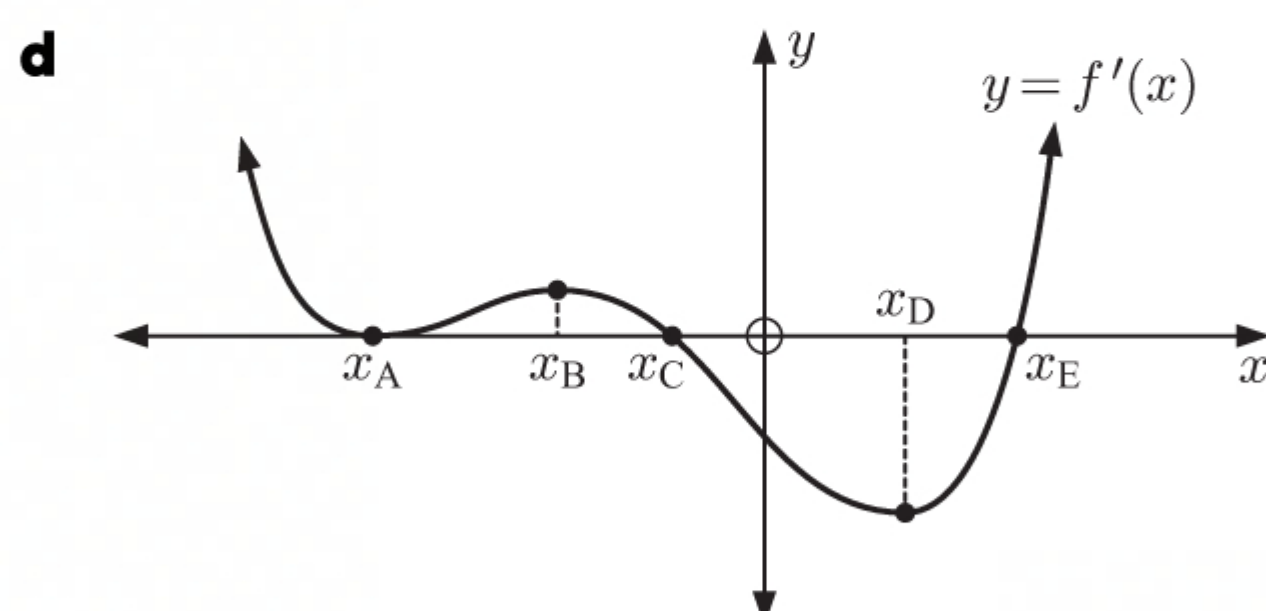
$$= \frac{8}{49} \sqrt{3}$$

A1

Total [15 marks]



- 10 a** There are 3 solutions to $f(x) = 0$. A1
- b i** $f'(x) = 0$ at A, C, and E. A2
- ii** $f''(x) = 0$ at A, B, and D. A2
- c i** $x_A \leq x \leq x_B, \quad x \geq x_D$ A1A1
- ii** $x \leq x_A, \quad x_B \leq x \leq x_D$ A1A1



A1 : zeros
A1 : local max/min
A1 : shape

Total [12 marks]

PAPER 2

Section A

- 1** $V = \frac{4}{3}\pi r^3$
 $= \frac{4}{3}\pi(2440)^3 \text{ km}^3$ M1A1
 $\approx 6.085 \times 10^{10} \text{ km}^3$ A1
 $\approx 6.085 \times 10^{10} \times (1000^3) \text{ m}^3$
 $\approx 6.085 \times 10^{19} \text{ m}^3$ A1

Total [4 marks]

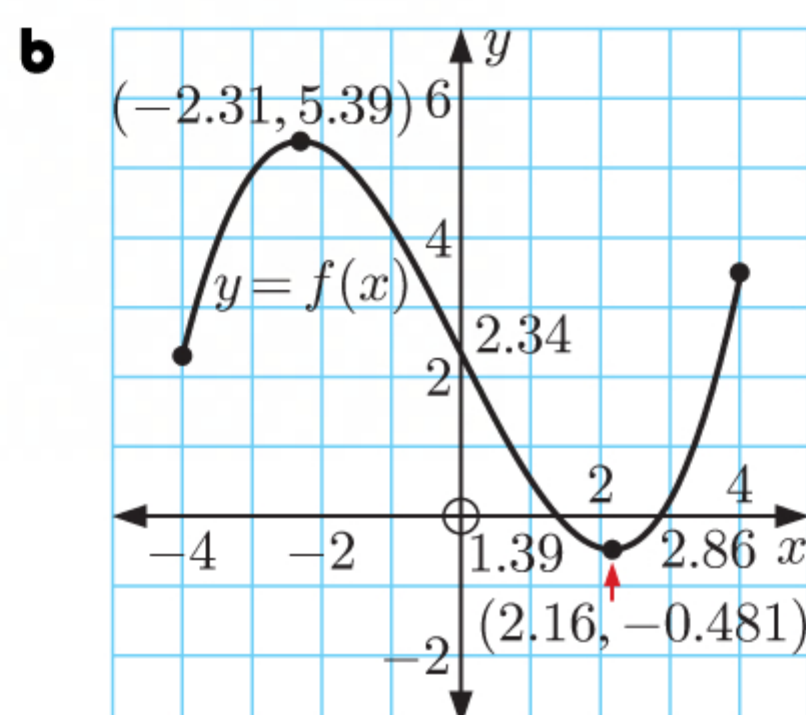
- 2** $S_6 = 402$ and $S_{12} = 1236$
 $\therefore 3(2u_1 + 5d) = 402$ and $6(2u_1 + 11d) = 1236$
 $\therefore 2u_1 + 5d = 134 \quad \dots (1)$ M1A1
 and $2u_1 + 11d = 206 \quad \dots (2)$ M1A1
 Solving (1) and (2) simultaneously gives $u_1 = 37, d = 12$. A1A1

Total [6 marks]

- 3 a** The linear regression equation is $W \approx 2.17N - 6.40$
- i** $m \approx 2.17, b \approx -6.40$ A1A1
- ii** $r \approx 0.990$ A1
- b** strong, positive A1A1
- c** When $n = 30, W \approx 2.17(30) - 6.40$ (M1)
 ≈ 58.7
 $\approx 59 \text{ eggs}$ A1

Total [7 marks]

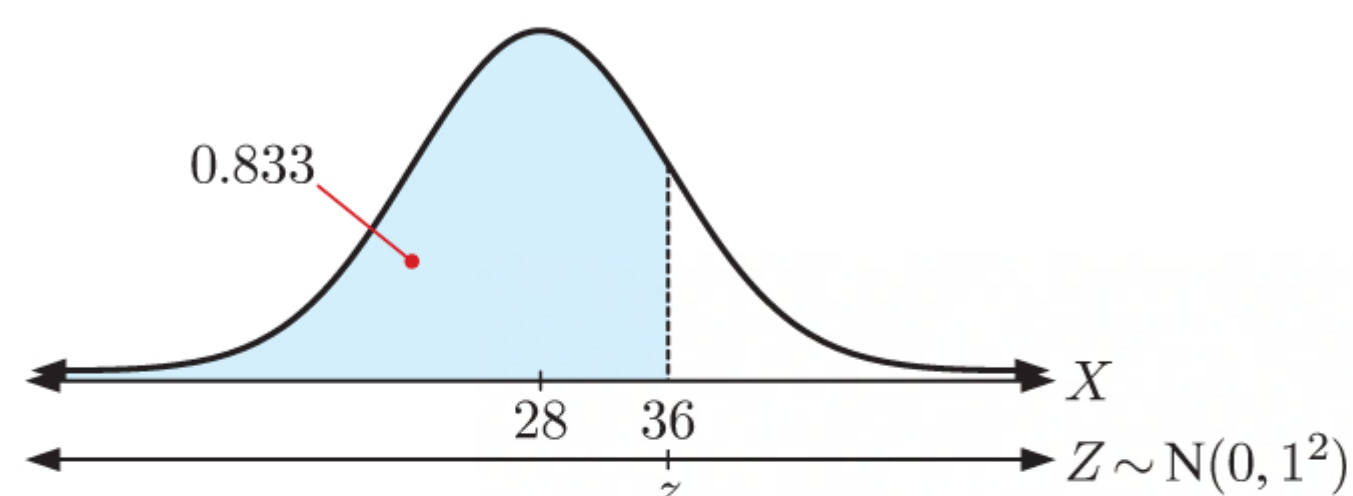
- 4** $f(x) = \frac{1}{8}x^3 - 2x + \ln(10 + e^{x-1}), \quad -4 \leq x \leq 4$
- a i** y -intercept at $(0, 2.34)$ A1
- ii** x -intercepts at $(1.39, 0)$ and $(2.86, 0)$ A1
- iii** local maximum at $(-2.31, 5.39)$ A1
 local minimum at $(2.16, -0.481)$



A1 : x -intercepts
 A1 : y -intercept
 A1 : turning points

Total [6 marks]

- 5** $X \sim N(28, \sigma^2)$
 $P(X < 36) = 0.833$



Using technology, $z \approx 0.966\ 09\ \dots$

M1

$$\therefore \frac{x - \mu}{\sigma} \approx 0.966\ 09$$

M1

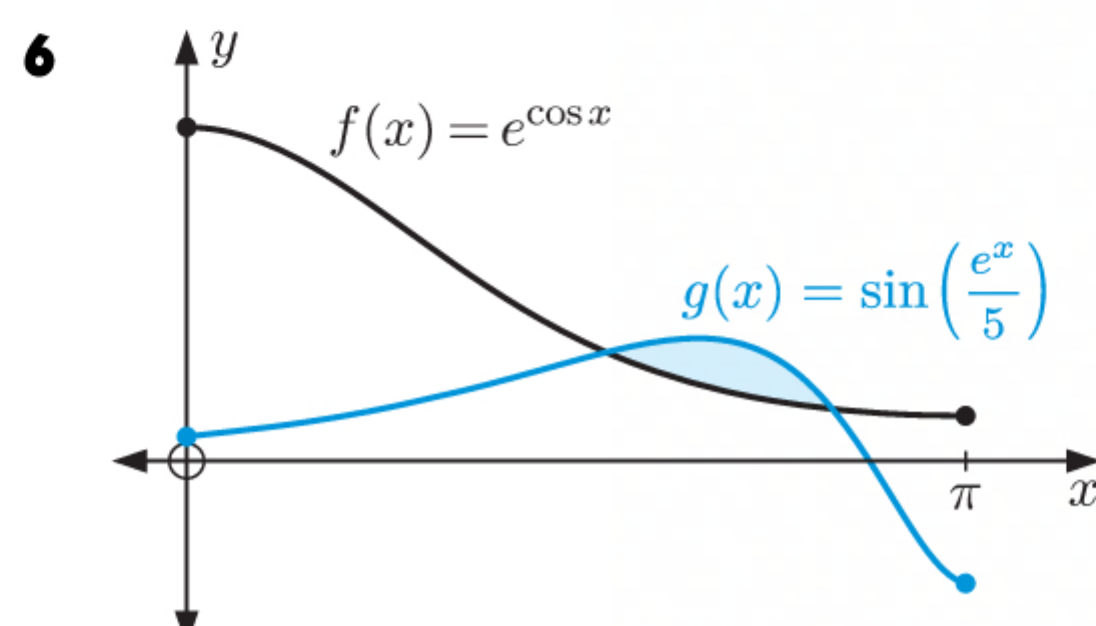
$$\therefore \sigma \approx \frac{36 - 28}{0.966\ 09}$$

A1

$$\therefore \sigma \approx 8.28$$

A1

Total [4 marks]



The graphs intersect at $x \approx 1.69$ and $x \approx 2.60$.

$$\begin{aligned} \therefore A &\approx \int_{1.69}^{2.60} [g(x) - f(x)] dx \\ &\approx 0.250 \quad \{\text{using technology}\} \end{aligned}$$

M1A1

A1

Total [6 marks]

- 7** $h(x) = x^2 - \frac{k}{2}x + (k - 3), \quad k > 0$

$h(x)$ does not intersect with the x -axis if $\Delta < 0$

$$\therefore \left(-\frac{k}{2}\right)^2 - 4(1)(k - 3) < 0$$

M1A1

$$\therefore \frac{k^2}{4} - 4k + 12 < 0$$

A1

$$\therefore k^2 - 16k + 48 < 0$$

$$\therefore (k - 12)(k - 4) < 0$$

A1

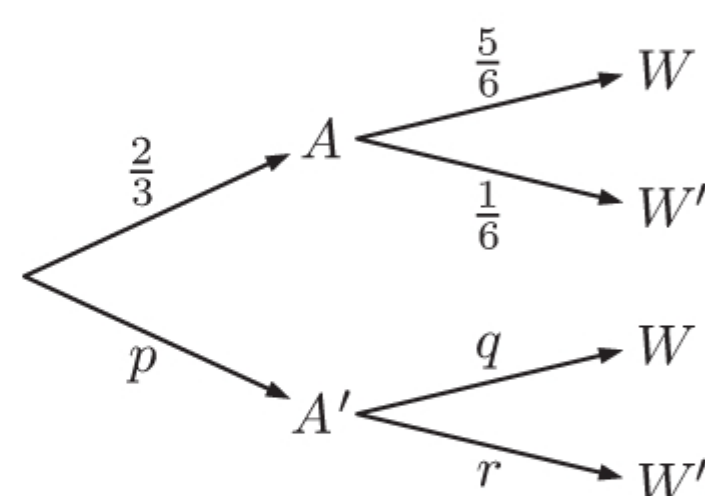
$$\therefore 4 < k < 12$$

A1A1

Total [6 marks]

Section B

8



- a** $p = \frac{1}{3}, q = \frac{2}{5}, r = \frac{3}{5}$

A1A1A1

$$\begin{aligned} \mathbf{b} \quad P(A \cap W') &= \frac{2}{3} \times \frac{1}{6} \\ &= \frac{1}{9} \end{aligned} \quad \begin{array}{l} \text{(M1)} \\ \text{A1} \end{array}$$

$$\begin{aligned} \mathbf{c} \quad P(W) &= P(A \cap W) + P(A' \cap W) \\ &= \frac{2}{3} \times \frac{5}{6} + \frac{1}{3} \times \frac{2}{5} \\ &= \frac{5}{9} + \frac{2}{15} \\ &= \frac{31}{45} \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{A1A1} \\ \text{A1} \end{array}$$

$$\begin{aligned} \mathbf{d} \quad P(A' \mid W) &= \frac{P(A' \cap W)}{P(W)} \\ &= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{31}{45}} \quad \{\text{using } \mathbf{c}\} \\ &= \frac{\left(\frac{2}{15}\right)}{\left(\frac{31}{45}\right)} \\ &= \frac{6}{31} \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{A1} \\ \text{A1} \end{array}$$

\mathbf{e} Let X represent the number of days Maria arrives to work on time.

$$X \sim B\left(5, \frac{31}{45}\right) \quad \text{M1}$$

$$P(X = 3) \approx 0.316 \quad \{\text{using technology}\} \quad \text{A1}$$

Total [14 marks]

9 a The number of tails $T \sim B\left(3, \frac{1}{2}\right)$

$$\therefore P(T = 0) = \frac{1}{8}, P(T = 1) = \frac{3}{8}, P(T = 2) = \frac{3}{8}, P(T = 3) = \frac{1}{8}. \quad \text{M1}$$

So, the probability distribution for X is: M1A1

X (\$)	0	4	m	10
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$Y = X - 5$, so the probability distribution for Y is:

M1A1

Y (\$)	-5	-1	$m - 5$	5
$P(Y = y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\therefore E(Y) = -5\left(\frac{1}{8}\right) - 1\left(\frac{3}{8}\right) + (m - 5)\left(\frac{3}{8}\right) + 5\left(\frac{1}{8}\right) \quad \text{M1}$$

$$= -\frac{5}{8} - \frac{3}{8} + \frac{3}{8}(m - 5) + \frac{5}{8} \quad \text{A1}$$

$$= \frac{3}{8}(m - 5 - 1)$$

$$= \frac{3}{8}(m - 6) \quad \text{AG}$$

\mathbf{b} The game is fair if $E(Y) = 0$

$$\therefore \frac{3}{8}(m - 6) = 0 \quad \text{M1}$$

$$\therefore m = 6 \quad \text{A1}$$

\mathbf{c} Tossing 5 tails when playing the game twice is a result of obtaining 2 tails in one game (profit \$1) and 3 tails in the other game (profit \$5). M1

$$\therefore \text{profit} = \$1 + \$5$$

$$= \$6 \quad \text{A1}$$

Total [11 marks]

10 $y = 4x - 3 - x^2$

$$\mathbf{a} \quad \frac{dy}{dx} = 4 - 2x \quad \text{A1A1}$$

$$\begin{aligned} \mathbf{b} \quad \text{At } x = 4, \quad \frac{dy}{dx} &= 4 - 2(4) \quad \text{and} \quad y = 4(4) - 3 - 4^2 \\ &= -4 \quad \quad \quad = -3 \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{AGA1} \end{array}$$

So L_1 has gradient -4 , and its point of contact with the curve is $(4, -3)$. M1

$$\therefore L_1 \text{ has equation } y - (-3) = -4(x - 4) \quad \text{M1}$$

$$\therefore y = -4x + 13 \quad \text{A1}$$

- c i** L_2 is perpendicular to L_1 , so L_2 has gradient $-\frac{1}{(-4)} = \frac{1}{4}$. **M1**
- The tangent to C has gradient $\frac{1}{4}$ when $4 - 2x = \frac{1}{4}$ **M1A1**
- $$\therefore -2x = -\frac{15}{4}$$
- $$\therefore x = \frac{15}{8}$$
- AG**
- ii** The y -coordinate of A is $4\left(\frac{15}{8}\right) - 3 - \left(\frac{15}{8}\right)^2 = \frac{63}{64}$ **M1**
- $\therefore L_2$ has equation $y - \frac{63}{64} = \frac{1}{4}\left(x - \frac{15}{8}\right)$ **A1**
- $$\therefore y = \frac{1}{4}x - \frac{15}{32} + \frac{63}{64}$$
- $$\therefore y = \frac{1}{4}x + \frac{33}{64}$$
- AG**
- d i**
$$\int_{\frac{15}{8}}^k \left(\left(\frac{1}{4}x + \frac{33}{64} \right) - (4x - 3 - x^2) \right) dx = \frac{43}{512}$$
 M1
- $$\therefore \int_{\frac{15}{8}}^k \left(x^2 - \frac{15}{4}x + \frac{225}{64} \right) dx = \frac{43}{512}$$
- $$\therefore \left[\frac{x^3}{3} - \frac{15}{8}x^2 + \frac{225}{64}x \right]_{\frac{15}{8}}^k = \frac{43}{512}$$
- M1**
- $$\therefore \frac{k^3}{3} - \frac{15}{8}k^2 + \frac{225}{64}k - \frac{1}{3}\left(\frac{15}{8}\right)^3 + \frac{15}{8}\left(\frac{15}{8}\right)^2 - \frac{225}{64}\left(\frac{15}{8}\right) = \frac{43}{512}$$
- A1**
- $$\therefore \frac{k^3}{3} - \frac{15}{8}k^2 + \frac{225}{64}k - \frac{1125}{512} = \frac{43}{512}$$
- $$\therefore \frac{k^3}{3} - \frac{15}{8}k^2 + \frac{225}{64}k - \frac{73}{32} = 0$$
- $$\therefore 64k^3 - 360k^2 + 675k - 438 = 0$$
- AG**
- ii** Using technology, $k \approx 2.51$ **A1**

Total [16 marks]

TRIAL EXAMINATION 3

PAPER 1

Section A

1 a $(n-3)(n)(n+3) = (n^2-3^2)(n) = (n^2-9)(n) = n^3-9n$ M1A1AG

b Let n be a multiple of 3

$\therefore (n-3)(n)(n+3) = n^3-9n$ represents the product of three consecutive multiples of 3. R1

Case 1: If n is even, it can be written as $n = 2k$ for $k \in \mathbb{Z}$.

$$\begin{aligned}\therefore n^3 - 9n &= (2k)^3 - 9(2k) \\ &= 8k^3 - 18k \\ &= 2(4k^3 - 9k) \quad \text{which is even.}\end{aligned}$$
M1A1

Case 2: If n is odd, it can be written as $n = 2k+1$ for $k \in \mathbb{Z}$.

$$\begin{aligned}\therefore n^3 - 9n &= (2k+1)^3 - 9(2k+1) \\ &= (8k^3 + 12k^2 + 6k + 1) - (18k + 9) \\ &= 8k^3 + 12k^2 - 12k - 8 \\ &= 2(4k^3 + 6k^2 - 6k - 4) \quad \text{which is even.}\end{aligned}$$
M1A1

Therefore, the product of three consecutive multiples of 3 is always even.

Total [7 marks]

2 a $P(A|B) = \frac{P(A \cap B)}{P(B)}$, so $P(B) = \frac{P(A \cap B)}{P(A|B)}$

$$\begin{aligned}&= \frac{\frac{1}{5}}{\frac{4}{9}} \\ &= \frac{1}{5} \times \frac{9}{4} \\ &= \frac{9}{20}\end{aligned}$$
M1A1

b $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned}&= \frac{11}{20} + \frac{9}{20} - \frac{1}{5} \\ &= \frac{16}{20} \\ &= \frac{4}{5}\end{aligned}$$
M1A1

c $P(A)P(B) = \frac{11}{20} \times \frac{9}{20} = \frac{99}{400} \neq \frac{1}{5} = P(A \cap B)$

Since $P(A \cap B) \neq P(A)P(B)$, the events A and B are not independent. M1A1

Total [6 marks]

3 $\log_4(x-2) + \frac{1}{2}\log_2(x) = \log_4(15) + \frac{3}{2}$

Using the change of base formula, $\log_4(x-2) + \frac{\log_4(x)}{2\log_4(2)} = \log_4(15) + \frac{3}{2}$ M1

$$\begin{aligned}\therefore \log_4(x-2) + \log_4(x) - \log_4(15) &= \frac{3}{2} \\ \therefore \log_4 \frac{(x-2)(x)}{15} &= \frac{3}{2} \\ \therefore \frac{(x-2)(x)}{15} &= 4^{\frac{3}{2}} \\ \therefore \frac{(x-2)(x)}{15} &= 8 \\ \therefore (x-2)(x) &= 120 \\ \therefore x^2 - 2x - 120 &= 0 \\ \therefore (x-12)(x+10) &= 0 \\ \therefore x &= 12 \text{ or } -10\end{aligned}$$
M1A1

However, $x = -10$ is not acceptable as the LHS of the original equation would be undefined. So, the only solution is $x = 12$. A1

Total [5 marks]

$$\begin{aligned}
\mathbf{4} \quad \mathbf{a} \quad \text{LHS} &= \frac{\sin x + \cos x}{\sin x - \cos x} \times \frac{\cos x - \sin x}{\cos x - \sin x} && \mathbf{M1} \\
&= \frac{\cos^2 x - \sin^2 x}{-\sin^2 x - \cos^2 x + 2 \sin x \cos x} && \mathbf{M1} \\
&= \frac{\cos 2x}{-(\sin^2 x + \cos^2 x) + \sin 2x} && \mathbf{M1} \\
&= \frac{\cos 2x}{-1 + \sin 2x} \\
&= -\frac{\cos 2x}{1 - \sin 2x} = \text{RHS} && \mathbf{A1AG}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \text{Using part a with } x = \frac{\pi}{12}, \quad \frac{\sin \frac{\pi}{12} + \cos \frac{\pi}{12}}{\sin \frac{\pi}{12} - \cos \frac{\pi}{12}} &= -\frac{\cos 2\left(\frac{\pi}{12}\right)}{1 - \sin 2\left(\frac{\pi}{12}\right)} && \mathbf{M1A1} \\
&= -\frac{\cos \frac{\pi}{6}}{1 - \sin \frac{\pi}{6}} \\
&= -\frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} \\
&= -\sqrt{3} && \mathbf{A1}
\end{aligned}$$

Total [7 marks]

$$\begin{aligned}
\mathbf{5} \quad \mathbf{a} \quad \text{The curves meet where } \sin x &= \cos x && \mathbf{M1} \\
\therefore \tan x &= 1 \\
\therefore x &= \frac{\pi}{4} \quad \left\{0 < x < \frac{\pi}{2}\right\} && \mathbf{A1} \\
\therefore A + B &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx && \mathbf{M1A1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad A + B &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx \\
&= \left[\sin x + \cos x \right]_0^{\frac{\pi}{4}} + \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} && \mathbf{M1A1} \\
&= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) + \left(-\cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right) && \mathbf{M1} \\
&= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}} \\
&= 2\sqrt{2} - 2 \text{ units}^2 && \mathbf{A1}
\end{aligned}$$

Total [8 marks]

$$\begin{aligned}
\mathbf{6} \quad \mathbf{a} \quad \int_2^6 (t^2 - t - 6) dt &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_2^6 && \mathbf{M1} \\
&= \left(\frac{6^3}{3} - \frac{6^2}{2} - 6(6) \right) - \left(\frac{2^3}{3} - \frac{2^2}{2} - 6(2) \right) && \mathbf{M1} \\
&= \left(\frac{216}{3} - \frac{36}{2} - 36 \right) - \left(\frac{8}{3} - \frac{4}{2} - 12 \right) \\
&= 18 - \left(-\frac{34}{3} \right) \\
&= \frac{88}{3} \text{ m} && \mathbf{A1}
\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad v(t) &= t^2 - t - 6 \\ &= (t+2)(t-3)\end{aligned}$$

So, $v(t) \leq 0$ for $-2 \leq t \leq 3$ and $v(t) \geq 0$ for $t \leq -2$ and $t \geq 3$. Thus, the total distance travelled during the time period $2 \leq t \leq 6$ is

$$\int_2^6 |t^2 - t - 6| dt = \int_2^3 (-(t^2 - t - 6)) dt + \int_3^6 (t^2 - t - 6) dt \quad \text{M1}$$

$$= \int_2^3 (-t^2 + t + 6) dt + \int_3^6 (t^2 - t - 6) dt \quad \text{M1}$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_2^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^6 \quad \text{M1}$$

$$= \left(-\frac{3^3}{3} + \frac{3^2}{2} + 6(3) \right) - \left(-\frac{2^3}{3} + \frac{2^2}{2} + 6(2) \right) + \left(\frac{6^3}{3} - \frac{6^2}{2} - 6(6) \right) - \left(\frac{3^3}{3} - \frac{3^2}{2} - 6(3) \right)$$

$$= 13.5 - \frac{34}{3} + 18 - (-13.5)$$

$$= \frac{101}{3} \text{ m} \quad \text{A1}$$

Total [7 marks]

Section B

$$\mathbf{7} \quad \mathbf{a} \quad \text{Range} = 14 - a = 12$$

$$\therefore a = 2 \quad \text{M1A1}$$

$$\mathbf{b} \quad \bar{x} = \frac{\sum x}{n} = \frac{90}{15} = 6 \text{ hours} \quad \text{M1A1}$$

$$\begin{aligned}\mathbf{c} \quad \text{Interquartile range} &= b - 4 = 5 \\ \therefore b &= 9 \quad \text{M1A1}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad \bar{x} &= \frac{\sum x}{10} = 5 \\ \therefore \sum x &= 5 \times 10 \\ &= 50 \text{ hours} \quad \text{M1A1}\end{aligned}$$

$$\mathbf{e} \quad \text{The total number of hours all 25 boys and girls spent playing computer games is} \quad \text{M1}$$

$$90 + 50 = 140 \quad \text{A1}$$

$$\text{mean hours} = \frac{140}{15 + 10} = 5.6 \quad \text{A1}$$

Total [11 marks]

$$\mathbf{8} \quad \mathbf{a} \quad u_2 = u_1 r = 8 \sin \theta \cos^3 \theta$$

$$u_4 = u_1 r^3 = \sin 2\theta$$

$$\therefore \frac{u_4}{u_2} = r^2 = \frac{\sin 2\theta}{8 \sin \theta \cos^3 \theta}$$

$$\therefore r = \pm \sqrt{\frac{\sin 2\theta}{8 \sin \theta \cos^3 \theta}} \quad \text{M1A1}$$

$$= \pm \sqrt{\frac{2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta)(4 \cos^2 \theta)}}$$

$$= \pm \sqrt{\frac{1}{4 \cos^2 \theta}}$$

$$= \pm \frac{1}{2 \cos \theta} \quad \text{M1A1}$$

$$\mathbf{b} \quad u_1 = \frac{u_2}{r}$$

$$= \frac{8 \sin \theta \cos^3 \theta}{\pm \frac{1}{2 \cos \theta}}$$

$$= (8 \sin \theta \cos^3 \theta)(\pm 2 \cos \theta) \quad \text{M1}$$

$$= \pm 16 \sin \theta \cos^4 \theta \quad \text{A1A1}$$

$$\begin{aligned}
 \text{c } u_8 &= u_1 \times r^7 \\
 &= \pm 16 \sin \theta \cos^4 \theta \times \left(\pm \frac{1}{2 \cos \theta} \right)^7 && \text{M1} \\
 &= \frac{\sin \theta}{8 \cos^3 \theta} && \text{A1}
 \end{aligned}$$

Total [9 marks]

$$9 \text{ a i } \Delta = b^2 - 4ac = 18^2 - 4(3)(c) = 324 - 12c \quad \text{M1A1}$$

$$\text{ii As the equation } f(x) = 0 \text{ has two equal roots, } \Delta = 0 \quad \text{M1}$$

$$\therefore 324 - 12c = 0$$

$$\therefore 324 = 12c$$

$$\therefore c = 27 \quad \text{A1}$$

$$\text{b } \frac{-b}{2a} = \frac{-18}{2(3)} = -3 \quad \text{M1A1}$$

The quadratic has axis of symmetry $x = -3$, so the x -coordinate of the vertex is -3 .

As the graph of $f(x)$ has its vertex on the x -axis, the y -coordinate of the vertex is 0. Therefore, the vertex has coordinates $(-3, 0)$. A1A1

$$\text{c } x = -3 \quad \text{A1}$$

$$\text{d i } a = 3 \quad \text{A1}$$

$$\text{ii } h = -3 \quad \text{A1}$$

$$\text{iii } k = 0 \quad \text{A1}$$

$$\text{e Transforming } f(x) \text{ by a reflection in the } x\text{-axis results in } -f(x) = -3(x+3)^2. \quad \text{M1A1}$$

Then, transforming $-f(x)$ by a vertical stretch with scale factor $\frac{1}{3}$ results in

$$\frac{1}{3} \times (-f(x)) = \frac{1}{3} \times (-3(x+3)^2) = -(x+3)^2$$

$$\therefore g(x) = -(x+3)^2 \quad \text{M1A1}$$

$$\text{f As } g(x) = -(x+3)^2, \text{ we have } g'(x) = -2(x+3) = -2x-6. \quad \text{M1A1}$$

$$\text{Now } g'(a) = 8$$

$$\therefore -2a - 6 = 8$$

$$\therefore a = -7 \quad \text{M1A1}$$

Total [20 marks]

PAPER 2

Section A

$$\begin{aligned}
 1 \text{ a Horizontal asymptote: } y &= -\frac{2}{4} \\
 &\therefore y = -\frac{1}{2} && \text{A1}
 \end{aligned}$$

$$\text{Vertical asymptote: } 4x - 5 = 0$$

$$\therefore x = \frac{5}{4} \quad \text{A1}$$

$$\text{b } g(x) \text{ has horizontal asymptote } y = 0. \quad \text{A1}$$

$$\therefore g^{-1}(x) \text{ has vertical asymptote } x = 0.$$

c Note that $h(x) = (h \circ g \circ g^{-1})(x)$
 $= (h \circ g)(g^{-1}(x))$

Find $g^{-1}(x)$:

Now if $g(x)$ is $y = \frac{5}{3x+3}$

then $g^{-1}(x)$ is $x = \frac{5}{3y+3}$

$$\therefore 3y+3 = \frac{5}{x}$$

$$\therefore y = \frac{5-3x}{3x}$$

$$\therefore g^{-1}(x) = \frac{5-3x}{3x}$$

$$\therefore h(x) = (h \circ g)\left(\frac{5-3x}{3x}\right)$$

$$= 3\left(\frac{5-3x}{3x}\right)$$

$$= \frac{5-3x}{x}$$

M1

A1

A1

Total [6 marks]

2 a $\frac{\$17\,575}{95} \times 100 = \$18\,500$

M1A1

b Total amount $= 17\,575 \left(1 + \frac{15}{12 \times 100}\right)^{12 \times \frac{1}{2}}$
 $= 18\,935.01$

M1A1

A1

Therefore, Jose pays $\$18\,935.01 - \$18\,500 = \$435.01$ more than the original price.

A1

Total [6 marks]

3 a Area of sector AOB $= \frac{1}{2} \times 6^2 \times \frac{\pi}{3} = 6\pi$

M1A1

b In right-angled triangle OCP, $CP = r$, $OP = 6 - r$, and $\widehat{COP} = \frac{\pi}{6}$.

M1

$$\sin \frac{\pi}{6} = \frac{r}{6-r}$$

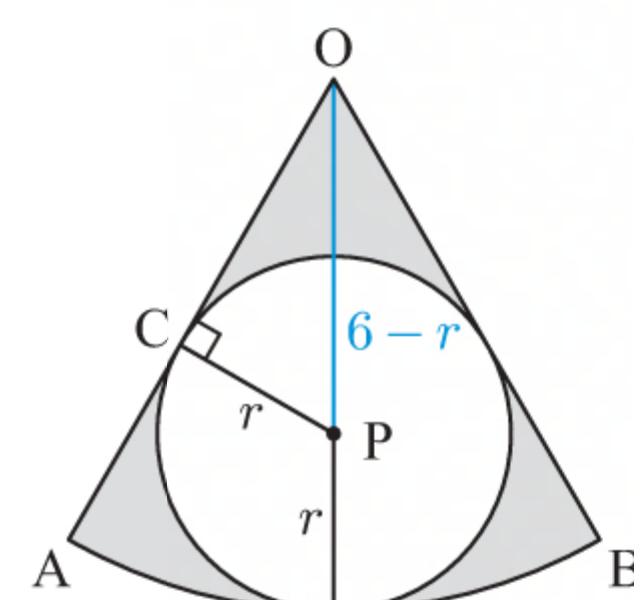
M1

$$\therefore \frac{1}{2} = \frac{r}{6-r}$$

$$\therefore 2r = 6 - r$$

$$\therefore r = 2$$

A1



c Area of circle with centre P $= \pi \times 2^2 = 4\pi$

M1

$$\therefore \text{area of shaded region} = 6\pi - 4\pi = 2\pi$$

M1A1

Total [8 marks]

4 a We know that $\pi r^2 h = 1.3$, so $h = \frac{1.3}{\pi r^2}$.

M1A1

Given that it is a fully closed container, its surface area is

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \left(\frac{1.3}{\pi r^2}\right) = 2\pi r^2 + \frac{2.6}{r}$$

M1AG

b $A(r) = 2\pi r^2 + 2.6r^{-1}$

$$\therefore \frac{dA}{dr} = 4\pi r - 2.6r^{-2}$$

M1A1

c The surface area is minimised when $\frac{dA}{dr} = 0$.

M1

$$\therefore 4\pi r = \frac{2.6}{r^2}$$

$$\therefore r^3 = \frac{2.6}{4\pi}$$

A1

$$\therefore r = \sqrt[3]{\frac{2.6}{4\pi}} \approx 0.591$$

A1

Total [8 marks]

- 5** 91% of the flights took under 12 hours 45 minutes = 765 minutes, so

$$P\left(Z < \frac{765 - \mu}{\sigma}\right) = 0.91 \quad \text{M1}$$

Similarly 95.5% of the flights took over 11 hours and 45 minutes = 705 minutes, so

$$P\left(Z > \frac{705 - \mu}{\sigma}\right) = 0.955$$

$$\therefore P\left(Z < \frac{705 - \mu}{\sigma}\right) = 0.045 \quad \text{M1}$$

$$\therefore \frac{765 - \mu}{\sigma} \approx 1.3408 \quad \text{and} \quad \frac{705 - \mu}{\sigma} \approx -1.6954$$

$$\therefore 765 - \mu \approx 1.3408\sigma \quad \dots (1) \quad \text{and} \quad 705 - \mu \approx -1.6954\sigma \quad \dots (2) \quad \text{A1A1}$$

Solving (1) and (2) simultaneously gives $\mu \approx 738.504$ and $\sigma \approx 19.762$ **M1**

Rounded to the nearest minute, we have $\mu \approx 739$ minutes and $\sigma \approx 20$ minutes **A1A1**

Total [7 marks]

- 6 a** This is a binomial distribution.

$$R \sim B(500, 0.15) \quad \text{A1A1}$$

b $\mu = 500 \times 0.15 = 75$ **A1**

c $P(R > 78) = 1 - P(R \leq 78) \approx 0.326$ **M1A1**

Total [5 marks]

Section B

7 a $f(0) = 2(0)^3 + 5(0)^2 + a(0) + 2 = 2$ **M1**

The coordinates of P are (0, 2). **A1**

b i $f'(x) = 6x^2 + 10x + a$ **A2**

ii $f'(0) = 6(0)^2 + 10(0) + a$
 $= a$ **M1A1**

$\therefore L$ has equation $y - 2 = a(x - 0)$
 $\therefore y = ax + 2$ **M1A1**

c $f(x)$ meets L where $2x^3 + 5x^2 + ax + 2 = ax + 2$
 $\therefore 2x^3 + 5x^2 = 0$
 $\therefore x^2(2x + 5) = 0$ **M1**
 $\therefore x = 0$ or $-\frac{5}{2}$

Thus, the x -coordinate of Q is $-\frac{5}{2}$. **A1**

d i $a(-1) + 2 = 0$
 $\therefore a = 2$ **M1A1**

ii When $x = -\frac{5}{2}$, $y = 2\left(-\frac{5}{2}\right) + 2 = -3$ **M1A1**

e The stationary points of $f(x)$ occur when $f'(x) = 0$
 $\therefore 6x^2 + 10x + 2 = 0$
 $\therefore 3x^2 + 5x + 1 = 0$ **M1A1**

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(1)}}{2(3)} = \frac{-5 \pm \sqrt{13}}{6} \quad \text{M1A1}$$

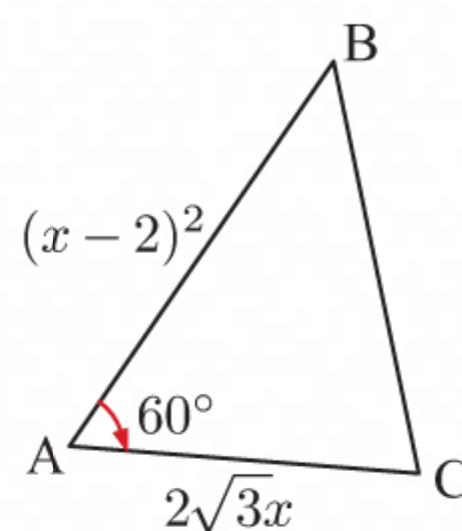
From the graph, the x -coordinate of R is $\frac{-5 - \sqrt{13}}{6} \approx -1.43$

Now $f(-1.43) \approx 2(-1.43)^3 + 5(-1.43)^2 + 2(-1.43) + 2$
 ≈ 3.52 **M1A1**

So, R has coordinates $(-1.43, 3.52)$.

Total [20 marks]

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad A &= \frac{1}{2}(x-2)^2(2\sqrt{3}x) \sin 60^\circ \\
 &= (x-2)^2(\sqrt{3}x) \left(\frac{\sqrt{3}}{2}\right) \\
 &= \frac{3}{2}x(x^2 - 4x + 4) \\
 &= \frac{3}{2}x^3 - 6x^2 + 6x
 \end{aligned}$$



M1A1AG

$$\mathbf{b} \quad \mathbf{i} \quad \frac{dA}{dx} = \frac{9}{2}x^2 - 12x + 6$$

A1A1A1

$$\begin{aligned}
 \mathbf{ii} \quad \frac{dA}{dx} &= 0 \text{ when } \frac{9}{2}x^2 - 12x + 6 = 0 \\
 &\therefore 9x^2 - 24x + 12 = 0 \\
 &\therefore 3x^2 - 8x + 4 = 0 \\
 &\therefore (x-2)(3x-2) = 0
 \end{aligned}$$

M1

$$\text{So, since } x \neq 2, \frac{dA}{dx} = 0 \text{ when } x = \frac{2}{3}.$$

A1AG

$$\mathbf{c} \quad \mathbf{i} \quad \frac{d^2A}{dx^2} = \frac{d}{dx} \left(\frac{9}{2}x^2 - 12x + 6 \right) = 9x - 12$$

M1A1

$$\mathbf{ii} \quad \text{For } x = \frac{2}{3},$$

$$\frac{d^2A}{dx^2} = 9\left(\frac{2}{3}\right) - 12 = 6 - 12 = -6$$

A1

$$\text{As } \frac{d^2A}{dx^2} < 0, x = \frac{2}{3} \text{ gives the maximum area for the triangle ABC.}$$

R1AG

$$\mathbf{d} \quad \text{When } x = \frac{2}{3}, \quad AB = \left(\frac{2}{3} - 2\right)^2 = \left(-\frac{4}{3}\right)^2 = \frac{16}{9} \text{ cm}$$

M1A1

$$\text{and } AC = 2\sqrt{3}\left(\frac{2}{3}\right) = \frac{4\sqrt{3}}{3} \text{ cm}$$

M1A1

$$\mathbf{e} \quad A_{\max} = \frac{1}{2} \left(\frac{16}{9} \right) \left(\frac{4\sqrt{3}}{3} \right) \sin 60^\circ = \frac{1}{2} \left(\frac{16}{9} \right) \left(\frac{4\sqrt{3}}{3} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{16}{9} \text{ cm}^2$$

M1A1

$$\mathbf{f} \quad \text{Using the cosine rule,}$$

$$(BC)^2 = \left(\frac{16}{9}\right)^2 + \left(\frac{4\sqrt{3}}{3}\right)^2 - 2\left(\frac{16}{9}\right)\left(\frac{4\sqrt{3}}{3}\right) \cos 60^\circ = \frac{256}{81} + \frac{16}{3} - \frac{64\sqrt{3}}{27} \approx 4.388$$

M1A1

$$\therefore BC \approx 2.09 \text{ cm}$$

A1

Total [20 marks]

TRIAL EXAMINATION 4

PAPER 1

Section A

1 a By the cosine rule, $(2\sqrt{7})^2 = 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos \widehat{BAC}$ **M1A1**

$$\therefore 48 \cos \widehat{BAC} = 16 + 36 - 28$$

$$\therefore \cos \widehat{BAC} = \frac{24}{48} = \frac{1}{2}$$
 A1

$$\therefore \widehat{BAC} = 60^\circ$$
 A1

b Area of triangle $= \frac{1}{2} \times 4 \times 6 \times \sin \widehat{BAC}$ **M1**

$$= \frac{1}{2} \times 4 \times 6 \times \sin 60^\circ$$

$$= 12 \times \frac{\sqrt{3}}{2}$$

$$= 6\sqrt{3} \text{ units}^2$$
 A1

Total [6 marks]

2 $P(Y) = P(X \cap Y) + P(X' \cap Y) = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$ **A1**

Since X and Y are independent, $P(X \cap Y) = P(X)P(Y)$

$$\therefore \frac{1}{5} = P(X) \times \frac{7}{10}$$

$$\therefore P(X) = \frac{1}{5} \times \frac{10}{7} = \frac{2}{7}$$
 M1A1

$$P(X \cup Y) = P(X) + P(X' \cap Y)$$
 M1

$$= \frac{2}{7} + \frac{1}{2}$$

$$= \frac{11}{14}$$
 A1

Total [5 marks]

3 Since $a + b + c$ is divisible by 3, $a + b + c = 3n$ for some $n \in \mathbb{Z}$ **M1**

$$\therefore 100a + 10b + c = 3n + 99a + 9b$$
 M1

$$\therefore "abc" = 3(n + 33a + 3b)$$
 A1

Since $n + 33a + 3b \in \mathbb{Z}$, " abc " is divisible by 3. **R1**

Total [4 marks]

4 a $f(x) = e^{\sin kx} + c$

$$\therefore f(0) = e^0 + c$$

$$= 1 + c$$
 A1

The tangent has y -intercept 3, so $1 + c = 3$

$$\therefore c = 2$$
 M1A1

b $f'(x) = e^{\sin kx}(k \cos kx)$

$$\therefore f'(0) = e^{\sin 0}(k \cos 0) = k$$
 M1A1

The tangent has gradient -1 , so $k = -1$. **A1**

Total [6 marks]

5 a g is $y = \frac{2x+1}{3}$

$$\therefore g^{-1} \text{ is } x = \frac{2y+1}{3}$$
 M1

$$\therefore 3x = 2y + 1$$

$$\therefore 2y = 3x - 1$$
 A1

$$\therefore y = \frac{3x-1}{2}$$

$$\therefore g^{-1}(x) = \frac{3x-1}{2}$$
 AG

$$\mathbf{b} \quad (f \circ g^{-1})(x) = 4$$

$$\therefore f\left(\frac{3x-1}{2}\right) = 4$$

$$\therefore 3 - \frac{3x-1}{2} = 4$$

$$\therefore 3 - \frac{3}{2}x + \frac{1}{2} = 4$$

$$\therefore -\frac{3}{2}x = \frac{1}{2}$$

$$\therefore x = -\frac{1}{3}$$

M1

A1

A1

Total [5 marks]

$$\mathbf{6} \quad 9^x + 18 = 3^{x+2}$$

$$\therefore 9^x - 3^{x+2} + 18 = 0$$

$$\therefore (3^x)^2 - 9 \times 3^x + 18 = 0$$

$$\therefore (3^x - 3)(3^x - 6) = 0$$

$$\therefore 3^x = 3 \quad \text{or} \quad 3^x = 6$$

$$\therefore x = 1 \quad \text{or} \quad x = \log_3 6$$

M1A1

A1

A1A1

Total [5 marks]

Section B

7 a i The ordered data set is:

6.8, 7.8, 8.2, 8.5, 8.7, 9.6, 9.9, 10.0, 10.1, 10.3, 11.3, 11.5, 12.1, 13.2, 14.2 (M1)

\therefore minimum = 6.8

$Q_1 = 8.5$

median = 10.0

$Q_3 = 11.5$

maximum = 14.2

A1

A1

A1

ii The IQR = $11.5 - 8.5 = 3$

$$\begin{aligned} \therefore \text{upper quartile} + 1.5 \times \text{IQR} &= 11.5 + 1.5 \times 3 \\ &= 16 > \text{maximum} \end{aligned}$$

M1

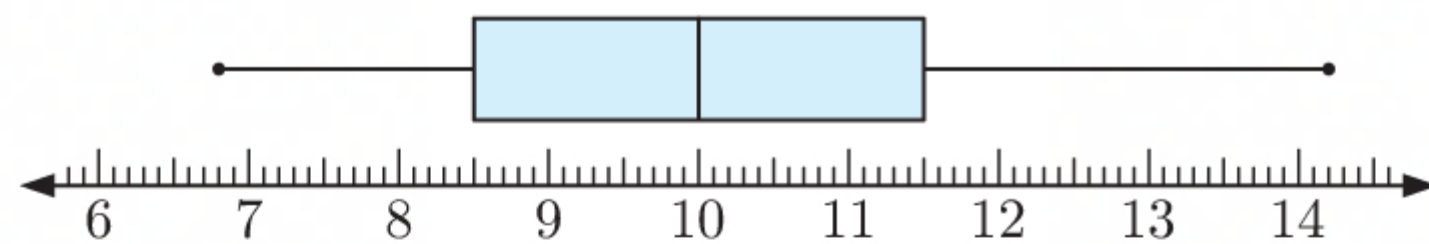
$$\begin{aligned} \text{and lower quartile} - 1.5 \times \text{IQR} &= 8.5 - 1.5 \times 3 \\ &= 4 < \text{minimum} \end{aligned}$$

M1

\therefore there are no outliers.

A1

iii



A1 : scale

A1 : box

A1 : whiskers

b i Range = $13.1 - 3 = 10.1$

A1

IQR = $11.3 - 8 = 3.3$

A1

ii Each statistic of the 5-number summary for sample B is lower than the corresponding statistic for sample A.

R1

The IQR for sample A is also less than the IQR for sample B.

R1

\therefore sample A had better and more reliable growing conditions.

A1

Total [15 marks]

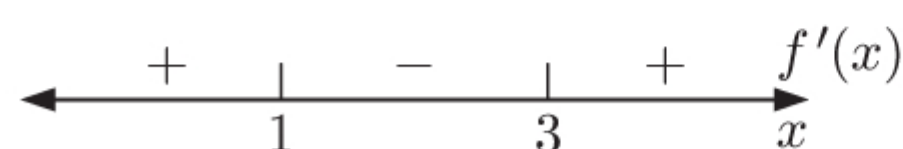
$$\mathbf{8 a} \quad f(x) = x^3 - 6x^2 + 9x - 2$$

$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x-1)(x-3)$$

A1



M1A1A1

$$\begin{aligned}\mathbf{b} \quad f(1) &= (1)^3 - 6(1)^2 + 9(1) - 2 = 2 \\ f(3) &= (3)^3 - 6(3)^2 + 9(3) - 2 = -2\end{aligned}$$

Using the sign diagram from **a**, there is a local maximum at $(1, 2)$ and a local minimum at $(3, -2)$. **A1A1**

$$\begin{aligned}\mathbf{c} \quad f''(x) &= 6x - 12 \\ &= 6(x - 2) \quad \mathbf{M1} \\ f(2) &= 8 - 24 + 18 - 2 = 0 \quad \mathbf{A1}\end{aligned}$$

\therefore the inflection point is at $(2, 0)$. **A1**

$$\begin{aligned}\mathbf{d} \quad \mathbf{i} \quad f'(2) &= 3(2)^2 - 12(2) + 9 = -3 \quad \mathbf{A1} \\ \therefore \text{ the gradient of the normal at the inflection point is } \frac{1}{3}, \text{ and it passes through } (2, 0) \\ \therefore \text{ its equation is } y &= \frac{1}{3}(x - 2). \quad \mathbf{M1A1}\end{aligned}$$

ii The normal meets the curve again when

$$x^3 - 6x^2 + 9x - 2 = \frac{1}{3}(x - 2) \quad \mathbf{M1}$$

We know $x - 2$ is a factor of the LHS, so we write

$$x^3 - 6x^2 + 9x - 2 = (x - 2)(x^2 + ax + 1) \quad \mathbf{M1}$$

Equating coefficients of x^2 , $-6 = a - 2$

Equating coefficients of x , $9 = 1 - 2a$ **M1**

Both equations give $a = -4$ **A1**

\therefore the normal meets the curve again when $(x - 2)(x^2 - 4x + 1) = \frac{1}{3}(x - 2)$

$$\therefore x^2 - 4x + 1 = \frac{1}{3}$$

$$\therefore x^2 - 4x + \frac{2}{3} = 0 \quad \mathbf{A1}$$

$$\therefore 3x^2 - 12x + 2 = 0$$

$$\begin{aligned}\therefore x &= \frac{12 \pm \sqrt{(-12)^2 - 4(3)(2)}}{6} \\ &= \frac{12 \pm \sqrt{144 - 24}}{6} \\ &= \frac{12 \pm 2\sqrt{30}}{6} \\ &= 2 \pm \frac{1}{3}\sqrt{30} \quad \mathbf{A1A1}\end{aligned}$$

Total [19 marks]

$$\begin{aligned}\mathbf{9} \quad \mathbf{a} \quad f(x) &= x \cos(kx^2), \text{ so } f(1) = \cos k \quad \mathbf{A1} \\ f'(x) &= \cos(kx^2) + x(-2kx \sin(kx^2)) \\ &= \cos(kx^2) - 2kx^2 \sin(kx^2) \quad \mathbf{M1A1}\end{aligned}$$

Now the tangent at $x = 1$ passes through $(0, 0)$ and $(1, \cos k)$, so

$$f'(1) = \cos k \quad \mathbf{R1}$$

$$\therefore \cos k - 2k \sin k = \cos k$$

$$\therefore 2k \sin k = 0 \quad \mathbf{A1}$$

$$\therefore k = \pi \quad \{\text{since } 0 < k < 2\pi\} \quad \mathbf{A1}$$

$$\mathbf{b} \quad f(x) = x \cos(\pi x^2)$$

The first positive x -intercept p occurs when $\pi x^2 = \frac{\pi}{2}$ **M1**

$$\therefore x^2 = \frac{1}{2}$$

$$\therefore p = \frac{1}{\sqrt{2}} \quad \{\text{since } p > 0\} \quad \mathbf{A1}$$

c Let $u = \pi x^2$, so $\frac{du}{dx} = 2\pi x$ M1A1

$$\begin{aligned}\therefore \int x \cos(\pi x^2) dx &= \frac{1}{2\pi} \int \cos(\pi x^2) \times 2\pi x dx \\ &= \frac{1}{2\pi} \int \cos u \frac{du}{dx} dx \\ &= \frac{1}{2\pi} \int \cos u du\end{aligned}$$
A1

$$\begin{aligned}&= \frac{1}{2\pi} \sin u + c \\ &= \frac{1}{2\pi} \sin(\pi x^2) + c\end{aligned}$$
A1

$$\begin{aligned}\therefore \text{the shaded area} &= \int_0^{\frac{1}{\sqrt{2}}} x \cos(\pi x^2) dx \\ &= \left[\frac{1}{2\pi} \sin(\pi x^2) \right]_0^{\frac{1}{\sqrt{2}}} \\ &= \frac{1}{2\pi} \sin \frac{\pi}{2} - \frac{1}{2\pi} \sin 0 \\ &= \frac{1}{2\pi} \text{ units}^2\end{aligned}$$
M1A1

Total [15 marks]

PAPER 2

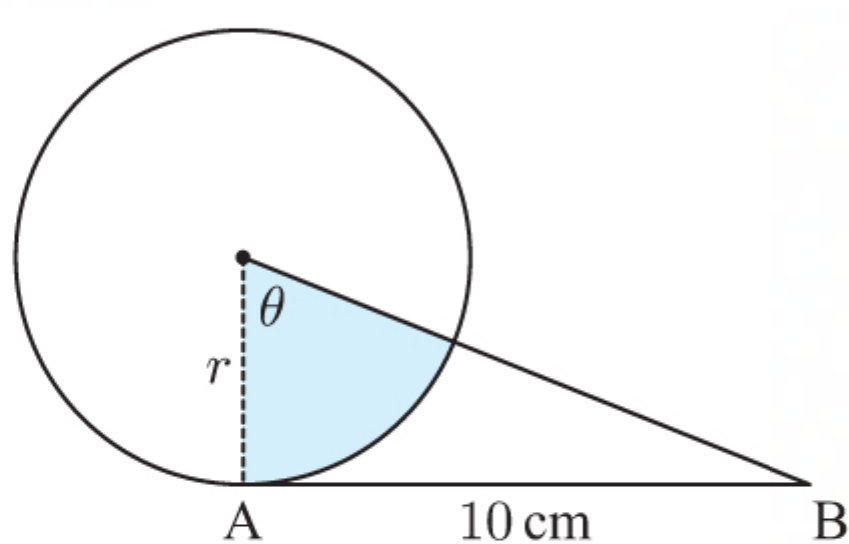
Section A

1 a Volume of Saturn $\approx \frac{4}{3}\pi r^3$
 $\approx \frac{4}{3}\pi(5.8232 \times 10^4)^3$
 $\approx 8.2713 \times 10^{14} \text{ km}^3$ which agrees with the NASA data. M1A1

b Radius of outer ring $R \approx 5.8232 \times 10^4 + 2.82 \times 10^5 \approx 3.40232 \times 10^5 \text{ km}$ A1
 Circumference of orbit $\approx 2\pi R$
 $\approx 2\pi \times (3.40232 \times 10^5)$ M1
 $\approx 2.14 \times 10^6 \text{ km}$ A1

Total [5 marks]

2



Since the shaded area is 20 cm^2 , $\frac{\theta}{2\pi} \times \pi r^2 = 20$
 $\therefore \theta r^2 = 40$ M1A1

Since AB is a tangent, $\tan \theta = \frac{10}{r}$ M1

$$\therefore r = \frac{10}{\tan \theta}$$
A1

$$\therefore \theta \times \frac{100}{\tan^2 \theta} = 40$$
M1

$$\therefore \frac{\theta}{\tan^2 \theta} = \frac{2}{5}$$

Using technology, $\theta \approx 1.01$ {since $0 < \theta < \frac{\pi}{2}$ } A1

and $r = \frac{10}{\tan \theta} \approx 6.30 \text{ cm}$ A1

Total [7 marks]

3 a Each quarter, the interest paid is $\text{€}10\,000 \times \frac{0.05}{4}$ A1

$$\begin{aligned}\therefore \text{after } n \text{ quarters, the investment is worth } &\text{€}10\,000 + n \times \text{€}10\,000 \times \frac{0.05}{4} \\ &= \text{€}10\,000(1 + 0.0125n)\end{aligned}$$
A1

b i Each quarter, the value of the investment is multiplied by $1 + \frac{0.044}{4} = 1.011$ A1

\therefore after n quarters, the investment is worth $\text{€}10\,000 \times 1.011^n$ A1

ii After 7 quarters, the compound interest investment would be worth $\text{€}10\,000 \times 1.011^7 = \text{€}10\,795.88$ A1

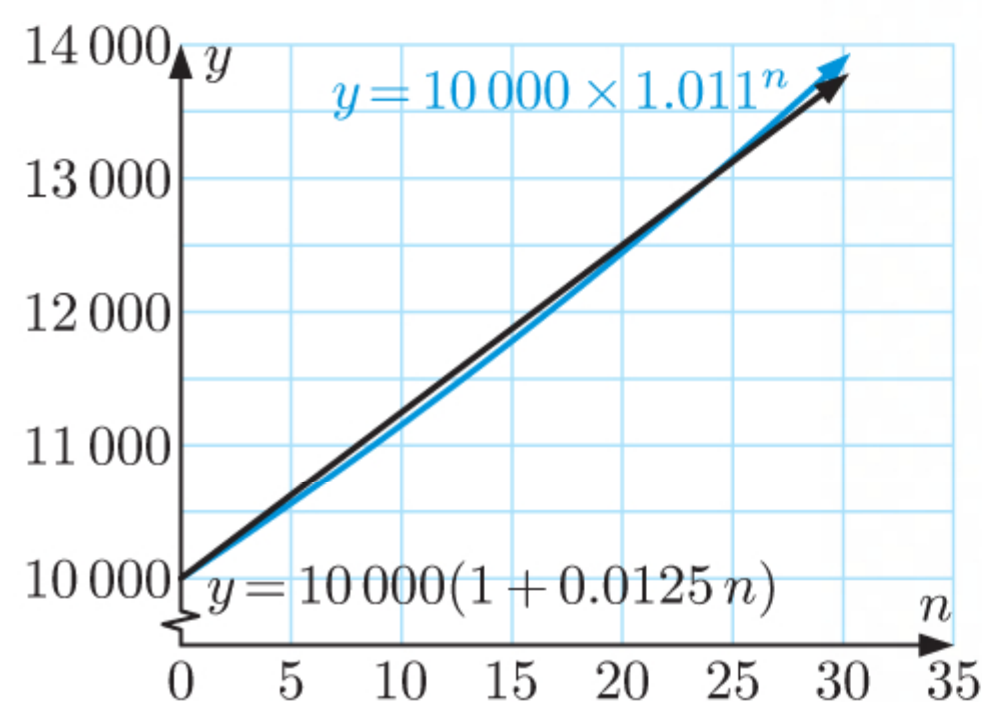
c We use technology to compare the graphs of

$$y = 10\,000(1 + 0.0125n)$$

and $y = 10\,000 \times 1.011^n$

The graphs intersect when $n \approx 23.85$

\therefore it will take 24 quarters, which is 6 years, for the compound interest investment to be the better option.



M1

A1

A1

Total [8 marks]

4 a There are $6 \times 6 = 36$ possible outcomes when rolling the dice.

Of these, the outcomes with sum 5 are: $(1, 4), (2, 3), (3, 2), (4, 1)$. A1

So, the probability that the sum is 5 is $\frac{4}{36} = \frac{1}{9}$. A1

b When the pair of dice is rolled 10 times, let X be the number of times that the sum of the dice is 5.

$\therefore X \sim B(10, \frac{1}{9})$ M1A1

Using technology, $P(X \geq 2) \approx 0.307$ A1

Total [5 marks]

5 a Using technology, $r \approx 0.974$ A1

b The time taken to get to work *depends* on the distance the worker needs to travel, so the time taken is the dependent variable, and the distance is the independent variable. R1

c i Since the workers have measured their distance to work by the path they move along rather than the straight-line distance, the values for x are less accurately measured than the values for y . It is therefore appropriate to use the x against y regression line. R1AG

ii Using technology, $x \approx 0.3489y - 2.4926$ A1A1

iii Letting $y = 50$, $x \approx 15.0$

Jody would expect the worker to live about 15.0 km from the office. A1

Total [6 marks]

6 a $v(t) = -32 \sin 2t \text{ cm s}^{-1}$

$\therefore a(t) = -64 \cos 2t \text{ cm s}^{-2}$ A1

$\therefore a(0) = -64 \text{ cm s}^{-2}$ A1

The initial acceleration is 64 cm s^{-2} in the direction of the pendulum's mean position.

b $s(t) = \int v(t) dt$

$$= \int (-32 \sin 2t) dt \text{ cm} \quad \text{M1}$$

$$= (16 \cos 2t + c) \text{ cm} \quad \text{A1}$$

The amplitude of the oscillation is 16 cm, so the arclength of the pendulum's path is 32 cm. R1

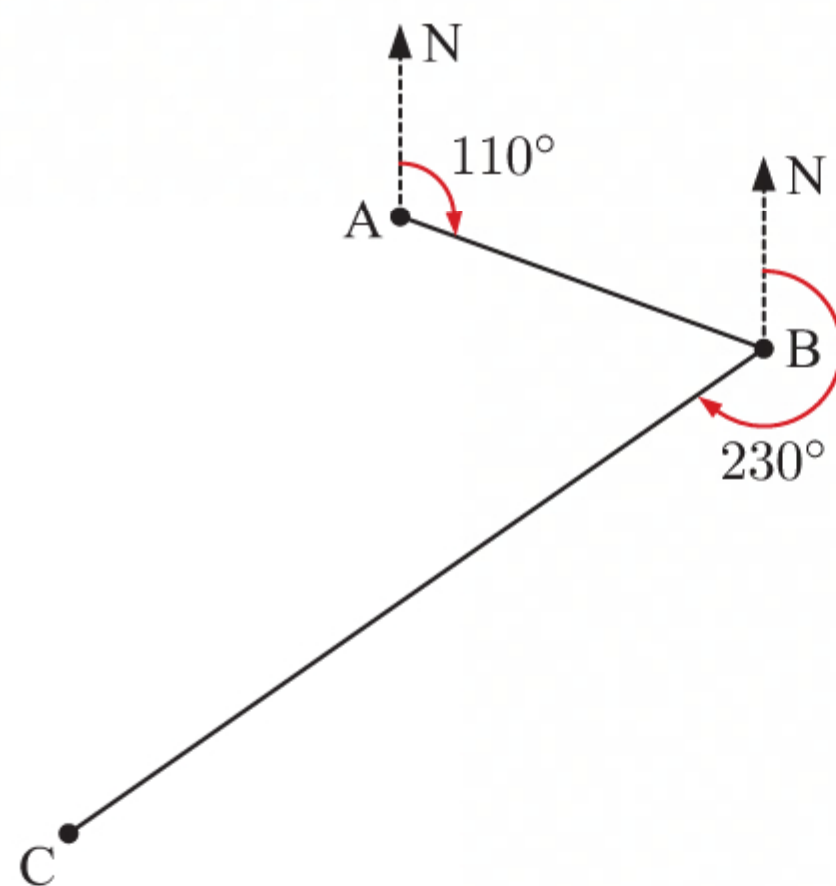
c To get from the maximum to minimum displacement takes $\frac{\pi}{2}$ seconds, so the average speed is M1A1

$$\frac{32}{(\frac{\pi}{2})} \text{ cm s}^{-1} \approx 20.4 \text{ cm s}^{-1}.$$

Total [7 marks]

Section B

7 a



$$\begin{aligned}\widehat{ABC} &= 360^\circ - 230^\circ - (180 - 110)^\circ \\ &= 60^\circ\end{aligned}$$

M1

A1

- b After t seconds, Alan has run $3t$ m.

A1

\therefore he is $(600 - 3t)$ m from B.

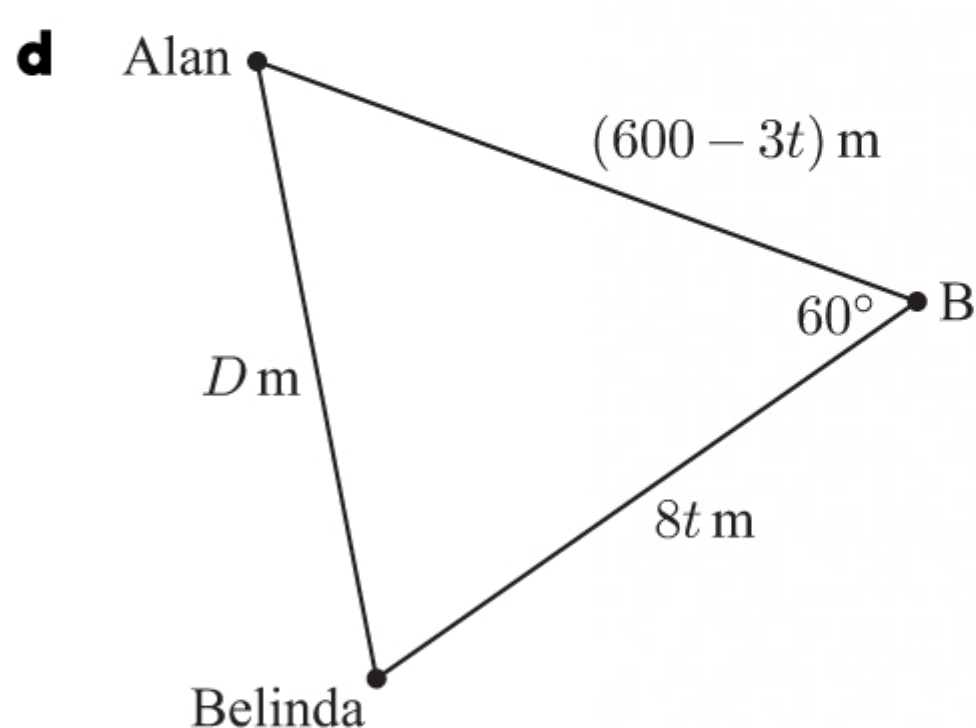
A1

- c After t seconds, Belinda has cycled $8t$ m.

A1

\therefore she is $8t$ m from B.

A1



Let the distance between Alan and Belinda after t seconds be D m.

Using the cosine rule,

$$\begin{aligned}D^2 &= (600 - 3t)^2 + (8t)^2 - 2(600 - 3t)(8t) \cos 60^\circ \\ &= 360\,000 - 3600t + 73t^2 - 8t(600 - 3t) \\ &= 97t^2 - 8400t + 360\,000\end{aligned}$$

M1A1

$$\therefore D = \sqrt{97t^2 - 8400t + 360\,000} \text{ m}$$

A1

- e D is minimised when D^2 is minimised.

(M1)

D^2 is a quadratic, so is minimised when $t = \frac{8400}{2 \times 97} \approx 43.3$ seconds.

A1

At this time, $D \approx 422$ m.

A1

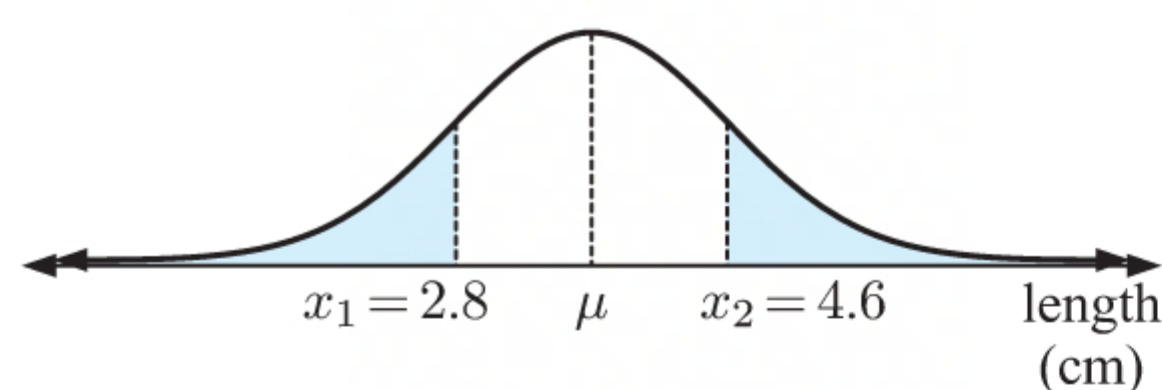
The minimum distance between Alan and Belinda is about 422 m. This occurs after about 43.3 seconds.

Total [12 marks]

- 8 a i Let X be the length of an acorn.

$$\therefore X \sim N(\mu, \sigma^2)$$

Let z_1 and z_2 be the z -scores corresponding to $x_1 = 2.8$ cm and $x_2 = 4.6$ cm.



$$\text{Now } P(X \leq x_1) = 0.15$$

$$\therefore P\left(Z \leq \frac{2.8 - \mu}{\sigma}\right) = 0.15$$

$$\therefore z_1 = \frac{2.8 - \mu}{\sigma} \approx -1.0364 \quad \{\text{technology}\}$$

M1A1

$$\therefore 2.8 - \mu \approx -1.0364\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.18$$

$$\therefore P\left(Z \geq \frac{4.6 - \mu}{\sigma}\right) = 0.18$$

$$\therefore z_2 = \frac{4.6 - \mu}{\sigma} \approx 0.9154 \quad \{\text{technology}\}$$

M1A1

$$\therefore 4.6 - \mu \approx 0.9154\sigma \quad \dots (2)$$

ii (2) – (1) gives $1.8 \approx 1.9518\sigma$

$$\therefore \sigma \approx 0.9222$$

A1

Substituting into (1), $2.8 - \mu \approx -1.0364 \times 0.9222$

$$\therefore \mu \approx 3.756$$

A1

So, the population mean $\mu \approx 3.76$ cm and standard deviation $\sigma \approx 0.922$ cm.

b $X \sim N(3.756, 0.922)$

M1

$$\therefore P(X > 4) \approx 0.396$$

A1

$$\therefore Y \sim B(12, 0.396)$$

M1A1

$$\therefore E(Y) \approx 12 \times 0.396$$

$$\approx 4.75 \text{ acorns}$$

A1

Total [11 marks]

9 a $f_2(0) = 0$

$$\therefore 2 \ln(\cos 0 + 1) + a = 0$$

M1

$$\therefore a = -2 \ln 2$$

A1

b i $f_2(k) = -4$

$$\therefore 2 \ln\left(\cos \frac{k\pi}{12} + 1\right) - 2 \ln 2 = -4$$

M1A1

$$\therefore \ln\left(\cos \frac{k\pi}{12} + 1\right) = \ln 2 - 2$$

$$= \ln 2 - \ln e^2$$

M1

$$= \ln\left(\frac{2}{e^2}\right)$$

A1

$$\therefore \cos \frac{k\pi}{12} = \frac{2}{e^2} - 1$$

AG

ii $\frac{k\pi}{12} \approx 2.388$

$$\therefore k \approx 9.12$$

A1

c $f_1(x) = \ln\left(\cos \frac{\pi x}{12} + 2\right)$

$$\therefore f_1'(x) = \frac{-\frac{\pi}{12} \sin \frac{\pi x}{12}}{\cos \frac{\pi x}{12} + 2}$$

M1A1

$$\therefore f_1''(x) = \frac{-\frac{\pi^2}{144} \cos \frac{\pi x}{12} \left(\cos \frac{\pi x}{12} + 2\right) + \frac{\pi}{12} \sin \frac{\pi x}{12} \left(-\frac{\pi}{12} \sin \frac{\pi x}{12}\right)}{\left(\cos \frac{\pi x}{12} + 2\right)^2}$$

M1A1

$$\therefore f_1''(x) = 0 \quad \text{when} \quad -\frac{\pi^2}{144} \left(\cos^2\left(\frac{\pi x}{12}\right) + \sin^2\left(\frac{\pi x}{12}\right) + 2 \cos \frac{\pi x}{12}\right) = 0$$

M1

$$\therefore -\frac{\pi^2}{144} (1 + 2 \cos \frac{\pi x}{12}) = 0$$

$$\therefore \cos \frac{\pi x}{12} = -\frac{1}{2}$$

A1

$$\therefore \frac{\pi x}{12} = \pm \frac{2\pi}{3} \quad \{\text{for } -12 \leq x \leq 12\}$$

$$\therefore x = \pm 8$$

A1

$$\text{Now } f'(-8) = \frac{\frac{\pi}{12} \sin\left(-\frac{2\pi}{3}\right)}{\cos\left(-\frac{2\pi}{3}\right) + 2} = \frac{-\frac{\pi}{12} \left(-\frac{\sqrt{3}}{2}\right)}{\frac{3}{2}} = \frac{\pi}{12\sqrt{3}}$$

A1

\therefore the maximum gradient of the road is $\frac{\pi}{12\sqrt{3}}$ at distance 8 m from the centre of the bridge, whichever side you are approaching from.

- d** Since the defining functions are both symmetric about the y -axis,

shaded area

$$= 2 \left(\int_0^k (f_1(x) - f_2(x)) dx + \int_k^{12} (f_1(x) - (-4)) dx \right) \quad \text{M1A1A1}$$

$$\approx 2 \left(\int_0^{9.12} (\ln(\cos \frac{\pi x}{12} + 2) - 2 \ln(\cos \frac{\pi x}{12} + 1) + 2 \ln 2) dx + \int_{9.12}^{12} (\ln(\cos \frac{\pi x}{12} + 2) + 4) dx \right)$$

$$\approx 2(17.66 + 11.77)$$

$$\approx 58.9 \text{ m}^2$$

A1

Total [19 marks]

TRIAL EXAMINATION 5

PAPER 1

Section A

1

x	A	5	7	9
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{9}$	p

a $p = 1 - \left(\frac{1}{3} + \frac{1}{9} + \frac{4}{9}\right)$
 $= 1 - \frac{8}{9}$
 $= \frac{1}{9}$

M1

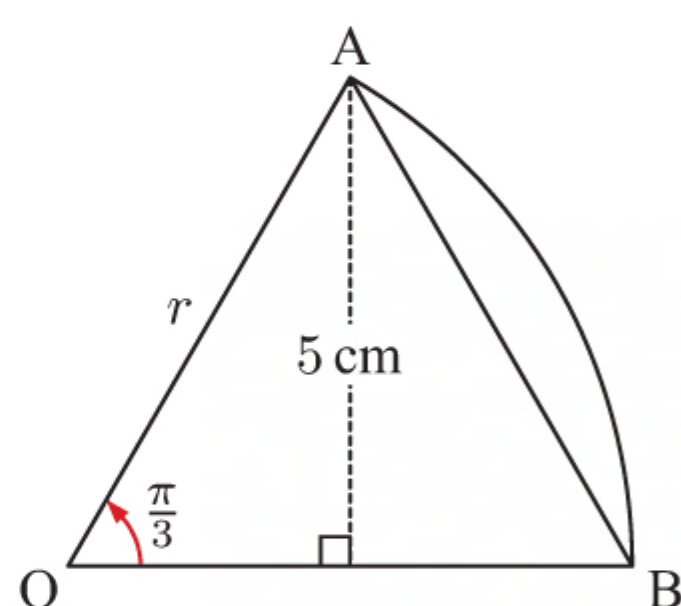
A1

b $E(X) = 6$
 $\therefore \frac{A}{3} + \frac{5}{9} + \frac{28}{9} + 1 = 6$
 $\therefore \frac{33 + 3A}{9} = 5$
 $\therefore 33 + 3A = 45$
 $\therefore 3A = 12$
 $\therefore A = 4$

M1A1

A1

Total [5 marks]

2 a

$$\sin \frac{\pi}{3} = \frac{5}{r}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{5}{r}$$

$$\therefore r = \frac{10}{\sqrt{3}}$$

$$\therefore r = \frac{10\sqrt{3}}{3} \text{ cm}$$

M1

A1

AG

b $P = 2r + r\theta$
 $= 2\left(\frac{10\sqrt{3}}{3}\right) + \frac{10\sqrt{3}}{3} \times \frac{\pi}{3}$
 $= \left(\frac{20\sqrt{3}}{3} + \frac{10\sqrt{3}\pi}{9}\right) \text{ cm}$

M1

A1

A1

Total [5 marks]

3 a $P(A | B) = \frac{P(A \cap B)}{P(B)}$
 $\therefore P(A \cap B) = \frac{1}{4} \times \frac{2}{5}$
 $= \frac{1}{10}$

M1

A1

b $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $= \frac{3}{10} + \frac{2}{5} - \frac{1}{10}$
 $= \frac{3 + 4 - 1}{10}$
 $= \frac{3}{5}$

M1

A1

c $P(A) \neq P(A | B) \therefore \text{not independent}$
 or $P(A) \times P(B) \neq P(A \cap B)$ as $\frac{3}{10} \times \frac{2}{5} \neq \frac{1}{10}$
 $\therefore \text{not independent}$

R1

R1

Total [5 marks]

$$4 \quad \mathbf{a} \quad \frac{4}{a} = 2$$

$$\therefore a = 2$$

A1

$$2\left(\frac{1}{2}\right) + b = 0$$

$$\therefore b = -1$$

A1

$$\mathbf{b} \quad y = \frac{4x-1}{2x-1}$$

$$\therefore \text{the inverse function is } x = \frac{4y-1}{2y-1}$$

M1

$$\therefore x(2y-1) = 4y-1$$

$$\therefore 2xy - x = 4y - 1$$

$$\therefore 2xy - 4y = x - 1$$

$$\therefore y(2x-4) = x-1$$

M1

$$\therefore y = \frac{x-1}{2x-4}$$

$$\therefore f^{-1}(x) = \frac{x-1}{2x-4}$$

A1

$$\mathbf{c} \quad f^{-1}(x) \text{ has asymptotes } x = 2 \text{ and } y = \frac{1}{2}.$$

$$\therefore \text{domain} = \{x \mid x \neq 2\}$$

A1

$$\text{range} = \{y \mid y \neq \frac{1}{2}\}$$

A1

Total [7 marks]

$$5 \quad \mathbf{a} \quad (2n+1)^3 = 1(2n)^3 + 3(2n)^2 + 3(2n) + 1$$

M1A1

$$= 8n^3 + 12n^2 + 6n + 1$$

A1

$$\mathbf{b} \quad k^3 + 3k^2 - k = (2n+1)^3 + 3(2n+1)^2 - (2n+1), \quad n \in \mathbb{Z}$$

M1

$$= 8n^3 + 12n^2 + 6n + 1 + 3(4n^2 + 4n + 1) - 2n - 1$$

$$= 8n^3 + 12n^2 + 6n + 1 + 12n^2 + 12n + 3 - 2n - 1$$

$$= 8n^3 + 24n^2 + 16n + 3$$

A1

$$= 8(n^3 + 3n^2 + 2n) + 3$$

A1

$$\text{where } A = n^3 + 3n^2 + 2n \in \mathbb{Z}$$

$$\therefore \text{if } k \text{ is an odd number, } k^3 + 3k^2 - k \text{ can be written as } 8A + 3 \text{ which is 3 more than a multiple of 8.}$$

R1

Total [7 marks]

$$6 \quad \text{Let } u = e^x - 1$$

M1

$$\therefore \frac{du}{dx} = e^x$$

A1

$$\therefore \int \frac{e^x}{(e^x-1)^2} dx = \int \frac{1}{u^2} \frac{du}{dx} dx$$

$$= \int \frac{1}{u^2} du$$

M1

$$= -\frac{1}{u} + c$$

A1

$$= -\frac{1}{e^x-1} + c$$

A1

Total [5 marks]

Section B

$$7 \quad \mathbf{a} \quad f(x) = kx(x+1)^2$$

$$\therefore f'(x) = k(x+1)^2 + 2(x+1)kx$$

M1A1

$$= k(x^2 + 2x + 1) + 2kx(x+1)$$

$$= kx^2 + 2kx + k + 2kx^2 + 2kx$$

A1

$$= 3kx^2 + 4kx + k$$

AG

- b** Turning points occur when $3kx^2 + 4kx + k = 0$ A1
 $\therefore k(3x^2 + 4x + 1) = 0$
 $\therefore k(3x + 1)(x + 1) = 0$ A1
 $\therefore x = -\frac{1}{3} \text{ or } -1 \quad \{k > 0\}$ AG

Turning points are at $x = -\frac{1}{3}$ and $x = -1$.

$$\begin{aligned} f\left(-\frac{1}{3}\right) &= -\frac{k}{3}\left(\frac{2}{3}\right)^2 \\ &= -\frac{4k}{27} \end{aligned}$$

$$f(-1) = 0$$

The turning points are $\left(-\frac{1}{3}, -\frac{4k}{27}\right)$ and $(-1, 0)$. A1A1

- c** $f'(x) = 3kx^2 + 4kx + k$
 $\therefore f''(x) = 6kx + 4k$ A1

$$\begin{aligned} f''\left(-\frac{1}{3}\right) &= -2k + 4k \\ &= 2k > 0 \quad \{k > 0\} \quad \therefore \text{minimum at } \left(-\frac{1}{3}, -\frac{4k}{27}\right) \end{aligned} \quad \text{A1}$$

$$\begin{aligned} f''(-1) &= -6k + 4k \\ &= -2k < 0 \quad \{k > 0\} \quad \therefore \text{maximum at } (-1, 0) \end{aligned} \quad \text{A1}$$

- d** $f(x)$ is strictly increasing when $f'(x) > 0$. (M1)

$f'(x)$ is a quadratic with $a > 0$.

$$\therefore f'(x) > 0 \text{ when } x < -1 \text{ and } x > -\frac{1}{3}. \quad \text{A1A1}$$

- e** $(-1, 0)$ is a local maximum, and the graph passes through $(0, 0)$

\therefore the graph lies below or on the x -axis for $-1 \leq x \leq 0$. (M1)

$$\begin{aligned} \therefore \text{area between } y = f(x) \text{ and the } x\text{-axis} &= -\int_{-1}^0 kx(x+1)^2 dx \quad \text{A1} \\ &= -\int_{-1}^0 (kx^3 + 2kx^2 + kx) dx \\ &= -\left[\frac{k}{4}x^4 + \frac{2k}{3}x^3 + \frac{k}{2}x^2\right]_{-1}^0 \quad \text{A1} \\ &= -\left(0 - \left(\frac{k}{4} - \frac{2k}{3} + \frac{k}{2}\right)\right) \\ &= \frac{k}{12} \quad \text{A1} \end{aligned}$$

$$\text{Now } \frac{k}{12} = 0.5, \text{ so } k = 6. \quad \text{A1}$$

Total [18 marks]

- 8 a** $\sin \theta = \frac{\sqrt{3}}{5}$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\therefore \frac{3}{25} + \cos^2 \theta = 1 \quad \text{M1}$$

$$\therefore \cos^2 \theta = \frac{22}{25} \quad \text{A1}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{22}}{5}$$

But θ is obtuse, so $\cos \theta = -\frac{\sqrt{22}}{5}$. A1

- b** $\cos 2\theta = 2\cos^2 \theta - 1$ M1

$$= 2\left(\frac{22}{25}\right) - 1$$

$$= \frac{44}{25} - 1$$

$$= \frac{19}{25} \quad \text{A1}$$

$$\begin{aligned}
 \text{c } f(x) &= 1 + \log_2(\tan x) + \log_2(1 - \sin^2 x) \\
 &= \log_2 2 + \log_2\left(\frac{\sin x}{\cos x}\right) + \log_2(\cos^2 x) && \text{M1A1} \\
 &= \log_2\left(2 \times \frac{\sin x}{\cos x} \times \cos^2 x\right) && \text{M1} \\
 &= \log_2(2 \sin x \cos x) && \text{A1} \\
 &= \log_2(\sin 2x) && \text{AG}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } 1 + \log_2(\tan x) + \log_2(1 - \sin^2 x) &= -1 \\
 \therefore \log_2(\sin 2x) &= -1 && \text{M1} \\
 \therefore \sin 2x &= 2^{-1} \\
 \therefore \sin 2x &= \frac{1}{2} && \text{A1} \\
 \therefore 2x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \quad \{0 < 2x < 4\pi\} && \text{M1} \\
 \therefore x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12} && \text{A1A1}
 \end{aligned}$$

Total [14 marks]

$$\text{9 a } r = \frac{\log x}{3 \log x} = \frac{1}{3} \quad \text{M1A1}$$

$$\begin{aligned}
 \text{b } S_G &= \frac{3 \log x}{1 - \frac{1}{3}} && \text{M1A1} \\
 &= \frac{3 \log x}{\frac{2}{3}} \\
 &= \frac{9}{2} \log x && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } d &= \log\left(\frac{x}{3}\right) - \log x && \text{M1} \\
 &= \log x - \log 3 - \log x && \text{M1} \\
 &= -\log 3 && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } u_1 &= \log x, \quad d = -\log 3 \\
 S_{10} &= \frac{10}{2}(2 \log x - 9 \log 3) && \text{M1A1} \\
 &= 10 \log x - 45 \log 3 && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } 10 \log x - 45 \log 3 &= \frac{2}{9} \times \frac{9}{2} \log x && \text{M1} \\
 \therefore 10 \log x - 45 \log 3 &= \log x \\
 \therefore 9 \log x &= 45 \log 3 && \text{A1} \\
 \therefore \log x &= 5 \log 3 \\
 \therefore \log x &= \log 3^5 \\
 \therefore x &= 3^5 && \text{A1}
 \end{aligned}$$

Total [14 marks]

PAPER 2

Section A

$$\text{1 a } X \sim \text{B}(15, 0.35) \quad \text{(M1)}$$

$$\text{P}(X \geq 8) = 1 - \text{P}(X \leq 7) \approx 0.113 \quad \text{M1A1}$$

$$\begin{aligned}
 \text{b } \mu &= 15 \times 0.35 = 5.25 \\
 \text{and } \sigma &= \sqrt{15 \times 0.35(1 - 0.35)} = 1.8472 \dots && \text{M1A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \text{P}(3.4027 \dots < X < 7.0972 \dots) &= \text{P}(X \leq 7) - \text{P}(X \leq 3) && \text{M1} \\
 &\approx 0.714 && \text{A1}
 \end{aligned}$$

Total [7 marks]

2 a $s(t) = \int v(t) dt$

$$= \int (2t + 3)(5t^{\frac{3}{2}} + 8) dt$$

$$= \int (10t^{\frac{5}{2}} + 16t + 15t^{\frac{3}{2}} + 24) dt$$

$$= \frac{20}{7}t^{\frac{7}{2}} + 8t^2 + 6t^{\frac{5}{2}} + 24t + c$$

When $t = 0$, $s = 0 \Rightarrow c = 0$

$$\therefore s(t) = \frac{20}{7}t^{\frac{7}{2}} + 8t^2 + 6t^{\frac{5}{2}} + 24t$$

b $a = \frac{dv}{dt} = 25t^{\frac{3}{2}} + 16 + \frac{45}{2}t^{\frac{1}{2}}$

Hence, when $t = 0$, $a = 16 \text{ m/s}^2$

M1

A1

M1

A1

M1

A1

Total [6 marks]

3 $P(X < 80) = 0.24$ and $P(X > 120) = \frac{1}{3}$

$$\therefore P\left(Z < \frac{80 - \mu}{\sigma}\right) = 0.24 \quad \text{and} \quad P\left(Z > \frac{120 - \mu}{\sigma}\right) = \frac{1}{3}$$

Hence, using the GDC, we have that $\frac{80 - \mu}{\sigma} = -0.70630 \dots$ and $\frac{120 - \mu}{\sigma} = 0.43072 \dots$

Solving simultaneously gives $\mu \approx 105$ minutes and $\sigma \approx 35.2$ minutes.

A1

M1

M1

M1A1

Total [5 marks]

4 a $V = \pi r^2 h = 240$

$$\therefore h = \frac{240}{\pi r^2}$$

Now $A = 2\pi r h + \pi r^2$

$$= 2\pi r \left(\frac{240}{\pi r^2}\right) + \pi r^2$$

$$= \frac{480}{r} + \pi r^2$$

b $\frac{dA}{dr} = -\frac{480}{r^2} + 2\pi r$

A is minimised when $\frac{dA}{dr} = 0$

$$\therefore -\frac{480}{r^2} + 2\pi r = 0$$

$$\therefore 2\pi r = \frac{480}{r^2}$$

$$\therefore r = \sqrt[3]{\frac{240}{\pi}}$$

$$\therefore h = \frac{240}{\pi \left(\frac{240}{\pi}\right)^{\frac{2}{3}}} = \sqrt[3]{\frac{240}{\pi}} = r$$

A is minimised when $r = h$.

M1

M1

A1

AG

M1

M1

A1

M1AG

Total [7 marks]

5 $3(4^x) - 10(2^x) = -3$

$$\therefore 3(2^2)^x - 10(2^x) + 3 = 0$$

$$\therefore 3(2^x)^2 - 10(2^x) + 3 = 0$$

Let $y = 2^x$:

$$\therefore 3y^2 - 10y + 3 = 0$$

$$\therefore (3y - 1)(y - 3) = 0$$

$$\therefore y = 2^x = \frac{1}{3} \text{ or } 3$$

$$\therefore x = \log_2\left(\frac{1}{3}\right) \text{ or } \log_2 3$$

(accept $x = -\frac{\log 3}{\log 2}$ or $\frac{\log 3}{\log 2}$)

M1

A1

M1

A1

A1

Total [5 marks]

- 6 a** $f'(x) = 3x^2e^{-x} - x^3e^{-x}$ M1
 $= x^2e^{-x}(3 - x)$ A1
 Stationary points occur when $f'(x) = 0$ M1
 $\therefore x^2e^{-x}(3 - x) = 0$
 $\therefore x = 0$ or $x = 3$ R1
 So, $f(x)$ has two stationary points. (AG)
- b** Using technology, the only solution is $x \approx 3.59$. M1A1

Total [6 marks]

Section B

- 7 a** $h(x) = 3g(x - 1) - 2$ (M1)(A1)
 $= 3e^{2(x-1)} - 2$ A1
- b** The graph crosses the x -axis when $3e^{2(x-1)} - 2 = 0$ M1
 $\therefore e^{2(x-1)} = \frac{2}{3}$
 $\therefore 2(x - 1) = \ln\left(\frac{2}{3}\right)$ M1
 $\therefore x = 1 + \frac{1}{2}\ln\left(\frac{2}{3}\right)$ A1
 $h(x)$ crosses the x -axis at $\left(1 + \frac{1}{2}\ln\left(\frac{2}{3}\right), 0\right)$. A1
- c** $f'(x) = 3x^2 - 1$, so $f'(2) = 11$ M1
 \therefore gradient of the normal at $x = 2$ is $-\frac{1}{11}$. A1
 $f(2) = 6$
 \therefore equation of the normal is $y - 6 = -\frac{1}{11}(x - 2)$ M1
 $\therefore y = -\frac{1}{11}x + \frac{68}{11}$ A1
- d** $gf(x) = g(x^3 - x) = e^{2(x^3-x)}$ A1
 Using technology, the graphs of $y = x^3 - x$ and $y = e^{2(x^3-x)} - 1$ intersect at $x \approx -1.27$, $x = -1$, (M1)
 $x = 0$, and $x = 1$.
 \therefore the total area of the enclosed regions $\approx \int_{-1.27}^1 \left| (x^3 - x) - (e^{2(x^3-x)} - 1) \right| dx$ M1
 $\approx 0.596 \text{ units}^2$ A1

Total [15 marks]

- 8 a** The x -intercepts occur at $\left(\frac{1}{2}, 0\right)$ and $(-3, 0)$. A1
 The y -intercept occurs at $(0, -3)$. A1
- b** Axis of symmetry lies halfway between the two x -intercepts. (M1)
 The equation of the axis of symmetry is $x = -\frac{5}{4}$. A1
- c** When $x = -\frac{5}{4}$, $y = \left(-\frac{7}{2}\right)\left(\frac{7}{4}\right) = -\frac{49}{8}$
 \therefore the coordinates of the turning point are $\left(-\frac{5}{4}, -\frac{49}{8}\right)$. M1A1
- d** Two equal roots exist when $\Delta = (a + 1)^2 - 4(1)(a) = 0$ M1A1
 $\therefore a^2 + 2a + 1 - 4a = 0$
 $\therefore a^2 - 2a + 1 = 0$
 $\therefore (a - 1)^2 = 0$
 $\therefore a = 1$ A1
- e** The graphs intersect when $(2x - 1)(x + 3) = x^2 + 2x + 1$ M1
 $\therefore 2x^2 + 5x - 3 = x^2 + 2x + 1$ A1
 $\therefore x^2 + 3x - 4 = 0$
 $\therefore (x + 4)(x - 1) = 0$ A1
 which has solutions $x = -4$ and $x = 1$. AG
 Using technology, $f(x) < g(x)$ when $-4 < x < 1$. M1A1

Total [14 marks]

$$\begin{aligned}
 \mathbf{9 \quad a} \quad \tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\
 &= \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} && \mathbf{M1} \\
 &= \frac{2 \frac{\sin \theta}{\cos \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} \quad \left\{ \text{dividing top and bottom by } \cos^2 \theta \right\} && \mathbf{A1} \\
 &= \frac{2 \tan \theta}{1 - \tan^2 \theta} && \mathbf{AG}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \tan 135^\circ &= \frac{2 \tan 67.5^\circ}{1 - \tan^2 67.5^\circ} \\
 \therefore -1 &= \frac{2 \tan 67.5^\circ}{1 - \tan^2 67.5^\circ} && \mathbf{M1}
 \end{aligned}$$

Letting $X = \tan 67.5^\circ$, $X^2 - 1 = 2X$

$$\therefore X^2 - 2X - 1 = 0 \quad \mathbf{A1}$$

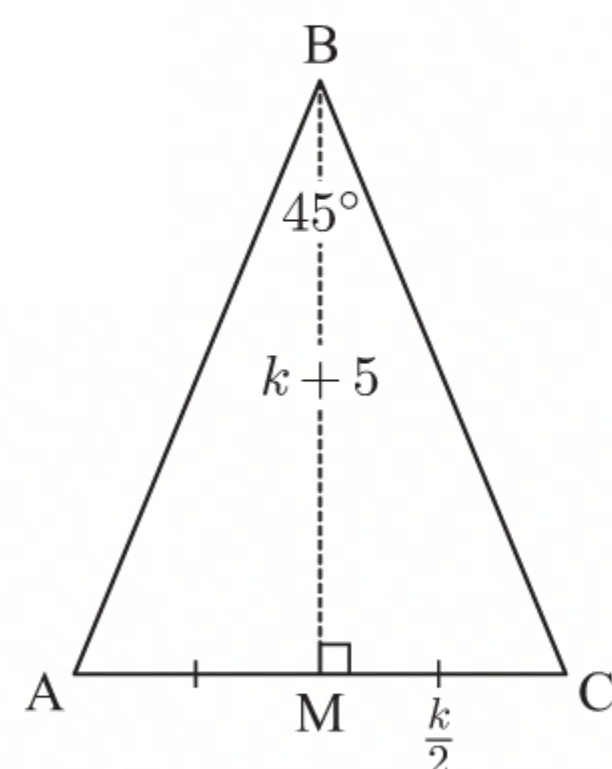
$$\therefore X = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore X = \frac{2 \pm 2\sqrt{2}}{2}$$

$$\therefore X = 1 \pm \sqrt{2} \quad \mathbf{A1}$$

But $\tan 67.5^\circ > 0$, so $\tan 67.5^\circ = 1 + \sqrt{2}$. **R1AG**

c i



Draw BM perpendicular to AC.

$$\therefore MC = \frac{k}{2} \quad \mathbf{M1}$$

$$\therefore \tan \widehat{BCA} = \frac{k+5}{\frac{k}{2}} \quad \mathbf{A1}$$

$$= \frac{2k+10}{k} \quad \mathbf{AG}$$

$$\mathbf{ii} \quad \widehat{BCA} = \left(\frac{180 - 45}{2} \right)^\circ = 67.5^\circ \quad \mathbf{M1}$$

$$\therefore \text{from } \mathbf{b}, \quad \frac{2k+10}{k} = 1 + \sqrt{2} \quad \mathbf{A1}$$

$$\therefore 2k + 10 = k + k\sqrt{2}$$

$$\therefore 10 = k(\sqrt{2} - 1)$$

$$\therefore k = \left(\frac{10}{\sqrt{2} - 1} \right) \times \left(\frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right) \quad \mathbf{M1}$$

$$= \frac{10\sqrt{2} + 10}{2 - 1}$$

$$= 10 + 10\sqrt{2} \quad \mathbf{A1}$$

$$\mathbf{iii} \quad \text{Area} = \frac{1}{2}k(k+5) \quad \mathbf{M1}$$

$$= \frac{1}{2}(10 + 10\sqrt{2})(15 + 10\sqrt{2})$$

$$= (5 + 5\sqrt{2})(15 + 10\sqrt{2}) \quad \mathbf{A1}$$

$$= 75 + 50\sqrt{2} + 75\sqrt{2} + 100$$

$$= (175 + 125\sqrt{2}) \text{ units}^2 \quad \mathbf{A1}$$

Total [15 marks]